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WHITNEY FORMS OF HIGHER DEGREE*

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The authors wish to dedicate this work to the memory of Professor Fausto Saleri

Abstract. Low-order Whitney elements are widely used for electromagnetic field problems. Higher-order approximations are receiving increasing interest, but their definition remains unduly complex. In this paper we propose a new simple construction for Whitney p-elements of polynomial degree higher than one that use only degrees of freedom associated to p-chains. We provide a basis for these elements on simplicial meshes and give a geometrical localization of all degrees of freedom. Properties of the higher-order Whitney complex are deeply investigated.

Key words. Whitney forms, simplicial meshes, high-order approximations

AMS subject classifications. 78M10, 65N30, 68U20

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1. Introduction.

Whitney elements on simplices [4, 15] are perhaps the most widely used finite elements in computational electromagnetics. They offer the simplest construction of polynomial discrete differential forms on simplicial complexes. Their associated degrees of freedom (dofs) have a very clear meaning as cochains and thus, give a recipe for discretizing physical balance laws, e.g., Maxwell's equations.

As interest grew for the use of high-order schemes, such as hk-finite element or spectral element methods (see [19] and [14] for a presentation of these methods), higher-order extensions of Whitney forms have become an important computational tool, appreciated for their better convergence and accuracy properties, as shown in [1, 2]. But, they are defined in different ways by different authors (see, e.g., [8, 9, 20]), with usually complex-looking formulas for the generating element basis functions, which make it difficult to decide whether the spaces thus generated coincide. The use of differential forms (see, e.g., [11, 12, 3]) has led to quite simple definitions of generalizations of both the spaces considered in [15] and their interdependency. However, it has remained unclear what kind of cochains such basis elements should be associated with: Can the corresponding dofs be assigned to precise geometrical elements of the mesh, just as, for instance, a degree of freedom for the space of Whitney 1-forms of polynomial degree one belongs to a specific edge? The current paper addresses this *localization* issue, that is, the relationship between dofs and measurable quantities, detailing the short presentation given in [16].

Why is this an issue? The existing constructions of high-order extensions of Whitney elements follow the traditional FEM path of using higher and higher "moments" to define the needed dofs [15]. As a result, such high-order finite p-elements in d dimensions may include dofs associated to q-simplices, with $p < q \leq d$, whose physical interpretation is obscure. Since a p-form can only be integrated over any p-dimensional manifold, what is the physical meaning of a dof associated to a q-simplex with $q > p$?

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To answer the question, we introduce an approach based on so-called small simplices, a set of subsimplices obtained by homothetic contractions of the original mesh ones, centered at mesh nodes (or more generally, when going up in degree, at particular points located on the edges, on the faces, and at the interior of each original simplex).

Each dof of high-order Whitney p-forms is then associated not with a single, specific small p-simplex, but with some linear combinations of small simplices, all of dimension p (the kind of objects called "p-chains" in homology). More precisely, the q-moments, with $p < q \leq d$, for finite p-elements of order k, are not integrals over domains of dimension q, but integrals of these p-elements of order k over suitable p-chains.

This statement has a dual side. If one takes as dofs for a p-form u its integrals over small p-simplices, then Whitney p-forms of order k which provide an approximation of u are suitable linear combinations of these p-elements of order k, with the same coefficients of the p-chains over the small p-simplices. (See [13] for another example of association of dofs with small p-cells—not p-simplices—with $p = 1$.)

Three key heuristic points underlie this construction: (i) High-order forms should satisfy a certain "partition of unity" property; (ii) They should pair up with integration domains of dimension p (which we shall build from "small" p-simplices, appropriate homothetic images of the mesh p-simplices); (iii) The spaces they span should constitute an exact sequence. On each tetrahedron, higher-order forms are here obtained as a product of Whitney forms of degree one [21] by suitable homogeneous monomials in the barycentric coordinate functions of the simplex.

Preliminary numerical tests in two dimensions with edge elements [17] show that the proposed shape functions can be used in practice and confirm optimal convergence rates with respect to both the maximal diameter h of the mesh elements and the approximation polynomial degree k. Other handy local bases for high-order elements are known [1, 2, 18] and widely used, which lead to better conditioned Galerkin matrices than ours, but our concern here is to provide an insight into the "geometrical nature" of high-order Whitney elements in the language of differential forms.

The paper is organized as follows. In section 2, some notations are introduced, the chain/cochain concepts are briefly recalled together with the definition of Whitney p-forms of degree one. Section 3 is the core of the paper, where the definition and some properties of the proposed Whitney p-forms of degree higher than one are stated. In this section we present a proof of exactness for the Whitney complex of polynomial degree one and higher. The paper ends with a few conclusions in section 4.

2. Algebraic tools and notation. In this section, we explain our notation and recall basic notions used in the exterior calculus of differential forms. We consider a three-dimensional domain Ω, but notions and proofs are valid in all dimensions. For all integrals, we omit specifying the integration variable when this can be done without ambiguity. We shall denote by $\int_\gamma u$ (resp., $\int_\Sigma u$) the circulation (resp., the flux) of a vector field u along the curve γ (resp., across the surface Σ). Moreover, we shall put emphasis on the maps $\gamma \to \int_\gamma u$ and $\Sigma \to \int_\Sigma u$, that is to say, the differential 1- and 2-form of degree 1 which one can associate with a given vector field u, and we use notations specific to exterior calculus, such as the exterior derivative d, as in the Stokes theorem (see [7] for a detailed exposition and [10] for some basic notions in algebraic topology).

Let d be the ambient dimension. A p-simplex s, $0 \leq p \leq d$, is the nondegenerate convex hull of $p + 1$ geometrically distinct points $n_0, \ldots, n_p$ of $\mathbb{R}^d$. The points

$n_0, \ldots, n_p$ are called vertices of s, and p is the dimension of the (oriented) p-simplex s, which we shall denote $s = \{n_0, \ldots, n_p\}$. Any $(p-1)$-simplex spanned by a subset of $\{n_0, \ldots, n_p\}$ is called $(p-1)$-face of $s = \{n_0, \ldots, n_p\}$. Labels n, e, f, v are used for nodes (0-simplices), edges (1-simplices), etc., each with its own orientation. Note that e (resp., f, v) is by definition an ordered couple (resp., triplet, quadruplet) of vertices, not merely a collection. For example, the edge $e = \{\ell, n\}$ is oriented from the node ℓ to n.

Given a domain $\Omega \subset \mathbb{R}^d$, a simplicial mesh $\mathtt{m}$ in Ω is a tessellation of $\overline{\Omega}$ by d-simplices, under the condition that any two of them may intersect along a common face of dimension $0 \le p \le (d-1)$. In dimension $d = 3$, which we shall assume when giving examples, this means along a common face, edge, or node, but in no other way. The sets of nodes, edges, faces, and volumes (i.e., tetrahedra) of the mesh $\mathtt{m}$ are denoted by $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{V}$, and the sets of nodes, edges, and faces of a volume v are denoted by $\mathcal{N}(v)$, $\mathcal{E}(v)$, $\mathcal{F}(v)$. When in need for more generic notation, we denote by $\mathcal{S}^p$ the set of p-simplices of $\mathtt{m}$, by $|\mathcal{S}^p|$ its cardinality, with similar notations when restricted to a given volume v.

A p-chain c, with $0 \le p \le d$, is an assignment to each p-simplex $s \in \mathcal{S}^p$ of a value $c^s \in \mathbb{R}$. This can be denoted by $c = \sum_{s \in \mathcal{S}^p} c^s\, s$. Let C_p be the set of all p-chains. If s is an oriented simplex, the elementary chain corresponding to s is the assignment $c^s = 1$ and $c^{s'} = 0$ for all $s' \neq s$. In what follows, we will use the same symbol s (or n, e, etc.) to denote the oriented simplex and the associated elementary chain. Note how this is consistent with the above expansion of c as a formal weighted sum of simplices. A p-chain can simply be stored as an array of dimension $|\mathcal{S}^p|$.

A p-cochain w (over $\mathtt{m}$) is the dual of a p-chain, that is to say, w is a linear mapping that takes p-chains (over $\mathtt{m}$) to $\mathbb{R}$. Since a chain is a linear combination of simplices, a cochain returns a linear combination of the values of that cochain on each involved simplex. For instance, given an array $\mathbf{b} = \{b_s : s \in \mathcal{S}^p\}$ of real numbers, we can define the p-cochain $c \to w(c) = \sum_{s \in \mathcal{S}^p} b_s\, c^s$ acting on p-chains c of the form $\sum_{s \in \mathcal{S}^p} c^s\, s$. So, the linear operation $w(c)$ translates into the duality product $\langle \mathbf{b}, \mathbf{c} \rangle$, where the vector $\mathbf{b}$ of size $|\mathcal{S}^p|$ represents the cochain w. A cochain corresponds to one value per simplex, and a p-cochain is evaluated on each oriented p-simplex. They are discrete analogues to differential forms. For instance, a 0-form can be evaluated at each point, a 1-form can be evaluated on each curve, a 2-form can be evaluated on each surface, etc. Now, if we restrict integration to take place only on the p-domain which is the union of the p-simplices in the mesh $\mathtt{m}$, we get a p-cochain. A differential p-form, let's call it b, generates a p-cochain of the above kind in a natural way: The map $c \to \sum_{s \in \mathcal{S}^p_m} c^s \int_s b$ is indeed a cochain, whose coefficients are the integrals $\int_s b$ of the differential form b on the p-simplices. They form an array $\mathbf{b}$, and the correspondence $b \to \mathbf{b}$ is called the de Rham map and denoted by $\mathcal{R}$ (mnemonic for "restriction"). Whitney forms go the other way: To an array $\mathbf{b}$ is associated the p-form $\sum_{s \in \mathcal{S}^p} b_s w^s$. The correspondence $\mathbf{b} \to \sum_{s \in \mathcal{S}^p} b_s w^s$ is the Whitney map, denoted here by $\mathcal{P}$ (mnemonic for "prolongation"). Such forms span a finite-dimensional subspace of the space of p-forms that we denote by W^p (with domain Ω and mesh $\mathtt{m}$ understood). Note that C_p and W^p are in duality via the bilinear bicontinuous map $\langle \cdot\, ,\, \cdot \rangle : W^p \times C_p \to \mathbb{R}$ defined by the pairing $\langle w\, ,\, c \rangle = \int_c w$.

The union of the $(p-1)$-faces of a p-simplex s is called boundary of s. The boundary operator ∂ takes a p-simplex s and returns the sum of all its $(p-1)$-faces f with coefficient 1 or -1 depending of whether the orientation of the $(p-1)$-face f matches or not with the orientation induced by that of the simplex s on f. Note that the boundary operator takes each p-simplex s and gives the signed sum of all its

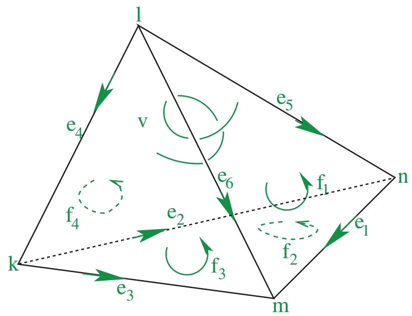

FIG. 2.1. *The oriented tetrahedron $v = \{l, n, m, k\}$ with oriented p-faces, $0 < p < d$.*

$(p-1)$-faces, that is, the boundary of a p-simplex s produces a $(p-1)$-chain, with coefficients equal to either 0, 1, or -1. The notion of boundary can be extended to p-chains by linearity, $\partial c = \partial(\sum_{s\in\mathcal{S}^p} c^s\, s) = \sum_{s\in\mathcal{S}^p} c^s\,\partial s$. Since the boundary operator is a linear mapping from the space of p-simplices to that of $(p-1)$-simplices, it can be represented by a matrix $\boldsymbol{\partial}$ of dimension $|\mathcal{S}^{p-1}| \times |\mathcal{S}^p|$, which is rather sparse, gathering the coefficients 0, -1, or $+1$. Note that in three dimensions, there are three nontrivial boundary operators: ∂_1 acting on edges, ∂_2 on triangles, ∂_3 on tetrahedra. The subscript is removed when there is no ambiguity, since the operator needed for a particular operation is indicated from the type of the operand (e.g., ∂_3 when ∂ applies to tetrahedra, etc.).

Example 2.1. By looking at Figure 2.1, the three incidence matrices are

$$\mathbf{G} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}, \mathbf{R} = \begin{pmatrix} -1 & 0 & 0 & 0 & -1 & 1 \\ -1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 \end{pmatrix}, \mathbf{D} = (1\ -1\ 1\ -1).$$

The boundary of the face $f_1 = \{l, m, n\}$ is $\partial_2 f_1 = -e_1 - e_5 + e_6$, which can be identified with the vector $(-1, 0, 0, 0, -1, 1)^t$ representing the coefficient in front of each edge. By repeating similar calculations for all simplices, one can readily conclude that the boundary operator ∂_2 (resp., ∂_1, ∂_3) is given by $\mathbf{R}^t$ (resp., $\mathbf{G}^t$, $\mathbf{D}^t$).

For $p > 0$, the exterior derivative of the $(p-1)$-form w is the p-form $\mathrm{d}w$ such that $\langle \mathrm{d}w,\ s\rangle = \langle w\ ,\ \partial s\rangle\ \forall s \in \mathcal{S}^p$. With this simple equation relating the evaluation of $\mathrm{d}w$ on a simplex s to the evaluation of w on the boundary of this simplex, the exterior derivative is readily defined. We can naturally extend the notion of evaluation of a differential form w on an arbitrary chain by linearity: $\int_{\sum_i c_i s_i} w = \sum_i c_i \int_{s_i} w$. Thus

$$\int_{\sum_i c_i s_i} \mathrm{d}w = \int_{\partial(\sum_i c_i s_i)} w = \int_{\sum_i c_i \partial s_i} w = \sum_i c_i \int_{\partial s_i} w.$$

The operator d is the dual of the boundary operator ∂. As a corollary of the boundary operator property $\partial \circ \partial = 0$, we have that $\mathrm{d} \circ \mathrm{d} = 0$. Since we used arrays of dimension $|\mathcal{S}^p|$ to represent a p-cochain, the operator d can be represented by a matrix $\mathbf{d}$ of dimension $|\mathcal{S}^p| \times |\mathcal{S}^{p-1}|$, $1 \le p \le d$. Again, we have one matrix for the exterior derivative operator for each simplex dimension. When a metric is introduced on the ambient affine space, the exterior derivative operator d stands for grad, curl, div, according to the value of p from 1 to 3, and it is represented by, respectively, $\mathbf{G}$, $\mathbf{R}$, $\mathbf{D}$, the incidence matrices of the mesh simplices. The symbol $\mathbf{d}_\sigma^s$ stands for the incidence matrix entry linking the $(p-1)$-simplex s to the p-simplex σ, $1 \le p \le d$. In particular, $\mathbf{d}_\sigma^s = 0$ if $s \notin \partial\sigma$, $\mathbf{d}_\sigma^s = 1$ if $s \in \partial\sigma$ and the orientation of s is in agreement with that induced by σ on s, $\mathbf{d}_\sigma^s = -1$ if $s \in \partial\sigma$ but the orientation of s is opposite to that induced by σ on s. Moreover, $\boldsymbol{\partial}$ is represented by $\mathbf{G}^t$ for $p = 1$, $\mathbf{R}^t$ for $p = 2$, and $\mathbf{D}^t$ for $p = 3$, as ∂ is the dual operator of d (see Figure 2.1). For instance, given $\mathbf{c} = \{c^e : e \in \mathcal{E}\}$, we have $\partial(\sum_{e\in\mathcal{E}} c^e\, e) = \sum_{n\in\mathcal{N}} (\boldsymbol{\partial}\,\mathbf{c})^n n$, with $\boldsymbol{\partial} = \mathbf{G}^t$ in this case.

A notational point before carrying on: When $e = \{m, n\}$ and $f = \{l, m, n\}$, we denote the node l by $f - e$. Thus $\lambda_l(x)$, the barycentric coordinate of a point x with respect to node l, can be also denoted by $\lambda_{f-e}(x)$. We can now state the following recursive definition of Whitney p-forms of polynomial degree one (as presented in [5]) with obvious generalization when $d > 3$.

Definition 2.1. *The differential Whitney p-form w^σ of polynomial degree 1 associated to the p-simplex σ is*

$$w^\sigma = \sum_{s\in\mathcal{S}^{p-1}} \mathbf{d}_\sigma^s\, \lambda_{\sigma-s} \mathrm{d}\, w^s, \qquad 1 \le p \le d, \tag{2.1}$$

with $w^n = \lambda_n$ for $p = 0$ and $W_1^p = span\{w^\sigma, \sigma \in \mathcal{S}^p\}$ $(\equiv W^p)$.

The forms w^e (resp., w^f, w^v) are indexed over the set of these couples (resp., triplets, quadruplets), thus we use e (resp., f, v) also as a label since it points to the same object in both cases. When a metric (i.e., a scalar product) is introduced on the ambient affine space, differential forms are in correspondence with scalar and vector fields (called "proxy fields"—metric dependent, of course). The coefficients of p-cochains just described are then seen to be the standard dofs of such scalar and vector fields, as obtained when Whitney finite elements are used to approximate them.

For the high-order case, multi-index notations are used, and the integer k will be no more a vertex label but a multi-index weight. Let $\mathbf{k}$, boldface, be the array $(k_0, \ldots, k_d)$ of $d+1$ integers $k_i \ge 0$, and denote by k its weight $\sum_{i=0}^d k_i$. The set of multi-indices $\mathbf{k}$ with $d+1$ components and of weight k is denoted $\mathcal{I}(d+1, k)$. We then adopt the following definition.

Definition 2.2. *Given $\mathbf{k} \in \mathcal{I}(d+1, k)$, we set $\lambda^{\mathbf{k}} = \Pi_{i=0}^d (\lambda_i)^{k_i}$.*

Let us denote by $\mathbb{P}_k(\Sigma)$ the vector space of polynomials defined on a domain $\Sigma \subset \mathbb{R}^d$ in d variables of degree $\le k$ and by $\widetilde{\mathbb{P}}_k(\Sigma)$ the subspace of $\mathbb{P}_k(\Sigma)$ of homogeneous polynomials of degree k. Homogeneous polynomials of degree k in the $d+1$ barycentric coordinates are in 1–to–1 correspondence with polynomials of degree $\le k$ in the d Cartesian ones. For this reason, we can say that $\mathbb{P}_k(v) = span(\lambda^{\mathbf{k}})_{\mathbf{k}\in\mathcal{I}(d+1,k)}$ on each volume v.

3. Whitney elements of higher degree. In order to define higher-order Whitney elements, we do not follow the traditional FEM path of using higher and higher moments to define the needed dofs. We follow a new approach based on the introduction of a set of subsimplices, called "small simplices," defined by means of a particular geometrical transformation in each mesh volume, the $\tilde{\mathbf{k}}$ map.

3.1. The $\tilde{\mathbf{k}}$ map. Let us focus on one mesh volume v and consider the following geometric transformation.

Definition 3.1. *To each multi-integer $\mathbf{k} \in \mathcal{I}(d+1, k)$ corresponds a map, denoted by $\tilde{\mathbf{k}}$, from v into itself. Let $\tilde{k}_i$ denote the affine function that maps $[0, 1]$ onto $[k_i/(k+$*

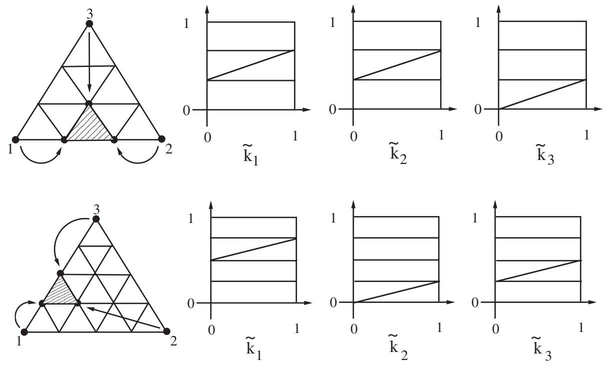

FIG. 3.1. *The mapping* $\tilde{\mathbf{k}}$ *associated to* $\mathbf{k} = (1, 1, 0)$ *with* $k = 2$ *(top), and* $\mathbf{k} = (2, 0, 1)$ *with* $k = 3$ *(bottom), and the (dashed) triangle* $\tilde{\mathbf{k}}(v)$.

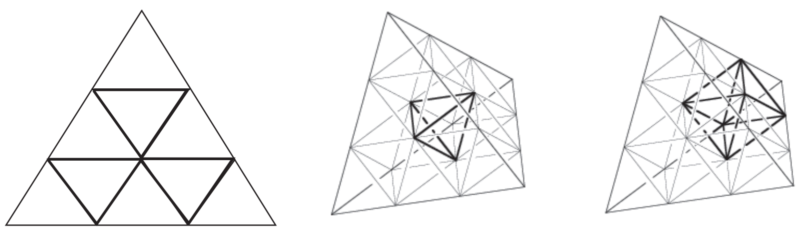

FIG. 3.2. *The images* $\tilde{\mathbf{k}}(v)$ *for a mesh triangle and tetrahedron* v *for* $k = 2$ *and (in thick line) examples of "holes"* s*, not congruent to* v*, such as inverted triangles, inverted tetrahedra, or octahedra.*

$1), (1 + k_i)/(k + 1)]$. *If* $\lambda_i(x)$*,* $0 \le i \le d$*, are the barycentric coordinates of point* $x \in v$ *with respect to the vertex* n_i *of* v*, its image* $\tilde{\mathbf{k}}(x)$ *has barycentric coordinates* $\tilde{k}_i(\lambda_i(x)) = (\lambda_i(x) + k_i)/(k + 1)$.

Geometrically, this map is a homothety, more precisely the transformation of space that contracts distances by a factor $k + 1$ with respect to the fixed point o of barycentric coordinates k_i/k (cf. Figure 3.1 for two examples). Let us set $r = 1/(k+1)$. If $\mathcal{H}(r, o)$ denotes the homothety of factor r with respect to the fixed point o, then the map $\tilde{\mathbf{k}}$ can also be seen as composition of elementary (i.e., with respect to the volume vertices) homotheties, as follows:

$$\tilde{\mathbf{k}} = \mathcal{H}\left(\frac{1}{k_0 + 1}, a_0\right) \circ \mathcal{H}\left(\frac{k_0 + 1}{k_0 + k_1 + 1}, a_1\right) \circ \cdots \circ \mathcal{H}\left(\frac{k_0 + \cdots + k_{d-1} + 1}{k + 1}, a_d\right).$$

Note that $\tilde{\mathbf{k}}(v)$ for all possible $\mathbf{k} \in \mathcal{I}(d + 1, k)$ are congruent by translation and homothetic to v. They don't pave v, and the holes left are not necessarily homothetic to v. As an example, take $k = 2$: for $d = 2$ (cf. Figure 3.2 (left)), the holes left are three small triangles not homothetic to v; for $d = 3$ (cf. Figure 3.2 (center and right)), the holes left are one central small inverted tetrahedron and four octahedra.

DEFINITION 3.2. *We call small* p*-simplices of* v*,* $0 \le p \le d$*, the images* $\tilde{\mathbf{k}}(S)$ *for all (big)* p*-simplices* $S \in \mathcal{S}^p(v)$ *and all* $\mathbf{k} \in \mathcal{I}(d + 1, k)$*, and denote them by* $s = \{\mathbf{k}, S\}$.

In short, all the p-simplices of $\tilde{\mathbf{k}}(v)$ for $\mathbf{k} \in \mathcal{I}(d + 1, k)$ are small p-simplices. As shown by Figure 3.3 (left), for $d = 3$ and $k = 1$, one has 4 small tetrahedra $\tilde{\mathbf{k}}(v)$ and

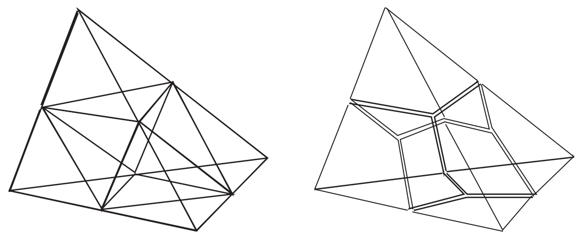

FIG. 3.3. *Small edges in one-to-one correspondence with the forms* $\lambda_n w^e$ *(left). Another way to define the small edges, according to* [13] *(right).*

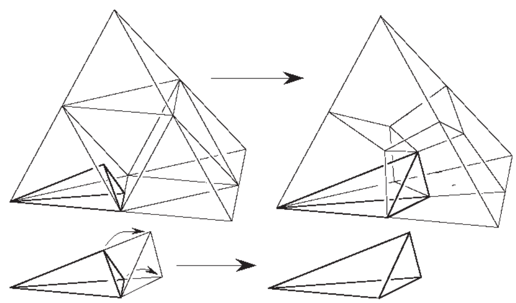

FIG. 3.4. *Different steps of the homotopy that shrinks each "hole" to a point. Each of the* 24 *tetrahedra similar to that on the bottom left goes onto its homologue on the bottom right by affine transformation. Two vertices are fixed whereas the other two are mapped as explained by the arrows.*

24 small edges. There are also 16 (and not 10) "small nodes"; the two homothetic images at each midedge should be considered distinct.

The right part of Figure 3.3 displays another way to define small cells, following [13], which it may be interesting to compare with ours. One passes from one to the other by shrinking each p-dimensional "hole" to its barycenter. Hence, in the displayed case, four solids—hexahedra—are deformations of the four small tetrahedra. The internal "hole" (here an octahedron) is thus reduced to four orthogonal segments, those connecting centers of opposite faces in the octahedron.

As suggested by Figure 3.4, the transformation thus involved is an association of local homotopies, one for each d-dimensional hole, that shrinks it to its barycenter, and at the same time shrinks the p-dimensional holes $0 \le p < d$, each to its own barycenter. Composing these local homotopies, one gets a global one which deflates all the holes, leaving a cell-complex which, being a subdivision of the initial one, has the same topological properties.

Since homology is preserved by homotopy [10], this suggests a link between the homology of the initial simplicial paving, i.e., of the domain Ω, and the homology of the complex of small simplices. This may look surprising, since the complex of small simplices is actually a union of *disjoint* simplicial complexes, one around each node, if $k = 1$. Moreover, each of these complexes, being the homothetic image of the cluster of simplices around the homothety center, has trivial homology. How these two homologies relate is therefore an interesting issue, which we briefly investigate in section 3.4 when we address the "exact sequence property" for higher-order Whitney forms.

3.2. Higher-order *p*-elements. Whitney p-elements of higher degree in each volume v are associated to the geometrical partition in v defined by the $\tilde{\mathbf{k}}$ map for all possible multi-indices $\mathbf{k} \in \mathcal{I}(d+1,k)$.

DEFINITION 3.3. *Whitney p-forms of polynomial degree $k+1$ in a volume v are the $w^s = \lambda^{\mathbf{k}} w^S$, where s is a pair $\{\mathbf{k}, S\}$, made of a multi-index $\mathbf{k} \in \mathcal{I}(d+1,k)$ and a (big) p-simplex $S \in \mathcal{S}^p(v)$ and w^S is the Whitney p-form of polynomial degree 1 associated to S (see Definition 2.1). The space of Whitney p-forms of polynomial degree $k+1$ in v is $W^p_{k+1}(v) = span\{w^s \,:\, s = \{\mathbf{k}, S\},\, \mathbf{k} \in \mathcal{I}(d+1,k),\, S \in \mathcal{S}^p(v)\}$.*

The recipe given in Definition 3.3 for Whitney p-forms of higher polynomial degree is simple: for W^p_2, attach to p-simplices $S \in \mathcal{S}^p$ products $\lambda_n\, w^S$, where n spans $\mathcal{N}$ and $w^S \in W^p_1$. For W^p_3, attach to p-simplices $S \in \mathcal{S}^p$ products $\lambda_n \lambda_m\, w^S$, where n, m span $\mathcal{N}$ and $w^S \in W^p_1$, etc. In other words, for W^p_2, consider all products $\lambda^{\mathbf{k}} w^S$, where $w^S \in W^p_1$ and $\{\mathbf{k}, S\}$ is a small p-simplex with $\mathbf{k} \in \mathcal{I}(d+1,1)$ and $S \in \mathcal{S}^p$, etc.

We now state some properties of the p-forms of Definition 3.3. Elements in W^p_{k+1} are defined as products between an element in W^p_1 and the continuous function $\lambda^{\mathbf{k}}$, $\mathbf{k} \in \mathcal{I}(d+1,k)$. Therefore, elements in W^p_{k+1} enjoy the same conformity properties as those in W^p_1.

Recall that barycentric functions sum to 1, thus forming a "partition of unity": $\sum_{n\in\mathcal{N}} w^n = 1$. A similar property holds for 1- and 2-forms. Going back, for a while, to the standard vector formalism to explain this, let us denote by $\mathbf{w}^e$ the vector field associated to w^e and $\mathbf{w}^f$ the one associated to w^f. By $\mathtt{vec}(e)$ and $\mathtt{vec}(f)$ we mean the vector that subtends edge e (the vector of modulus length(e) parallel to the edge e) and the vectorial area of face f (the vector of modulus area(f) orthogonal to f, pointing to the side prescribed by the right-hand rule).

PROPOSITION 3.4. *At all points x, for all vectors $\mathbf{v}$,*

$$\sum_{e\in\mathcal{E}} (\mathbf{w}^e(x)\cdot\mathbf{v})\,\mathtt{vec}(e) = \mathbf{v}, \qquad \sum_{f\in\mathcal{F}} (\mathbf{w}^f(x)\cdot\mathbf{v})\,\mathtt{vec}(f) = \mathbf{v}. \tag{3.1}$$

The p-forms w^s of Definition 3.3 constitute, in a similar way, a partition of unity.

Proof. The first relation (3.1) results from the *chain* equality $xy = \sum_{e\in\mathcal{E}} \langle w^e, xy\rangle e$ by letting $\mathtt{vec}$ act on it. We replace w^e by $\mathbf{w}^e$, then xy by $\mathtt{vec}(xy)$, and e by $\mathtt{vec}(e)$. Similarly, the second relation (3.1) results from the chain identity $xyz = \sum_{f\in\mathcal{F}} \langle w^f, xyz\rangle f$ by replacing w^f by $\mathbf{w}^f$, then xyz by its vector area $\mathtt{vec}(xyz)$, and f by its vector area $\mathtt{vec}(f)$. Relations for the high-order p-forms are obtained by iterating on the weight k of the multi-index $\mathbf{k}$, the multiplication by $\lambda^{\mathbf{k}}$, and the sum over k on both sides of (3.1). ☐

PROPOSITION 3.5. *For any p-simplex, a relation among the Whitney forms associated to its faces holds. For any edge e, face f, and tetrahedron v, we have, respectively,*

$$\sum_{n\in\mathcal{N}} \mathbf{G}^n_e\, \lambda_{e-n}\, w^n = 0, \qquad \sum_{e\in\mathcal{E}} \mathbf{R}^e_f\, \lambda_{f-e}\, w^e = 0, \qquad \sum_{f\in\mathcal{F}} \mathbf{D}^f_v\, \lambda_{v-f}\, w^f = 0. \tag{3.2}$$

The p-forms w^s of Definition 3.3 are generators of W^p_{k+1} but are not linearly independent.

Proof. The first identity is obvious: we get $-\lambda_n\,\lambda_m + \lambda_m\,\lambda_n = 0$ for the edge $e = \{m, n\}$. To prove the second identity, we replace w^e by its expression given in (2.1), and we get $\sum_e \mathbf{R}^e_f\, \lambda_{f-e}\, w^e = \sum_{n,e} \lambda_{f-e}\lambda_{e-n}\, \mathbf{R}^e_f\, \mathbf{G}^n_e\, \mathrm{d}w^n = 0$ since $\mathbf{RG} = \mathbf{0}$ and $\lambda_{f-e}\lambda_{e-n}$ is the same for all e in ∂f. The third identity can be proved similarly, thanks to the fact that $\mathbf{DR} = \mathbf{0}$. Due to relations (3.2), there exists a combination with nonzero coefficients of the forms $\lambda^{\mathbf{k}} w^S$, with $\mathbf{k} \in \mathcal{I}(d+1,1)$ and $S \in \mathcal{S}^p$, that gives zero. Iterating on the weight k of the multi-index $\mathbf{k}$, we have the linear dependency of the high-order forms. ∎

Dofs on a simplex for a given finite element approximation space W usually satisfy three conditions: (1) unisolvence, that is, there must be a one-to-one correspondence between the values of the dofs and the function in the finite element space W; (2) invariance, that is, dofs should be invariant under canonical transformations of differential forms accompanying a transformation of the reference simplex. This ensures that the arbitrary choice of a reference element has no impact on the final finite element space W; (3) locality, that is, the global dofs obtained from the local ones provide sufficient "cement" between adjacent elements to enforce conformity of the finite element approximation in W.

The above three conditions are verified by the Whitney elements of polynomial degree one. The dof $\mathbf{v}_S$ is the integral of the p-form $\sum_{S'} \mathbf{v}_{S'} w^{S'}$ over the p-simplex S, and this guarantees the invariance property. The square matrix $A_S^{S'} = \langle w^S, S' \rangle$ is the identity. Concerning unisolvence, if one takes p-simplices as basis elements for the vector space of p-chains, Whitney p-forms make a basis for its dual. We will say "p-simplices S and Whitney p-forms are in duality" to express this. Note that the integrals are evaluated over suitable subsimplex (edges for $p = 1$, faces for $p = 2$, etc.) on the surface of an element, and this guarantees conformity of the corresponding finite element approximation.

It is less evident to see that the above three conditions are also verified by the higher-order forms defined in Definition 3.3. Indeed, the small p-simplex $\{\mathbf{k}, S\}$ and p-form $\lambda^{\mathbf{k}}\, w^S$ are no more in duality. The square matrix $A_{\{\mathbf{k},S\}}^{\{\mathbf{k}',S'\}} = \langle \lambda^{\mathbf{k}}\, w^S, \{\mathbf{k}', S'\} \rangle$ is not the identity (cf. Table 3.1 for an example in two dimensions with $k = 1$ and $p = 1$). For the unisolvence, the relevant fact is that the subdomains of dimension p such that integrals over them of $\sum_{\mathbf{k},S} \mathbf{v}_{\mathbf{k}S} \lambda^{\mathbf{k}}\, w^S$ determine the $\mathbf{v}_{\mathbf{k}S}$ are in one-to-one correspondence with the forms $\lambda^{\mathbf{k}}\, w^S$. These p-dimensional subdomains do not coincide with the small p-simplices, but with linear combinations of small p-simplices with coefficients given by the pseudoinverse matrix A^+. Note that contrary to what occurs with standard FEM paths towards higher orders, the p-forms $\lambda^{\mathbf{k}}\, w^S$ are integrated over the above subdomains, which are manifolds of dimension exactly equal to p. If on the one hand, we lose the duality property (in part due to the simplicity of our high-order form definition), we keep on integrating a p-form over a p-chain. We still have as dofs the integrals of differential forms over simplices, yielding the invariance property. The high-order Whitney p-forms are here defined as products of Whitney p-forms of polynomial degree one and homogeneous polynomials in the barycentric coordinates. The forms in W_{k+1}^p thus conserve the same kind of continuity of those in W_1^p, namely, they are continuous along the direction of the (small) p-simplex (i.e., tangential continuity for 1-forms, normal continuity for 2-forms). For the locality condition, ampler work is needed, as follows.

Let us analyze in detail the basis we found out for W_{k+1}^1 when $d = 3$ in a volume $v = \{n, l, m, i\}$. The set $\mathcal{B}$ of basis functions for W_{k+1}^1 can be partitioned in three subsets, namely $\mathcal{B}_e$, $\mathcal{B}_f$, and $\mathcal{B}_v$. Note that $\mathcal{B}_f$ is empty for $k = 0$ and $\mathcal{B}_v$ is empty for $k = 0, 1$. So, we assume that $k = 2$, i.e., the minimum value for k for which the three subsets $\mathcal{B}_e$, $\mathcal{B}_f$, and $\mathcal{B}_v$ are not empty. Then,

$$\begin{aligned}
\mathcal{B}_e &= \textstyle\bigcup_{\{n,m\}\in\mathcal{E}(v)} \{\lambda_r\, \lambda_s\, w^{\{n,m\}}, \quad r, s \in \{n, m\}\}, \quad \#\mathcal{B}_e = 18,\\
\mathcal{B}_v &= \textstyle\bigcup_{\{n,m\}\in\mathcal{E}(v)} \{\lambda_r\, \lambda_s\, w^{\{n,m\}}, \quad r, s \notin \{n, m\}\}, \quad \#\mathcal{B}_v = 6,\\
\mathcal{B}_f &= \textstyle\bigcup_{\{n,m\}\in\mathcal{E}(v)} \{\lambda_r\, \lambda_s\, w^{\{n,m\}} \notin \mathcal{B}_e \cup \mathcal{B}_v\}, \qquad \#\mathcal{B}_f = 36.
\end{aligned}$$

TABLE 3.1

The elements of the matrix $A_{\{\mathbf{k},E\}}^{\{\mathbf{k}',E'\}}$ (all multiplied by 16) over the triangle $\{n,l,m\}$ for $k=1$ (cf. Figure 3.5, left, for the small-edge indexing and orientation). This 9×9 matrix is not regular: its rank is 8, and the vector $(4,1,1,1,4,1,1,1,4)$ generates its kernel.

$A_{\{\mathbf{k},E\}}^{\{\mathbf{k}',E'\}}$	$m,\{n,l\}$	$n,\{n,l\}$	$l,\{n,l\}$	$m,\{l,m\}$	$n,\{l,m\}$	$l,\{l,m\}$	$m,\{m,n\}$	$n,\{m,n\}$	$l,\{m,n\}$
$\lambda^m w^{\{n,l\}}$	2	0	0	0	-1	0	0	0	-1
$\lambda^n w^{\{n,l\}}$	1	6	2	0	-2	0	0	0	-1
$\lambda^l w^{\{n,l\}}$	1	2	6	0	-1	0	0	0	-2
$\lambda^m w^{\{l,m\}}$	-2	0	0	6	1	2	0	0	-1
$\lambda^n w^{\{l,m\}}$	-1	0	0	0	2	0	0	0	-1
$\lambda^l w^{\{l,m\}}$	-1	0	0	2	1	6	0	0	-2
$\lambda^m w^{\{m,n\}}$	-2	0	0	0	-1	0	6	2	1
$\lambda^n w^{\{m,n\}}$	-1	0	0	0	-2	0	2	6	1
$\lambda^l w^{\{m,n\}}$	-1	0	0	0	-1	0	0	0	2

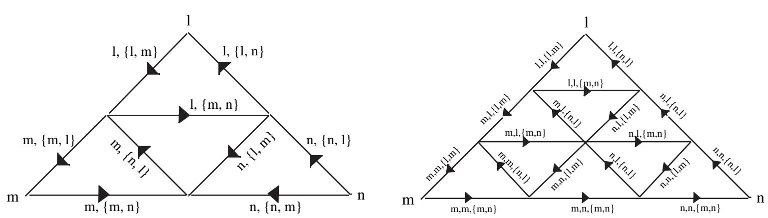

FIG. 3.5. *Indexing and orientation of the small edges in the triangle $\{n,l,m\}$ for the forms spanning W_2^1 (left) and W_3^1 (right) respectively. The symbol "$n,\{m,n\}$" denotes the small edge $\{\mathbf{k},\{m,n\}\}$ where $\mathbf{k}=(1,0,0)$ and is linked to the form $\lambda_n w^{\{m,n\}}$, etc. The symbol "$m,m,\{l,m\}$" denotes the small edge $\{\mathbf{k},\{l,m\}\}$ with $\mathbf{k}=(2,0,0)$ and is linked to the form $\lambda_m^2 w^{\{l,m\}}$, etc.*

Functions in $\mathcal{B}$ verify 15 relations, three per face plus three for the internal tetrahedron not similar to t. The set $\mathcal{B}_v$ gives "interior" basis functions in the sense that the moments on the (big) edges and faces (see [15], Definition 4) are zero for these functions. Indeed, functions in $\mathcal{B}_v$ can be expressed as a difference of two terms, each of which has a factor of the form $\lambda_k\lambda_l\lambda_n\,\nabla\lambda_m$. This factor is zero on faces containing the vertex m, while on face $\{k,l,n\}$, its tangential component is zero due to the presence of $\nabla\lambda_m$. Hence, functions in $\mathcal{B}_v$ have zero face and edge moments. The d-moments for p-elements of order k (see, e.g., [15] Definition 4, for $p=1$) are integrals of the p-forms $\lambda^{\mathbf{k}}w^S$ in $\mathcal{B}_v$ over p-chains linear combinations of the associated small p-simplices $\{\mathbf{k},S\}$ with coefficients given by the pseudoinverse matrix A^+.

The set $\mathcal{B}_f$ gives "face" basis functions in the sense that these functions have edge dofs equal to zero. Indeed, these functions are of the form $\lambda_n^\alpha\lambda_m^\beta\lambda_l^\gamma(\lambda_l\,w^{\{m,n\}})$ for some nonnegative integer powers α, β, and γ, with $\alpha+\beta+\gamma=1$. It is easily verified that the tangential components of the function $\lambda_l\,w^{\{m,n\}}$ are zero on all faces except face $\{l,m,n\}$. Its tangential component is zero on all edges of the tetrahedron. The $(d-1)$-moments for p-elements of order k (see, e.g., [15] Definition 4, for $p=1$) are again integrals of the p-forms $\lambda^{\mathbf{k}}w^S$ in $\mathcal{B}_f$ over p-chains suitable linear combinations of the associated small p-simplices $\{\mathbf{k},S\}$.

The remaining functions, those in $\mathcal{B}_e$, are "edge" basis functions. These functions take the form $\lambda_n^\alpha \lambda_m^\beta w^{\{m,n\}}$. Their tangential component is nonzero on the edge $\{m, n\}$ but vanishes on every other edge of the tetrahedron. Note that if one picks up a face, let us say $f = \{l, m, n\}$, the component $v_{|f}$ of a vector v is uniquely defined by the dofs associated to the edges of f and to f itself.

3.3. Properties of the first-order Whitney complex. The properties (nature of the dofs, continuity, partition of unity, etc.) that concern spaces W_1^p as taken one by one, for different values of p, are well known and were recalled in section 3.2. We now address properties of the structure made by all the W_1^p when taken together, namely, the first-order "Whitney complex," in view of the higher-order case.

Let W^p be the space generated by w^S with $\dim(S) = p$ and X^p the space of dof-arrays $(u_S)_{S\in\mathcal{S}^p}$ over the simplices S with $\dim(S) = p$. We have two operators $\mathcal{P}$ from dof-arrays to forms and $\mathcal{R}$ from forms to dof-arrays:

$$\begin{array}{llcc} \mathcal{P}: & X^p & \to & W^p \\ & \mathbf{u} & \to & \mathcal{P}\mathbf{u} = \sum_S u_S\, w^S, \\[1em] \mathcal{R}: & W^p & \to & X^p \\ & u & \to & (\mathcal{R}\, u) = \{\langle u, S\rangle \;:\; S \in \mathcal{S}^p\}. \end{array}$$

Note that $\mathcal{R}\,\mathcal{P} = id$ and $\mathcal{P}\,\mathcal{R} \to id$ when the mesh size h (i.e., the maximum diameter of all tetrahedra) tends to zero (see [6], section 1.6).

LEMMA 3.6. *Whitney spaces verify the inclusions* $\mathrm{d}\, W^{p-1} \subset W^p$*,* $p = 1$ *to* 3*. In particular,* $\mathrm{d}\, w^s = \sum_\sigma \mathbf{d}_\sigma^s\, w^\sigma$*.*

(Recall that $\mathbf{d}_\sigma^s$ is the incidence number linking the $(p-1)$-simplex s to the p-simplex σ.)

Proof. A general proof, for all d, can be given, thanks to the recursive Definition 2.1 of Whitney forms of polynomial degree one, but in three dimensions we gain clarity by reverting to the standard vector formalism. For $p = 1$, we have to show that $\nabla\, w^m = \sum_e \mathbf{G}_e^m\, w^e$. For the node m, if $\mathbf{G}_e^m \neq 0$, either $e = \{m, n\}$ or $e = \{n, m\}$, but in both cases, $\mathbf{G}_e^m\, w^e = w^n \nabla\, w^m - w^m \nabla\, w^n$, by definition of the incidence numbers $\mathbf{G}_e^m$. Therefore,

$$\begin{aligned} \sum_e \mathbf{G}_e^m\, w^e &= \sum_n (w^n \nabla\, w^m - w^m \nabla\, w^n) \\ &= \left(\sum_n w^n\right) \nabla\, w^m - w^m\, \nabla \left(\sum_n w^n\right) = \nabla\, w^m \end{aligned}$$

since $\sum_n w^n = 1$, hence $\nabla\, w^m \in W^1$, and hence the inclusion $d\, W^0 \subset W^1$ by linearity.

For $p = 2$, we must show that $\nabla \times w^e = \sum_f \mathbf{R}_f^e\, w^f$. If $e = \{m, n\}$, one has $\nabla \times w^e = 2\, \nabla\, w^m \times \nabla\, w^n$. Moreover, $\mathbf{R}_f^e \neq 0$ yields $f = \{\ell, m, n\}$ or $f = \{\ell, n, m\}$ for some ℓ. In both cases,

$$\mathbf{R}_f^e\, w^f = 2\left(w^\ell\, \nabla\, w^m \times \nabla\, w^n + w^m\, \nabla\, w^n \times \nabla\, w^\ell + w^n\, \nabla\, w^\ell \times \nabla\, w^m\right).$$

Therefore, summing over all faces insisting on e,

$$\begin{aligned}\sum_f \mathbf{R}^e_f\, w^f &= 2\,\sum_\ell \big(w^\ell\,\nabla\, w^m \times \nabla\, w^n + \\ &\qquad w^m\,\nabla\, w^n \times \nabla\, w^\ell + w^n\,\nabla\, w^\ell \times \nabla\, w^m\big) \\ &= 2\,\left(\sum_\ell w^\ell\right)\,\nabla\, w^m \times \nabla\, w^n + \\ &\qquad 2\,w^m\,\nabla\, w^n \times \nabla\,\left(\sum_\ell w^\ell\right) + 2\,w^n\,\nabla\,\left(\sum_\ell w^\ell\right) \times \nabla\, w^m \\ &= 2\,\nabla\, w^m \times \nabla\, w^n,\end{aligned}$$

being $\sum_\ell w^\ell = 1$, hence $\nabla \times w^e \in W^2$.

For $p = 3$, we need to show that $\nabla \cdot w^f = \sum_v \mathbf{D}^f_v\, w^v$, that is to say, $\nabla \cdot w^f \in W^3$. Let us remark that for $f = \{\ell, m, n\}$ and $v = \{\ell, k, m, n\}$, we have

$$\begin{aligned}w^f &= 2\,\left(w^\ell \nabla\, w^m \times \nabla\, w^n + w^m \nabla\, w^n \times \nabla\, w^\ell + w^n \nabla\, w^\ell \times \nabla\, w^m\right), \\ \nabla \cdot w^f &= 2\left(\nabla w^\ell \cdot \nabla\, w^m \times \nabla\, w^n + \nabla w^m \cdot \nabla\, w^n \times \nabla\, w^\ell + \nabla w^n \cdot \nabla\, w^\ell \times \nabla\, w^m\right) \\ &= 6 \det\,\left(\nabla w^\ell, \nabla\, w^m, \nabla\, w^n\right) = \frac{1}{vol(v)} = w^v.\end{aligned}$$

Two compensating changes of sign occur if $v = \{k, \ell, m, n\}$, the other orientation. ◻

LEMMA 3.7. *The following diagram is commutative:*

$$\begin{array}{ccc} W^p & \xrightarrow{\mathrm{d}} & W^{p+1} \\ \mathcal{R} \downarrow\uparrow \mathcal{P} & & \mathcal{P} \uparrow\downarrow \mathcal{R} \\ X^p & \xrightarrow{\mathbf{d}} & X^{p+1} \end{array}$$

Proof. The proof is done for $p = 1$ but can be generalized to other values of p. First, $\mathbf{d}\,\mathcal{R} = \mathcal{R}\,\mathrm{d}$ by the following steps:

$$\begin{aligned}(\mathbf{d}\,\mathcal{R}\,u)_f &= \sum_e \mathbf{R}^e_f (\mathcal{R}\,u)_e = \sum_e \mathbf{R}^e_f\,\langle u, e\rangle \\ &= \left\langle u, \sum_e \mathbf{R}^e_f\, e\right\rangle = \langle u, \partial f\rangle = \langle \mathrm{d}u, f\rangle = \quad (\mathcal{R}\,\mathrm{d}\,u)_f.\end{aligned}$$

Second, $\mathcal{P}\,\mathbf{d} = \mathrm{d}\,\mathcal{P}$:

$$\begin{aligned}\mathrm{d}\,\mathcal{P}\,\mathbf{u} &= \mathrm{d}\left(\sum_e u_e\, w^e\right) = \mathrm{d}\left(\sum_e \langle u, e\rangle w^e\right) \\ &= \sum_e \langle u, e\rangle \mathrm{d}w^e = \sum_e \langle u, e\rangle \sum_f \mathbf{R}^e_f\, w^f \\ &= \sum_f \left\langle u, \sum_e \mathbf{R}^e_f\, e\right\rangle w^f = \sum_f \langle \mathrm{d}u, f\rangle w^f = \quad \mathcal{P}\,\mathbf{d}\,\mathbf{u}.\end{aligned}$$

Note the two key points there: the Stokes theorem and Lemma 3.6. ◻

To deal with the exact sequence issue for the first-order Whitney complex, let us introduce some vocabulary. A family of vector spaces $X^0, \ldots, X^d$ (all on the same scalar field) and of linear maps A^p from X^{p-1} to X^p, $p = 1$ to d, form a sequence, denoted by (X, A), which is exact at level of X^p if the image of A^p fills in the kernel of A^{p+1}, in case $1 \leq p \leq d-1$, of X^0 if A^1 is injective and of X^d if A^d is surjective. It is customary to discuss sequences with help of diagrams of the following form:

$$\{0\} \longrightarrow X^0 \xrightarrow{A^1} X^1 \xrightarrow{A^2} \cdots \longrightarrow X^{d-1} \xrightarrow{A^d} X^d \longrightarrow \{0\},$$

where $\{0\}$ is the space of dimension 0.

PROPOSITION 3.8. *If the set-union of all tetrahedra in the mesh is contractible, i.e., topologically trivial, the sequence* $(X, \mathbf{d})$ *is exact at all levels from* 1 *to* $d-1$*, in dimension* d*.*

Proof. We have to prove that $\mathbf{d}\,\mathbf{u} = 0$, with $\mathbf{u} \in X^p$, implies the existence of $\mathbf{z} \in X^{p-1}$ such that $\mathbf{u} = \mathbf{d}\,\mathbf{z}$. So,

$$\mathbf{d}\,\mathbf{u} = 0, \qquad \mathbf{d}\,\mathcal{R}\,\mathcal{P}\,\mathbf{u} = 0, \qquad \mathcal{R}\,\mathrm{d}\,\mathcal{P}\,\mathbf{u} = 0.$$

Since $\mathcal{P}\,\mathbf{u} \in W^p$ and $\mathrm{d}W^p$ is a subset of W^{p+1} on which $\mathcal{R}$ is injective, the relation $\mathcal{R}\,\mathrm{d}\,\mathcal{P}\,\mathbf{u} = 0$ yields $\mathrm{d}\,\mathcal{P}\,\mathbf{u} = 0$. Thus, the set-union of all tetrahedra in the mesh is contractible; it exists $v \in W^{p-1}$ such that $\mathcal{P}\,\mathbf{u} = \mathrm{d}\,v$ so that $\mathcal{R}\,\mathcal{P}\,\mathbf{u} = \mathcal{R}\,\mathrm{d}\,v = \mathbf{d}\,\mathcal{R}\,v$, that is, $\mathbf{u} = \mathbf{d}\,\mathcal{R}\,v$. Setting $\mathbf{z} = \mathcal{R}\,v$ ends the proof. ∎

Thanks to (2.1), one can show that $\mathrm{d}W^{p-1} \subset W^p$. As a consequence, we have that $\mathrm{d}\,\mathcal{P} = \mathcal{P}\,\mathbf{d}$. Moreover, if the mesh $\mathtt{m}$ is such that $\ker \mathbf{d}_p = \operatorname{cod} \mathbf{d}_{p-1}$ and this occurs when the domain Ω is topologically trivial, then $\ker(\mathrm{d}\,; W^p) = \mathrm{d}\,W^{p-1}$, i.e., the Whitney spaces form an exact sequence.

3.4. Properties of the high-order Whitney complex. For our purpose here, a *sequence* (C, ∂) is a collection of homomorphisms $\partial_p : C_p \to C_{p-1}$ between vector spaces, whose elements are called *chains*, with the property $\partial_{p-1}\partial_p = 0$. A chain c such that $\partial c = 0$ is a *cycle*, *bounds* if $c = \partial\gamma$ for some γ in C_{p+1}. The sequence is *exact* if $\ker(\partial_{p-1}) = \operatorname{cod}(\partial_p)$ (the codomain, or image, of C_p by ∂_p), i.e., if all cycles bound. Two chains c and c' in C_p are *homologous*, denoted by $c \sim c'$, if $c - c'$ bounds, i.e., if there exists γ in C_{p+1} such that $c - c' = \partial\gamma$. Equivalence classes for this relation are called *homology classes*. We refer to the corresponding family of quotient spaces $H_p = \ker(\partial_p; C_p)/\operatorname{cod}(\partial_{p+1})$ as "the homology" of (C, ∂). These notions pass to the sequence of dual spaces the obvious way, generating "the cohomology" of the latter. A *chain map* χ between two sequences (C, ∂) and (C', ∂') is a family of maps $\chi_p : C_p \to C'_p$ such that $\chi_{p-1}\partial_p = \partial'_p\chi_p$. Thanks to this, χ passes to homology classes and induces homomorphisms $\chi_p : H_p \to H'_p$.

We have encountered several examples so far: spaces of simplicial chains, dof arrays, the ∂s being either boundary operators, incidence matrices, or their transposes, and the Whitney or de Rham map are many examples of chain maps.

Let now K_p be subspaces of the C_ps with $\partial K_p \subset K_{p-1}$, and let's agree that $c = c'$ mod K, for c and c' both in C_p, means $c - c' \in K_p$. Note that $\partial c = \partial c'$ ensues. We say that two such chains are *homologous modulo* K, denoted $c \sim c'$ mod K, if there exists γ in C_{p+1} such that $c - c' = \partial\gamma$ mod K. Maps ∂_p pass to quotient spaces C_p/K_p, yielding a new sequence $(C/K, \partial)$, the homology of which is called *relative to* K.

The standard example of relative homology is when K_p is made from simplicial chains supported on the closure of some subdomain Ω_K of Ω, in which case a chain c is a *cycle* mod K if its boundary lies in Ω_K and *bounds* mod K if there is γ such that $c\partial\gamma$ lies in Ω_K. But here, we have something else in view. First, an obvious result.

PROPOSITION 3.9. *If* $\chi : (C, \partial) \to (C', \partial')$ *is a* surjective *chain map, the homology of* (C, ∂) *relative to* $\ker(\chi)$ *coincides with the homology of* (C', ∂')*.*

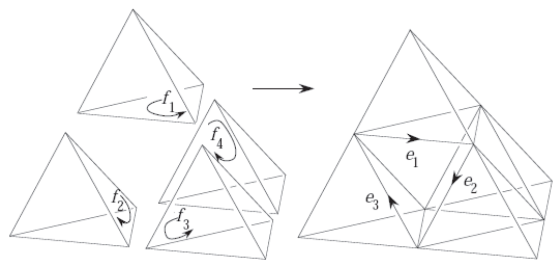

FIG. 3.6. *Left: Complex of small simplices and how the separated clusters of small simplices can be stuck together, by plugging the holes between them, thus giving (right) a paving of the domain, here reduced to a single "big" volume. The four generators of K_1 are the "small" 1-chain $c = e_1 + e_2 + e_3$ and the three other similar ones, and the small 2-chain $f_1 + f_2 - f_3 - f_4$ generates K_2. Note that $\partial(f_1 + f_2 - f_3 - f_4)$ thus belongs to K_1.*

Next, let (C', ∂') be the cell-complex composed of the small simplices *and* the holes, whose homology is that of Ω, which they pave. For C_p, take only the small simplices and for subspaces K_p those generated by chains that belong to the boundary of some hole. For instance (Figure 3.6 should help follow this, in the case $k = 1$), generators of K_1, one for each face of the original mesh, are the boundaries of the small inner triangles; generators of K_2, one for each original tetrahedron, are chains of faces made of the four small faces that belong to the boundary of the core octahedron (the three-dimensional hole). The required property $\partial K_2 \subset K_1$ stems from the fact that the *other* four faces of this boundary happen to be the two-dimensional holes, by construction.

So we may conclude that the relative homology, "modulo the hole boundaries," of the complex of small simplices is the same as the homology of the original mesh. We shall use this result, or rather its dual counterpart, in cohomology.

Let's now examine whether the higher-degree forms make an exact sequence when Ω is contractible. This is a long development, in which the difficult part will be to prove the inclusion $\mathrm{d}W^p_{k+1} \subset W^{p+1}_{k+1}$, where W^p_{k+1} is the space generated by the $\lambda^{\mathbf{k}} w^S$s with $\dim(S) = p$.

PROPOSITION 3.10. *Let σ be a $(p-1)$-simplex and s a p-simplex. Then*

$$\mathbf{d}^\sigma_s w^s = \lambda_{s-\sigma} \mathrm{d}w^\sigma - p\, \mathrm{d}\lambda_{s-\sigma} \wedge w^\sigma. \tag{3.3}$$

Proof. This is more clearly done by reverting to the standard vector formalism. For $p = 1$, we have to show that $\mathbf{G}^m_e\, w^e = w^n \nabla w^m - w^m \nabla w^n$. For the node m, if $\mathbf{G}^m_e \neq 0$, either $e = \{m, n\}$ or $e = \{n, m\}$, but in both cases, $\mathbf{G}^m_e\, w^e = w^n \nabla w^m - w^m \nabla w^n$, by definition of the incidence numbers $\mathbf{G}^m_e$.

For $p = 2$, we must show that $\mathbf{R}^e_f\, w^f = \lambda_{f-e} \nabla \times w^e - 2 \nabla \lambda_{f-e} \times w^e$. If $e = \{m, n\}$, one has $\nabla \times w^e = 2 \nabla w^m \times \nabla w^n$. Moreover, $\mathbf{R}^e_f \neq 0$ yields $f = \{\ell, m, n\}$ or $f = \{\ell, n, m\}$ for some ℓ. In both cases, $f - e = \ell$ and

$$\begin{aligned} \mathbf{R}^e_f\, w^f &= 2 \left(w^\ell \nabla w^m \times \nabla w^n + w^m \nabla w^n \times \nabla w^\ell + w^n \nabla w^\ell \times \nabla w^m \right) \\ &= 2\, w^\ell \nabla \times w^e - 2 \nabla w^\ell \times (w^m \nabla w^n - w^n \nabla w^m), \end{aligned}$$

i.e., $\mathbf{R}^e_f\, w^f = 2\, w^\ell \nabla \times w^e - 2 \nabla w^\ell \times w^e$, with $w^\ell = \lambda_\ell$.

For $p = 3$, we need to show that $\mathbf{D}^f_v w^v = \lambda_{v-f} \nabla \cdot w^f - 3 \nabla \lambda_{v-f} \cdot w^f$. Let us remark that for $f = \{\ell, m, n\}$ and $v = \{\ell, m, n, i\}$, we have $\mathbf{D}^f_v = 1$ and $\nabla \cdot w^f = w^v$.

Knowing that

$$3\,\nabla\, w^i \cdot w^f = -\,\left(w^\ell + w^m + w^n\right)\, w^v,$$

it is easy to see that $w^i\, w^v - 3\,\nabla\, w^i \cdot w^f = (w^i + (w^\ell + w^m + w^n))w^v = w^v$. □

Proposition 3.11. *If node n belongs to the p-simplex s, then*

$$\lambda_n \mathrm{d}w^s = (p+1)\,\mathrm{d}\lambda_n \wedge w^s. \tag{3.4}$$

Proof. Let σ be the $(p-1)$-simplex opposite to n, whose joint with n is s, oriented in such a way that $\mathbf{d}_s^\sigma = 1$. Applying d to (3.3), we find that $\mathrm{d}w^s = (p+1)\,\mathrm{d}\lambda_n \wedge \mathrm{d}w^\sigma$. Left multiplying (3.3) by $\mathrm{d}\lambda_n$, we find

$$\begin{aligned}\mathrm{d}\lambda_n \wedge w^s &= \lambda_n\,\mathrm{d}\lambda_n \wedge \mathrm{d}w^\sigma - p\,\mathrm{d}\lambda_n \wedge (\mathrm{d}\lambda_n \wedge w^\sigma)\\ &= \lambda_n\,\mathrm{d}\lambda_n \wedge \mathrm{d}w^\sigma.\end{aligned}$$

Hence, (3.4). □

Corollary 3.12. *If node n belongs to the p-simplex s, then*

$$(p+1)\,\mathrm{d}(\lambda_n w^s) - (p+2)\,\lambda_n\,\mathrm{d}w^s = 0. \tag{3.5}$$

Proof. If s reduces to node n, then $p = 0$ and $\mathrm{d}(\lambda_n w^n) = 2\,\lambda_n\,\mathrm{d}w^n$, thanks to (3.4). If $s = e = \{m, n\}$, then $p = 1$ and $2\,\mathrm{d}(\lambda_n w^e) = 2[\mathrm{d}\lambda_n \wedge (\lambda_m\,\mathrm{d}w^n - \lambda_n\,\mathrm{d}w^m) + 2\,\lambda_n\,\mathrm{d}w^m \wedge \mathrm{d}w^n] = 6\,\lambda_n\,\mathrm{d}w^m \wedge \mathrm{d}w^n$, while $3\,\lambda_n \mathrm{d}(\lambda_m\,\mathrm{d}w^n - \lambda_n\,\mathrm{d}w^m) = 6\,\lambda_n\,\mathrm{d}w^m \wedge \mathrm{d}w^n$ as well. Similarly for $p = 2$ and $p = 3$. Otherwise,

$$\begin{aligned}(p+2)\,\lambda^n\,\mathrm{d}w^s &= (p+2)\,\mathrm{d}(\lambda_n w^s) - (p+2)\,\mathrm{d}\lambda_n \wedge w^s\\ &= (p+2)\,\mathrm{d}(\lambda_n w^s) - \mathrm{d}\lambda_n \wedge w^s - (p+1)\,\mathrm{d}\lambda_n \wedge w^s\\ &= (p+2)\,\mathrm{d}(\lambda_n w^s) - \mathrm{d}\lambda_n \wedge w^s - \lambda_n\,\mathrm{d}w^s = (p+1)\,\mathrm{d}(\lambda_n w^s),\end{aligned}$$

by using (3.4) and $d(\lambda\, w) = d\lambda \wedge w + \lambda\, dw$. □

Proposition 3.13. *If node n does not belong to the p-simplex σ, then*

$$(p+1)\,\mathrm{d}(\lambda_n w^\sigma) - (p+2)\,\lambda_n\,\mathrm{d}w^\sigma \in W_1^{p+1}. \tag{3.6}$$

Proof. Either $\lambda_n w^\sigma = 0$, and then $\lambda_n\,\mathrm{d}w^\sigma = 0$ as well, or the join of n and σ is a $(p+1)$-simplex s. Then $n = s - \sigma$, and

$$\begin{aligned}(p+1)\,\mathrm{d}(\lambda_n w^\sigma) &= (p+1)\,\mathrm{d}\lambda_n \wedge w^\sigma + (p+1)\,\lambda_n\,\mathrm{d}w^\sigma\\ &= (p+2)\,\lambda_n\,\mathrm{d}w^\sigma \pm w^s\end{aligned}$$

results from (3.3). □

Proposition 3.14. *If s is a p-simplex, then $\mathrm{d}\lambda_n \wedge w^s$ belongs to W_2^{p+1} for all nodes n.*

Proof. If $n \in s$, use (3.4): $\mathrm{d}w^s \in W_1^{p+1}$, so $\lambda_n\,\mathrm{d}w^s \in W_2^{p+1}$, so $\mathrm{d}\lambda_n \wedge w^s \in W_2^{p+1}$. If $n \notin s$, note that (3.6) implies (on substituting s for σ) that $(p+1)\mathrm{d}\lambda_n \wedge w^s = \lambda_n\,\mathrm{d}w^s + \omega$ for some $\omega \in W_1^{p+1}$. Again, $\mathrm{d}\lambda_n \wedge w^s \in W_2^{p+1}$. □

Corollary 3.15. $\mathrm{d}W_2^p \subset W_2^{p+1}$.

Proof. An element b of W_2^p is a sum of terms of the form $\lambda_n\,\mathrm{d}w^s$, where n is a node and s a p-simplex. When $n \in s$, then $\mathrm{d}(\lambda_n w^s) = (p+1)^{-1}(p+2)\lambda_n\,\mathrm{d}w^s$, by (3.5), and $\mathrm{d}w^s \in W_1^{p+1}$, so $\lambda_n\,\mathrm{d}w^s \in W_2^{p+1}$. When $n \notin s$, then (3.6) yields $d(\lambda^n\, w^s) = (p+1)^{-1}(p+2)\lambda_n\,\mathrm{d}w^s + \omega$, for some $\omega \in W_1^{p+1} \subset W_2^{p+1}$, so $d(\lambda_n\, w^s) \in W_2^{p+1}$. □

Proposition 3.16. $\mathrm{d}W_{k+1}^p \subset W_{k+1}^{p+1}$.

Proof. Consider the p-form $\lambda^{\mathbf{k}}\, \mathrm{d}w^s$, where s is a p-simplex and $\mathbf{k}$ a multi-index of weight k. We need to show that $\mathrm{d}(\lambda^{\mathbf{k}}\, \mathrm{d}w^s) \in W^{p+1}_{k+1}$. For this, notice that $\lambda^{\mathbf{k}}\, \mathrm{d}w^s \in W^{p+1}_{k+1}$ and that $\mathrm{d}(\lambda^{\mathbf{k}}\, \mathrm{d}w^s) - \lambda^{\mathbf{k}}\, \mathrm{d}w^s$ is a sum of terms of the form $\lambda^{\mathbf{k}'}\, \mathrm{d}\lambda_n \wedge w^s$, where $\mathbf{k}'$ has weight $k-1$. By Proposition 3.14, these products are in W^{p+1}_{k+1} indeed. ∎

Let us now attempt to follow the pattern of the proof of Proposition 3.8 to conclude about the exact sequence property. Before proceeding with the diagram chase, we need to investigate the structure of the commutative diagram, now more complicated because of matrix A being singular. Let X^p_{k+1} be the space generated by arrays of dofs $\{u_{\mathbf{k}S} : \mathbf{k}, S\}$ indexed over the small simplices $\{\mathbf{k}, S\}$ with $\dim(S) = p$. We still have two operators, $\mathcal{P}$ from vectors of dofs to forms and $\mathcal{R}$ from forms to vectors of dofs:

$$\begin{array}{rccc} \mathcal{P}: & X^p_{k+1} & \to & W^p_{k+1} \\ & \mathbf{u} & \to & \mathcal{P}\mathbf{u} = \displaystyle\sum_{\mathbf{k},S} u_{\mathbf{k}S}\, \lambda^{\mathbf{k}} w^S, \\ \mathcal{R}: & W^p_{k+1} & \to & X^p_{k+1} \\ & u & \to & (\mathcal{R}\, u) = \{\langle u, \{\mathbf{k}, S\}\rangle : |\mathbf{k}| = k,\, S \in \mathcal{S}^p\}. \end{array}$$

Operator $\mathcal{P}$ is surjective, each $\lambda^{\mathbf{k}} w^S$ being the image of the array $\mathbf{u}$ all components of which are zero, apart from $u_{\mathbf{k}S} = 1$. It is not injective, since the functions $\lambda^{\mathbf{k}} w^S$ are not linearly independent. Operator $\mathcal{R}$, on the other hand, is injective but not surjective. Let us denote by $\langle \mathbf{z}, \mathcal{R}u\rangle$, with $\mathbf{z} \in X^p_{k+1}$, the integral of u over the chain $\sum_{k,S} z_{\mathbf{k}S}\{k, S\}$. Some choices of $\mathbf{z}$ nullify the quantity $\langle \mathbf{z}, \mathcal{R}u\rangle$ for all $u \in W^p_{k+1}$. (For example, for $d = 2$, $p = 1$, and $k = 1$, take $\mathbf{z} = (4, 1, 1, 4, 1, 1, 4, 1, 1)$ that, with reference to Figure 3.5 (left), means once the boundary of the big triangle minus four times that of the small internal triangle.) Note that now $\mathcal{R}\,\mathcal{P} = A$, with $A = (A^{\{\mathbf{k}', S'\}}_{\{\mathbf{k}, S\}})$ a singular matrix. Let us therefore introduce a pseudoinverse A^+ such that $AA^+A = A$. (For definiteness, it may be the Moore–Penrose pseudoinverse, characterized by $AA^+A = A$, $A^+AA^+ = A^+$, and AA^+ symmetric, but we shall not invoke other properties than $AA^+A = A$.)

Last, let us denote by $\boldsymbol{\delta}$ the transpose of the incidence matrix linking small p-simplices and set $\tilde{\mathbf{d}} = A^+\boldsymbol{\delta}A$. Since the boundary of the small simplex $\{\mathbf{k}, \Sigma\}$ is the chain $\sum \mathbf{d}^{\Sigma}_{S}\{\mathbf{k}, S\}$, owing to the homothetic relation between small and big simplices, the entries of $\boldsymbol{\delta}$ are easily derived from those of $\mathbf{d}$. No need for us to make them explicit, what counts is that the sequence $(X_{k+1}, \boldsymbol{\delta})$, involving the spaces X^p_{k+1} and applications $\boldsymbol{\delta}_p$, is exact. Indeed, this is the cohomology sequence of the complex of small simplices, which as we remarked earlier (section 3.1) is the disjoint union of homothetic transforms, with centers at points with barycentric coordinates k_i/k of clusters of volumes around these points.

Lemma 3.17. *$\mathcal{P}A^+\mathcal{R} = id$.*

Proof. From $AA^+A = A$ and $\mathcal{R}\,\mathcal{P} = A$, we have $\mathcal{R}\,\mathcal{P}A^+\mathcal{R}\,\mathcal{P} = \mathcal{R}\,\mathcal{P}$. Operator $\mathcal{R}$ is injective, then $\mathcal{P}A^+\mathcal{R}\,\mathcal{P} = \mathcal{P}$, whereas $\mathcal{P}$ is surjective, so $\mathcal{P}A^+\mathcal{R} = id$. ∎

Lemma 3.18. *$\boldsymbol{\delta}\mathcal{R} = \mathcal{R}\mathrm{d}$ and $\mathcal{P}\tilde{\mathbf{d}} = \mathrm{d}\mathcal{P}$.*

Proof. The first identity results from Stokes theorem, as in the proof of Lemma 3.7. For the second identity, we have that $\mathcal{P}\tilde{\mathbf{d}} = \mathcal{P}A^+\boldsymbol{\delta}A = \mathcal{P}A^+\boldsymbol{\delta}\mathcal{R}\,\mathcal{P} = \mathcal{P}A^+\mathcal{R}\mathrm{d}\mathcal{P} = \mathrm{d}\mathcal{P}$, thanks to the first identity and Lemma 3.17. ∎

Now that we know all sequences in display on Figure 3.7 and which maps between them are chain maps, we may compare cohomologies. Our aim is to show that the cohomology of (W, d) is that of Ω. As remarked earlier, the latter coincides with

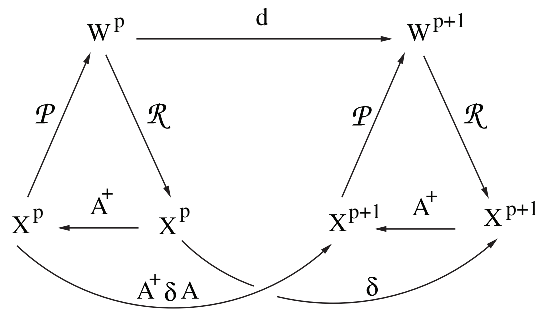

Fig. 3.7. *The complexes of Whitney forms of order $k+1$ (top) and of "$(k+1)$-small" simplicial cochains (bottom), showing commutation relations (recall that $A = \mathcal{RP}$): $\mathrm{d}\mathcal{P} = \mathcal{P}A^+\boldsymbol{\delta}A$, $\mathcal{R}\mathrm{d} = \boldsymbol{\delta}\mathcal{R}$, $\mathcal{P}A^+\mathcal{R} = id$. Beware however, that neither $\mathcal{RP}A^+$ nor $A^+\mathcal{RP}$ are the identity. Note that W^p stands for W^p_{k+1} and the same with W^{p+1}, X^p, and X^{p+1}. Similarly, A^+, A, and $\boldsymbol{\delta}$ are generic symbols for matrices A_p, A^+_p, etc.*

the homology of the complex of small simplices modulo the hole boundaries, that is to say, in algebraic terms, the complex $(X_{k+1}, \boldsymbol{\delta}^t)$ relative to A, where elements of X_{k+1} are interpreted as chains of small simplices. By transposition, this is also the cohomology of $(X_{k+1}, \boldsymbol{\delta})$ relative to A^t, with elements of X_{k+1} now interpreted as *co*chains. Remark that $\ker(A^t)$ complements $\mathrm{cod}(A)$, that is to say, the image of W_{k+1} by the de Rham map, in X_{k+1}.

The last step, identifying this with the cohomology of (W_{k+1}, d) comes now. To simplify a bit, assume Ω is contractible, so that $(X_{k+1}, \boldsymbol{\delta})$ mod A^t is exact. Then,

Proposition 3.19. *The spaces W^p_{k+1} and the maps* d *from W^p_{k+1} to W^{p+1}_{k+1} form an exact sequence (W_{k+1}, d).*

Proof. Let $b \in W^p_{k+1}$ be such that $\mathrm{d}b = 0$, and $\bar{\mathbf{b}} = \mathcal{R}b$, an element of X^p_{k+1}. Then $\boldsymbol{\delta}\bar{\mathbf{b}} = \boldsymbol{\delta}\mathcal{R}b = \mathcal{R}\mathrm{d}b = 0$, which implies the weaker property that $\boldsymbol{\delta}\bar{\mathbf{b}} = 0 \bmod A^t$. By exactness mod A^t, one has $\bar{\mathbf{b}} = \boldsymbol{\delta}\bar{\mathbf{a}}$, where $\bar{\mathbf{a}}$ can be modified by adding any element of $\ker(A^t)$. Let's use this "gauge freedom" to place $\bar{\mathbf{a}}$ in $\mathrm{cod}(A)$, that is to say, in the image of $\mathcal{R}$. This means that it exists $z \in W^p_{k+1}$ such that $\bar{\mathbf{a}} = \mathcal{R}z = \mathcal{RP}A^+\mathcal{R}z = AA^+\bar{\mathbf{a}}$. Then, $b = \mathcal{P}A^+\mathcal{R}b = \mathcal{P}A^+\boldsymbol{\delta}\bar{\mathbf{a}} = \mathcal{P}A^+\boldsymbol{\delta}AA^+\bar{\mathbf{a}} = \mathcal{P}\tilde{\mathbf{d}}A^+\bar{\mathbf{a}} = \mathrm{d}\mathcal{P}A^+\bar{\mathbf{a}}$, thus $b = \mathrm{d}a$ by setting $a = PA^+\bar{\mathbf{a}}$. □

4. Conclusions. This work has been concerned with the definition of shape functions for high-order Whitney finite element spaces. The case of lowest-order has been well understood in the literature, and dofs are generally associated with suitable moments over simplices of the triangulation. When moving to higher orders, the situation is somehow more complicated, and various possible definitions of shape functions have been introduced in the literature. Such definitions generally involve dofs which have no obvious physical meaning.

Here, the main goal consisted in designing shape functions for Whitney p-elements which used only dofs associated with integration subdomains of dimension p. This task has been performed with the introduction of so-called small simplices, which were defined by means of a particular homothety. Dofs are then the integrals over suitable p-dimensional subdomains, linear combinations of small p-simplices. Higher degree Whitney forms, whatever they are, must keep the "exact sequence" property. Here we have explicitely shown that this can be achieved without forfeiting the natural association of dofs with geometric elements of dimension p, namely, linear combinations of the "small simplices" discussed here.

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