MS24 - Structure preserving discretizations and high order finite elements for differential forms (Snorre H. Christiansen, Ragnar Winther, Ana Alonso Rodrıguez, Francesca Rapetti):

Proposed speakers:
Daniele Di Pietro
Richard Falk
Andrew Gilette
Kaibo Hu
Martin Licht
Francesca Rapetti
Claire Scheid
Ruben Specogna

Submitted:
Richard Falk
Andrew Gillette
Kaibo Hu
Martin Licht
Francesca Rapetti
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Roberta Tittarelli

8 of 8 submitted.

Accepted speakers:

A New Approach to Numerical Computation of the Hausdorff Dimension of Invariant Sets of Iterated Function Systems

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We present a new approach to the computation of the Hausdorff dimension of the invariant set of an iterated function system (IFS) by combining finite element approximation theory with theoretical results about the properties of the eigenfunctions of a class of positive, linear Perron-Frobenius operators L_s. Under appropriate assumptions on the IFS, the Hausdorff dimension of the invariant set of an IFS is the value $s = s_*$ for which the spectral radius λ_s of L_s is equal to one. To approximate this eigenvalue, we first use a collocation method employing continuous piecewise linear or bilinear functions to reduce the infinite dimensional eigenvalue problem to a finite dimensional one. The key property of the approximation scheme is that it preserves the positive structure of the operator L_s. Using the theory of positive linear operators and explicit a priori bounds on the derivatives of the strictly positive eigenfunction v_s corresponding to λ_s, we can modify the matrix produced by the collocation scheme to produce matrices A_s and B_s whose spectral radii bracket λ_s. These spectral radii are easily found by a standard variant of the power method, since the spectral radius of A_s (and of B_s) is the only eigenvalue with that modulus. In this way, we obtain rigorous upper and lower bounds on the Hausdorff dimension s_*, and these bounds converge to s_* as the mesh size approaches zero. Although the present theory is not applicable when higher order piecewise polynomials are used, we demonstrate by numerical examples the promise of this approach. Applications to the computation of the Hausdorff dimension of invariant sets arising from classical continued fraction expansions in one dimension and also to complex continued fractions are described.

*Mini-Symposium: Structure preserving and high order discretization

Decompositions of (Trimmed) Serendipity Spaces

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We recently introduced the family of trimmed serendipity finite element differential form spaces, defined on cubical meshes in any number of dimensions, for any polynomial degree, and for any form order in [2]. The relation between the trimmed serendipity and (non-trimmed) serendipity family developed by Arnold and Awanou in [1] is analogous to the relation between the trimmed and (non-trimmed) polynomial finite element differential form families on simplicial meshes from finite element exterior calculus. The first part of this talk will present key definitions and results regarding trimmed and non-trimmed serendipity spaces. The second part of the talk will examine different ways to decompose these spaces into direct sums and how these decompositions inform basis function construction.

References

[1] D. Arnold, G. Awanou. Finite element differential forms on cubical meshes. *Mathematics of Computation*, 83(288):1551–1570, 2014.

[2] A. Gillette, T. Kloefkorn. Trimmed serendipity finite element differential forms. *Revised version under review*, Preprint: arXiv:1607.00571, 2016.

*Mini-Symposium: Structure preserving discretizations and high order finite elements for differential forms

Well-Conditioned Frames for Finite Element Methods

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We discuss representations of high order C^0 finite element spaces on simplicial meshes in any dimension. For high order methods the conditioning of the basis is likely to be important. However, so far there seems to be no generally accepted concept of "a well-conditioned bases", or a general strategy for how to obtain such representations. In the talk we will argue that the L^2 condition number is a proper measure of the conditioning of the basis. This condition number is independent of the elliptic problem to be solved by the method, and it will lead to constructions which balance desired properties of the stiffness matrix and the preconditioner. In fact, we will not restrict the discussion to representations by bases, but instead allow representations by frames. In particular, we will construct frames for the finite element spaces that lead to L^2 condition numbers which are bounded independently of the polynomial degree. A key tool to obtain this result is the properties of the bubble transform, introduced previously in [1].

References

[1] Richard S. Falk and Ragnar Winther. *The bubble transform: A new tool for analysis of finite element methods*. Foundations of Computational Mathematics 16.1 (2016): 297-328.

*Mini-Symposium: Structure preserving and high order discretization

New Developments in Commuting Finite Element Projections

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Commuting projections from Sobolev and distributional de Rham complexes onto discrete de Rham complexes appear in pure and applied mathematics. In differential geometry, they are established tools in determining the cohomology of differential complexes. In numerical analysis, commuting projections and commuting interpolants have received increasing interest during the past years. For example, commuting projections from Sobolev de Rham complexes onto finite element de Rham complexes are a very versatile tool in proving the stability of mixed finite element methods for the numerical analysis of partial differential equations that arise in computational electromagnetism.

In this talk, we address several new directions of research on commuting projection operators in finite element exterior calculus. On the one hand, we generalize the smoothed projections of Arnold, Falk, and Winther to weakly Lipschitz domains, which is a class of domains strictly larger than the class of strongly Lipschitz domains. A well-known example is the so-called *crossed bricks* domain. On the other hand, we address mixed boundary conditions in finite element exterior calculus, where essential boundary conditions are imposed on one part of the boundary and natural boundary conditions are imposed on the complementary boundary part.

In addition to numerical analysis for partial differential equations over domains, we close a gap in the literature and develop smoothed projections that enable finite element exterior calculus on manifolds. Essentially, this relates recent results in surface finite element methods to a class commuting mollification operators that has already been known to de Rham.

References

[1] M. Licht. Smoothed Projections over Weakly Lipschitz Domains. Submitted.

[2] M. Licht. Smoothed Projections and Mixed Boundary Conditions. Submitted.

[3] S. Christiansen, Martin Licht. Finite Element Exterior Calculus over Manifolds. In Preparation.

The discrete relations between fields and potentials with high order Whitney forms

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Besides the list of nodes and of their positions, the mesh data structure also contains incidence matrices, saying which node belongs to which edge, which edge bounds which face, etc., and there is a notion of (inner) orientation of the simplices to consider. In short, an edge, face, etc. is not only a two-node, three-node, etc. sub- set of the set of nodes, but such a set plus an orientation of the simplex it subtends. These matrices are very meaningfull. Besides containing all the information about the topology of the domain, for the lowest approximation polynomial degree, they help connecting the dofs describing potentials to dofs describing fields. As an example, the relation $\mathbf{E} = -\mathrm{grad}\, V$ at the continuous level becomes $\mathbf{e} = -G\,\mathbf{v}$ where G coincides with the node-to-edge incidence matrix and $\mathbf{e}$ (resp. $\mathbf{v}$) is the vector of edge circulations (resp. values at nodes) of the electric field $\mathbf{E}$ (resp. the scalar electric potential V). When fields and potentials are approximated by forms of higher polynomial degree, the discrete equivalent of the field/potential relation is more structured. The involved matrices present a structure by blocks, each block taking into account of the transmission of dofs associated to a geometrical dimension. We wish to investigate the block-structure of these matrices, when fields and potentials are approximated by high order Whitney forms [4], with dofs given either by the well-known moments [3, 1] or by the more recent weights on the small simplices [4, 2].

References

[1] M. Bonazzoli, F. Rapetti, *High order finite elements in numerical electromagnetism: degrees of freedom and generators in duality*, Numerical Algorithms, 74/1 (2017) 111-136.

[2] S. H. Christiansen, F. Rapetti, *On high order finite element spaces of differential forms*, Mathematics of Computation, 85/298 (2016) 517-548.

[3] J.C. Nédélec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Math., 35/2 (1980) 315-341.

[4] F. Rapetti, A. Bossavit, *Whitney forms of higher degree*, SIAM J. on Numerical Analysis, 47/3 (2009) 2369-2386.

*Mini-Symposium: Structure preserving and high order discretization

A structure preserving numerical discretization framework for the Maxwell Klein Gordon equations in 2D.

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As an attempt to develop structure preserving numerical schemes for some non linear wave equations arising in theoretical physics, we focus on the case of the Maxwell Klein Gordon (MKG) equations in dimension 2. The equations are derived from a Lagrangian through a variational principle. The Lagrangian is invariant through some gauge transformations. As a consequence, Euler-Lagrange equations contain constraint equations that are preserved by the solutions making the theory consistent. One should thus strive to preserve this symmetry at the discrete level.

The coupling between the electromagnetic field and the complex Klein Gordon scalar field arises through a covariant derivative involving two parts: the gradient of the electromagnetic field and a product of the electromagnetic and the Klein Gordon fields. Classical discretization methods, such as standard Finite Difference methods or Finite Element methods, usually approximate these two parts separately. By doing this, the gauge symmetry is broken. To recover gauge symmetry at the discrete level, we propose to take advantage from Lattice Gauge Theory introduced in [1]. It approximates the covariant derivative in a consistent way and at the same time preserves the local gauge symmetry and as a result, preserves the constraint. This approach has been already successfully applied in e.g. [2, 3].

We propose to continue the analysis of this type of method for MKG and investigate the possibility of including time discretization. Hence, we develop a fully discrete scheme for the MKG equations and prove its convergence. The strategy of proof, based on a discrete energy principle, is developed in the more general context of mechanical Lagrangian and is next applied to the particular case of MKG equations.

References

[1] K. G. Wilson, *Confinement of quarks*, Phys. Rev. D, Volume 10, Number 8, pp. 2445–2459, Oct. 1974.

[2] S. H. Christiansen and T. G. Halvorsen, *Discretizing the Maxwell-Klein-Gordon equation by the Lattice Gauge Theory formalism*, IMA Journal of Numerical Analysis, (2011) 31 (1): 1-24.

[3] S. H. Christiansen and T. G. Halvorsen, *Convergence of Lattice Gauge Theory for Maxwell's equation*, BIT Numerical Mathematics, (2009), Volume 49, Issue 4, pp 645-667.

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T-Ω formulation with higher order hierarchical basis functions to solve eddy current problems in non simply connected conductors

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The standard T-Ω Finite Element formulation [1] for solving eddy currents problems based on Whitney edge elements and its geometric counterpart described in [2] provide a current density which is only uniform inside each mesh element. Higher order basis functions are therefore very attractive since they yield greater accuracy for a given computational cost and smoother current density vector field. Among the various possibilities to obtain a high order of convergence, the hierarchical basis functions introduced in [1] are particularly appealing. They allow to have a good control over the distribution of degrees of freedom (dofs) given that different orders can coexist on the same mesh. Yet, the authors of [1] assume that the conducting region is simply connected.

This contribution extends the formulation in [1] so that it can deal with conductors of arbitrary topology. To this aim we supplement the classical hierarchical basis functions with non-local basis functions spanning the first de Rham cohomology group of the insulating region [3]. Such non-local basis functions may be efficiently found in negligible time with the Dłotko–Specogna (DS) algorithm [2]. We also show several shortcomings of the technique used in [4], which is the only previous attempt to solve the same issue we were able to find in literature.

References

[1] J.P. Webb, B. Forghani, T-Ω *method using hierarchal edge elements*, IEE Proc., Vol. 142, No. 2, p. 133-141, 1995.

[2] P. Dłotko, R. Specogna, *Physics inspired algorithms for (co)homology computations of three-dimensional combinatorial manifolds with boundary*, Computer Phys. Commun., Vol. 184, No. 10, pp. 2257-2266, 2013.

[3] A. Khebir, P. Dłotko, B. Kapidani, A. Kouki, R. Specogna, T-Ω *formulation with higher order hierarchical basis functions for non simply connected conductors*, in 17th IEEE Conf. Electromagnetic Field Comput. (CEFC 2016), DOI: https://doi.org/10.1109/CEFC.2016.7816343, 13-16 Nov. 2016, Miami, FL. Extended version submitted to a journal.

[4] B. He, P. Zhou, D. Lin, C. Lu, *The Second Order Finite Element Analysis of Eddy Currents Based on the* T-Ω *Method*, PIER M, Vol. 41, 159-166, 2015.

*Mini-Symposium: Structure preserving and high order discretization

A residual a posteriori error estimator for the Hybrid High-Order Method

We study residual-type error estimators for Hybrid High-Order (HHO) discretizations of diffusive problems. The HHO method, recently introduced in [1, 3], has several advantageous features: it supports fairly general meshes including polyhedral elements and nonmatching interfaces, it allows arbitrary approximation orders, and it has a moderate computational cost thanks to hybridization and static condensation. When diffusive problems with sufficiently smooth solutions are considered, the HHO method corresponding to a polynomial degree $k \geq 0$ displays an optimal order of convergence of $(k+1)$ in the energy norm and a superconvergence in $(k+2)$ in the L^2-norm. For non-smooth solutions, on the other hand, achieving optimal convergence requires the use of a posteriori-driven mesh adaptivity.

We present here residual-based a posteriori error estimates for the energy-norm of the error, and we prove upper and (local) lower bounds. The construction relies on the residual-based approach of [2], where an upper bound without undetermined constants is proved. The error estimators are used to drive an adaptive algorithm including local mesh refinement. Two mesh adaptation strategies are presented. The first strategy classically consists in regenerating a locally refined standard mesh based on the error distribution predicted by the a posteriori error estimators. The second strategy, on the other hand, exploits the possibility to use polyhedral elements: the computation is performed on a polyedral mesh obtained by adaptive agglomeration from a fine mesh composed of standard elements. This approach has the advantage of avoiding remeshing, thus waiving a sizeable contribution to the computational cost in industrial applications. Numerical tests are presented to confirm theorical predicitions and show the efficiency of the estimator for physical applications.

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A residual-type error estimator for the Hybrid High-Order discretizations is presented. It is devised such that the upper bound of the energy-norm of the error is guaranteed and the lower bound is local. Numerical tests confirm theorical predicitions and show its efficiency for adaptive mesh refinement.

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References

[1] D. A. Di Pietro and A. Ern, *A hybrid high-order locking-free method for linear elasticity on general meshes* Comput. Methods Appl. Mech. Engrg., 283:1–21, 2015.

[2] D. A. Di Pietro and R. Specogna, *An a posteriori-driven adaptive Mixed High-Order method with application to electrostatics* J. Comput. Phys., 326(1):35–55, 2016.

[3] D. A. Di Pietro and R. Tittarelli, *An introduction to Hybrid High-Order methods* Submitted. Preprint hal-01490524, 2017.