COHOMOLOGICAL CHARACTERIZATION OF RELATIVE HYPERBOLICITY AND COMBINATION THEOREM

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ABSTRACT. We give a cohomological characterization of Gromov relative hyperbolicity. As an application we prove a converse to the combination theorem for graphs of relatively hyperbolic groups given in [9]. We build upon and follow the ideas of the work of S. Gersten [11] about the same topics in the classical Gromov hyperbolic setting.

INTRODUCTION

The celebrated Gromov hyperbolic groups [13] form a central class of groups in Geometric Group Theory. In [11], Gersten gives a cohomological characterization of the Gromov hyperbolicity which was up to then a purely geometrical notion. As an application, Gersten proved the converse of the Bestvina-Feighn combination theorem [3] for graphs of hyperbolic groups. More precisely, given a finite graph $\mathcal{G}$ of hyperbolic groups and assuming the quasi convexity of the edge groups in the vertex groups, he proved the necessity of the so-called *annular flare condition* for the fundamental group of $\mathcal{G}$ being hyperbolic.

Since then, relative hyperbolicity has appeared in Geometric Group Theory, and is the object of a lot of interest nowadays. Although already present in [13], it really grew with Farb's formulation [8]. Among all the definitions which now coexist [5, 17], two are not equivalent [19, 6]. We will call them weak and strong: Gromov relative hyperbolicity is the strong one [19] and, of course, implies weak relative hyperbolicity [6]. In order to give an illustration of these two notions, let us just recall two classical examples:

- The fundamental group of a hyperbolic, finite volume manifold with cusps is strongly hyperbolic relatively to the family of peripheral subgroups. On the other hand, in dimension $n \geq 3$ such a fundamental group is not Gromov hyperbolic (in the absolute sense).
- The group $\mathbb{Z} \oplus \mathbb{Z}$ is hyperbolic relatively to any of the $\mathbb{Z}$-factors in the weak sense but not in the strong sense. In the same line of

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idea, the mapping-class groups of compact surfaces are weakly relatively hyperbolic [15] but not strongly relatively hyperbolic in a non-trivial way (as soon as the surfaces have sufficiently high complexity) [2].

We refer the reader to Section 1 for the definitions about relative hyperbolicity. A general combination theorem for graphs of relatively hyperbolic groups, similar to the Bestvina-Feighn theorem for graphs of hyperbolic groups, has been proven in [9]. For previous results in this direction, see [7, 1].

The purpose of this note is to adapt and extend the above cited results of [11] to the setting of the strong relative hyperbolicity. Our first step is to get a well-suited notion of the ℓ_∞-cohomology of a group G relative to a family of subgroups $\mathfrak{H}$. In order to define the relative cohomology of a pair $(G, \mathfrak{H})$, we follow the classical approach [4]. Our first result is stated as follows:

Theorem 1. *The second relative ℓ_∞-cohomology of a group G relative to a family of subgroups $\mathfrak{H}$, denoted by $H^2_{(\infty)}(G, \mathfrak{H})$, is well-defined as soon as G admits a finite presentation relative to $\mathfrak{H}$. If G is strongly hyperbolic relative to $\mathfrak{H}$, then $H^2_{(\infty)}(G, \mathfrak{H})$ strongly vanishes.*

For *strong vanishing*, see Definition 3.6. This theorem is false in the setting of weak relative hyperbolicity, see Example 3.1.

The strong vanishing is necessary to get the announced application about the combination theorem. The application we give below concerns only semi-direct products of strongly relatively hyperbolic groups with a free group. The reason is that the parabolic subgroups, i.e. the subgroups which are up to conjugacy in the relative part, of the fundamental group of an arbitrary graph of relatively hyperbolic groups are somewhat tedious to describe. This would lead to a heavy formulation, without introducing new interesting phenomena, which all appear in the semi-direct product case. This semi-direct product case is in some sense a "generic" non-acylindrical case, and the most sophisticated one which might appear as the fundamental group of a graph of groups. If one wishes to treat semi-direct products with groups which are not free, one is led to work on 2-complexes of groups.

The uniform free groups of relatively hyperbolic automorphisms which appear below were defined in [9]. Definitions are recalled in Section 5.

Theorem 2. *Let G be a group which is strongly hyperbolic relative to a finite collection of finitely generated subgroups, denoted by $\mathfrak{H}$. Let $\mathrm{Aut}(G, \mathfrak{H})$ be the group of relative automorphisms of $(G, \mathfrak{H})$. Let $\alpha\colon \mathbb{F}_n \to \mathrm{Aut}(G, \mathfrak{H})$ be a monomorphism from the rank n free group into $\mathrm{Aut}(G, \mathfrak{H})$. Then the following properties are equivalent:*

(a) $G \rtimes_\alpha \mathbb{F}_n$ is strongly hyperbolic relative to a $\mathbb{F}_n$-extension of $\mathfrak{H}$ (which we denote by $\mathfrak{H}_\alpha$),

(b) $H^2_{(\infty)}(G \rtimes_\alpha \mathbb{F}_n, \mathfrak{H}_\alpha)$ *strongly vanishes,*

(c) $\mathbb{F}_n$ *is a uniform free group of relatively hyperbolic automorphisms of* $(G, \mathfrak{H})$.

We refer the reader to Remark 6.5 for a discussion about the various assumptions of finiteness (finiteness of the family $\mathfrak{H}$, finite generation of the subgroups in $\mathfrak{H}$) which appear in Theorem 2. In particular, the implication $(c) \Rightarrow (a)$ is where we really need the finite generation of the parabolic subgroups, but this implication is the object of [9], not of the present paper.

Among many other topics, Groves and Manning [14] are also interested in homological characterizations of the strong relative hyperbolicity and generalize some results from [11]. However, their approach is different than our's in the sense that Groves and Manning consider absolute cycles with compact support (instead of relative cycles with non-compact support). In an other paper [16] Mineyev and Yaman are dealing with a similar subject. They consider both usual and bounded cohomology.

1. Relative hyperbolicity

In [11] the proof of the vanishing theorem for the ℓ_∞-cohomology for hyperbolic groups uses in a crucial way the existence of a linear isoperimetric inequality. This designates Osin approach (see [17]) of relative hyperbolicity as the ideal candidate for our purpose.

We recall some basic definitions from [17, Ch. 2]: let G be a group, $\mathfrak{H} = (H_\lambda)_{\lambda\in\Lambda}$ a family of subgroups of G and $X \subset G$. We say that X is a *relative generating set of G with respect to $\mathfrak{H}$* if G is generated by $\big(\cup_{\lambda\in\Lambda} H_\lambda\big) \cup X$. In the sequel we will always assume that X is symmetric. In this situation G is a quotient of the free product

$$F = \big(*_{\lambda\in\Lambda} \widetilde{H}_\lambda \big) * F(X)$$

where the groups $\widetilde{H}_\lambda$ are isomorphic copies of H_λ and $F(X)$ is the free group with the basis X. Let us denote by $\mathcal{H}$ the disjoint union

$$\mathcal{H} = \bigsqcup_{\lambda\in\Lambda} \widetilde{H}_\lambda \smallsetminus \{1\}$$

and by $(\mathcal{H} \cup X)^*$ the free monoid generated by $\mathcal{H} \cup X$. For every $\lambda \in \Lambda$, we denote by $\mathcal{S}_\lambda$ the set of all words over the alphabet $\widetilde{H}_\lambda \smallsetminus \{1\}$ that represent the identity in F. The isomorphism $\widetilde{H}_\lambda \to H_\lambda$ and the identity map on X can be uniquely extended to a surjective homomorphism $\epsilon\colon F \to G$. We say that G has the *relative presentation*

$$\langle X, \mathcal{H} \mid S = 1,\ S \in \mathcal{S} = \sqcup_{\lambda\in\Lambda}\mathcal{S}_\lambda,\ R = 1,\ R \in \mathcal{R}\rangle \tag{1}$$

with respect to $\mathfrak{H}$, where $\mathcal{R} \subset (\mathcal{H}\cup X)^*$, if $\mathrm{Ker}(\epsilon)$ is the normal closure of $\mathcal{R}$ in F. In the sequel we will write $G = \langle X, H_\lambda, \lambda \in \Lambda \mid R = 1, R \in \mathcal{R}\rangle$ or $G = \langle X, \mathcal{H} \mid \mathcal{S}, \mathcal{R}\rangle$ for short.

The relative presentation (1) is called *finite* if both sets X and $\mathcal{R}$ are finite. We say that G is *finitely presented relative to* $\mathfrak{H}$ if there is a finite relative presentation of G with respect to $\mathfrak{H}$.

Remark 1.1. Assume that G admits a finite relative presentation with respect to a finite family of finitely generated subgroups. Then G is finitely generated.

We denote by $l_{\mathcal{H}\cup X}(\gamma)$ the $\mathfrak{H}$*-relative length* of $\gamma \in G$, defined as the word-length of γ with respect to the system of generators $X \cup \mathcal{H}$ (that is the minimal number of elements in $X \cup \mathcal{H}$ needed to write $\gamma \in G$). This is nothing else than the length of a geodesic from the identity to γ in the Cayley graph $\Gamma(G, X \cup \mathcal{H})$ of G with respect to $X \cup \mathcal{H}$.

Let c be a cycle in $\Gamma(G, X \cup \mathcal{H})$, where we call *cycle* a loop in a graph. Consider a *filling* Δ for c, with respect to a finite relative presentation (1). That is $\partial\Delta$ corresponds to c and decomposes in subcells whose boundary either corresponds to a relator from $\mathcal{S}_\lambda$ for some $\lambda \in \Lambda$ or to a relator from $\mathcal{R}$. The $\mathcal{R}$*-relative area of* Δ is the number of $\mathcal{R}$-cells that it contains. We denote by $\mathit{Area}^{\mathcal{R}}(c)$ the $\mathcal{R}$*-relative area of* c which is the minimal $\mathcal{R}$-relative area of a filling with boundary c.

Two functions $f, g\colon \mathbb{N} \to \mathbb{N}$ are *asymptotically equivalent* if there exist constants C, K, L, C_0, K_0, L_0 such that $f(n) \leq g(Cn + K) + Ln$ and $g(n) \leq f(C_0 n + K_0) + L_0 n$.

Definition 1.2. [17] Let G be a group which admits a finite relative presentation $G = \langle X, \mathcal{H} \mid \mathcal{S}, \mathcal{R}\rangle$ with respect to $\mathfrak{H}$.

A *relative isoperimetric function* for this presentation is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for any $n \in \mathbb{N}$, for any cycle $c \in \Gamma(G, X \cup \mathcal{H})$ with length less or equal to n, $\mathit{Area}^{\mathcal{R}}(c) \leq f(n)$.

The *relative Dehn function for the presentation* $G = \langle X, \mathcal{H} \mid \mathcal{S}, \mathcal{R}\rangle$ is the smallest relative isoperimetric function for this presentation.

The group G is *strongly hyperbolic relative to* $\mathfrak{H}$ if G is finitely presented with respect to $\mathfrak{H}$ and the relative Dehn-function of G for a finite relative presentation is linear.

Parabolic subgroups are subgroups conjugate to subgroups in $\mathfrak{H}$.

Remark 1.3. Note that two relative Dehn-functions (as defined above) of two finite relative presentations of G with respect to $\mathfrak{H}$ are asymptotically equivalent. Note also that in general not every finite presentation (1) admits a finite relative Dehn function.

Remark 1.4. Let G be a group which is strongly hyperbolic relative to a family $\mathfrak{H}$. Substituting any subgroup in $\mathfrak{H}$ by a conjugate yields a new family with respect to which G is still strongly relatively hyperbolic.

2. Relative ℓ_∞-cohomology

Let (X, Y) be a CW-pair i.e. X is a CW-complex and $Y \subset X$ is a subcomplex. We denote by X^k the k-skeleton of X. Note that

$Y^k = X^k \cap Y$. We shall write $(X,Y)^k$ for the relative k-skeleton i.e. $(X,Y)^k = X^k \cup Y$.

The cellular chain complex $(C_*(X,Y),\partial)$ is defined in the usual way (see [20]). Here and in the sequel we will work with integer coefficients. The chain group $C_k(X,Y) = C_k(X)/C_k(Y)$ is usually thought of as the $\mathbb{Z}$-module generated by the k-cells of $X \setminus Y$. We denote by $C^k_{(\infty)}(X,Y) \subset C^k(X,Y)$ the subgroup of bounded relative k-cochains. These cochains correspond exactly to the k-cochains $h\colon C_k(X) \to \mathbb{Z}$ which vanish on the k-cells of Y and which admit an uniform upper-bound M_h over all k-cells e^k from X^k, i.e. $h(e^k) = 0$ for all $e^k \in Y^k$ and $|h(e^k)| \leq M_h$ holds for all $e^k \in X^k$. We denote by $||h||_\infty$ the supremum of h on the k-cells.

The pair (X,Y) has *bounded geometry* in dimension n if for all $k \leq n$ there is a bound M_k on the ℓ_1-norms of the chains ∂e^k, where e^k is a k-cell of $X \setminus Y$. Consider a CW-pair (X,Y) with bounded geometry in dimension n. It is easy to see that $\delta(C^{k-1}_{(\infty)}(X,Y)) \subset C^k_{(\infty)}(X,Y)$ for $k \leq n$ and hence we can form $H^k_{(\infty)}(X,Y) = Z^k_{(\infty)}/B^k_{(\infty)}$ if $k \leq n$.

We are mainly interested in the following situation: let (K,L) be a CW-pair such that K is connected and $K \setminus L$ has only finitely many k-cells in dimension $k \leq n$. We denote by $\pi\colon \widetilde{K} \to K$ the universal covering. In this situation the relative complex $(\widetilde{K},\overline{L})$, $\overline{L} := \pi^{-1}(L)$, has bounded geometry in dimension n and the groups $H^k_{(\infty)}(\widetilde{K},\overline{L})$ are well defined for $k \leq n$.

Let K' denote an other CW-complex such that $L \subset K'$. Suppose that K' collapses to K relative to L by an *elementary collapse* i.e. $K' = K \cup e^{k-1} \cup e^k$ where the cells e^{k-1} and e^k are not in K. As in [10, §10] it is easy to see that there is a canonical isomorphism

$$H^l_{(\infty)}(\widetilde{K}',\overline{L}) \cong H^l_{(\infty)}(\widetilde{K},\overline{L})$$

for all $l \leq n$.

3. Relative ℓ_∞-cohomology of $(G,\mathfrak{H})$

Let G be a group, let $\mathfrak{H} = (H_\lambda)_{\lambda\in\Lambda}$ be a family of subgroups of G and $\mathcal{P} = \langle X,\mathcal{H} \mid \mathcal{S},\mathcal{R}\rangle$ a finite relative presentation of G. We will construct a relative CW-complex $K_{\mathcal{P}}$ associated to the presentation $\mathcal{P}$.

For each $\lambda \in \Lambda$ let L_λ be an aspherical CW–complex with one 0-cell e^0_λ such that $\pi_1(L_\lambda, e^0_\lambda) = \widetilde{H}_\lambda$.

Remark 3.1. The condition that the L_λ's are aspherical is not really essential. What we need is that for two different choices L_λ and L'_λ the isomorphism of the fundamental groups is induced by a cellular map. So we might choose the L_λ's to be 2-dimensional complexes (see [18, Lemma 1.5]).

We let L denote the disjoint union $L = \sqcup_{\lambda\in\Lambda} L_\lambda$. Associated to the presentation $\mathcal{P}$ and L there is a *canonical CW-pair* $(K_{\mathcal{P}}, L)$ constructed from L as follows:

0-cells: add one 0-cell e^0;

1-cells: add two types of 1-cells: $\{e^1_\lambda \mid \lambda \in \Lambda\}$ and $\{e^1_x \mid x \in X\}$. The 1-cell e^1_λ is attached to e^0 and e^0_λ. The 1-cells e^1_x are attached to the 0-cell e^0. We shall orient the cells e^1_λ such that $\partial e^1_\lambda = e^0_\lambda - e^0$;

2-cells: add one 2-cell e^2_R for each relation $R \in \mathcal{R}$. The attaching map for a 2-cell is given by the corresponding relation, after the following modification: each maximal subword w of the relation which consists of letters from $\widetilde{H}_\lambda$ is substituted by the word $e^1_\lambda w \bar{e}^1_\lambda$ ($\bar{e}^1_\lambda$ designates the edge e^1_λ with the opposite orientation).

Note that for each $\lambda \in \Lambda$ the subspace $L_\lambda \subset K_{\mathcal{P}}$ is a subcomplex.

By the construction we have $\pi_1(K_{\mathcal{P}}, e^0) = G$ and $\pi_1(L_\lambda) = \widetilde{H}_\lambda$. Note also that $K^1_{\mathcal{P}} \cup L \subset K_{\mathcal{P}}$ is a connected subcomplex and that $\pi_1(K^1_{\mathcal{P}} \cup L, e^0) = F$ (the free product $\big(*_{\lambda\in\Lambda} \widetilde{H}_\lambda\big) * F(X)$).

Remark 3.2. The notation $\pi_1(L_\lambda, e^0_\lambda) = \widetilde{H}_\lambda$ really means that the fundamental group $\pi_1(L_\lambda, e^0_\lambda) \hookrightarrow \pi_1(K, e^0)$ is conjugate to the subgroup H_λ of G. The conjugacy depends on the chosen path from e^0 to e^0_λ. Since the parabolic subgroups in $\mathfrak{H}$ can be substituted by any of their conjugates when dealing with relative hyperbolicity, this ambiguity causes no harm for our purpose.

Let K be an aspherical CW-complex such that $K^2 = K_{\mathcal{P}}$. Such a K can be obtained from $K_{\mathcal{P}}$ by attaching k-cells, $k \geq 3$, in order to kill the higher homotopy groups. Hence we obtain a triple $L \subset K_{\mathcal{P}} \subset K$. We consider the universal covering $\pi\colon \widetilde{K} \to K$. Note that $\pi^{-1}(K_{\mathcal{P}})$ is connected and is hence the universal covering of $K_{\mathcal{P}}$. Define $\overline{L_\lambda} = \pi^{-1}(L_\lambda)$ and $\overline{L} = \pi^{-1}(L) = \sqcup_{\lambda\in\Lambda} \overline{L_\lambda}$.

Now, let $(L'_\lambda)_{\lambda\in\Lambda}$ be a second family as above. Then for each $\lambda \in \Lambda$ there exists a cellular map $f_\lambda\colon L_\lambda \to L'_\lambda$ which induces an isomorphism between the fundamental groups. Note that we do not require that f_λ is a homotopy equivalence. Starting from the relative presentation $\mathcal{P}$ we obtain CW-complexes $K'_{\mathcal{P}}$ and K' by the construction described above. It is obvious that there exists a cellular map $f\colon K \to K'$ extending f_λ, $\lambda \in \Lambda$, which is a homeomorphism on each k-cell $e^k \in K \smallsetminus L$, $k = 0, 1, 2$. The lift $\tilde{f}\colon \widetilde{K} \to \widetilde{K'}$ induces an isomorphism $\tilde{f}_*\colon C_k(\widetilde{K}, \overline{L}) \to C_k(\widetilde{K'}, \overline{L'})$, $k = 0, 1, 2$. More precisely, for $k = 0, 1, 2$ the image of each k-cell of $\widetilde{K} \smallsetminus \overline{L}$ is a k-cell of $\widetilde{K'} \smallsetminus \overline{L'}$. Hence $\tilde{f}$ induces an isomorphism $\tilde{f}^*\colon C^k_{(\infty)}(\widetilde{K'}, \overline{L'}) \to C^k_{(\infty)}(\widetilde{K}, \overline{L})$ for $k = 0, 1, 2$. By the symmetry of the construction we obtain in the same way a cellular map $g\colon K' \to K$ and its lift $\tilde{g}\colon \widetilde{K'} \to \widetilde{K}$ can be chosen such that $(\tilde{f}_*)^{-1} =$

$\tilde{g}_*\colon C_k(\widetilde{K'},\overline{L'}) \to C_k(\widetilde{K},\overline{L})$ for $k=0,1,2$. A direct calculation gives that $\tilde{f}^*\colon H^k_{(\infty)}(\widetilde{K'},\overline{L'}) \to H^k_{(\infty)}(\widetilde{K},\overline{L})$ is an isomorphism for $k=0,1,2$. All which precedes motivates calling an aspherical pair (K,L) as above a *canonical CW-pair for the finite relative presentation* $\mathcal{P}$.

Remark 3.3. From Section 2, if (K,L) is a canonical CW-pair, collapsing an edge e^1_λ yields a new (non-canonical) CW-pair (K',L') such that $H^k_{(\infty)}(\widetilde{K},\overline{L}) = H^k_{(\infty)}(\widetilde{K'},\overline{L'})$, $k=0,1,2$. Beware however that it is not possible to consecutively collapse two of the edges e^1_λ since they form an edge-path with both endpoints in L.

Lemma 3.4. *Let G be a group which admits two finite presentations $\mathcal{P} = \langle X,\mathcal{H} \mid \mathcal{S},\mathcal{R}\rangle$ and $\mathcal{P}' = \langle X',\mathcal{H} \mid \mathcal{S},\mathcal{R}'\rangle$ relative to a family of subgroups $\mathfrak{H}$. Let (K,L) (resp. (K',L')) be a canonical CW-pair for $\mathcal{P}$ (resp. for $\mathcal{P}'$) as defined above. Then $H^2_{(\infty)}(\widetilde{K},\overline{L}) = H^2_{(\infty)}(\widetilde{K'},\overline{L'})$.*

Proof. As in [17, Ch. 2], the two relative presentations are related by a finite sequence of Tietze transformations. As in [10], a Tietze transformation is geometrically realized by an elementary expansion or an elementary collapse. On the other hand, one can prove that the group $H^2_{(\infty)}(\widetilde{K},\overline{L})$ does not change under such an elementary operation: this is proven in [10, Theorem 10.1] for absolute l_∞-cohomology and this still holds here in the relative case because the transformations involved do not affect neither the complex L nor the edges e^1_λ (see Section 2). $\square$

Lemma 3.4 justifies the following definition:

Definition 3.5. Let G be a group which admits a finite presentation $\langle X,\mathcal{H} \mid \mathcal{S},\mathcal{R}\rangle$ relative to a family of subgroups $\mathfrak{H}$. The second ℓ_∞-cohomology group of G relative to $\mathfrak{H}$, denoted by $H^2_{(\infty)}(G,\mathfrak{H})$, is equal to $H^2_{(\infty)}(\widetilde{K},\overline{L})$, where (K,L) is a canonical CW-pair for $\mathcal{P}$ as defined above.

Definition 3.6. Let G be a group which admits a finite presentation $\langle X,\mathcal{H} \mid \mathcal{S},\mathcal{R}\rangle$ relative to a family of subgroups $\mathfrak{H}$. The second ℓ_∞-cohomology group of G relative to $\mathfrak{H}$ *strongly vanishes* if, for some (and hence any) associated canonical CW-pair (K,L), the sequence

$$0 \to H^1_{(\infty)}(\widetilde{K},\overline{L}) \to C^1_{(\infty)}(\widetilde{K},\overline{L})/B^1_{(\infty)}(\widetilde{K},\overline{L}) \xrightarrow{\delta} Z^2_{(\infty)}(\widetilde{K},\overline{L}) \to 0$$

is a short exact-sequence and there is a bounded section $\sigma\colon Z^2_{(\infty)}(\widetilde{K},\overline{L}) \to C^1_{(\infty)}(\widetilde{K},\overline{L})$ of δ i.e. $\delta\circ\sigma = id$ and there exists C such that $\|\sigma(z)\|_\infty \le C\|z\|_\infty$.

3.1. **Example.** We give an easy example of a weakly relatively hyperbolic group for which the ℓ_∞-cohomology does not vanish. Recall that a group G is weakly hyperbolic relative to $\mathfrak{H}$ if and only if the Cayley graph $\Gamma(G, X\cup\mathcal{H})$ is a hyperbolic metric space (see Section 1).

Let $G = \mathbb{Z} = \langle h_1, h_2 \mid h_1 h_2 = 1\rangle$ be a finite presentation of the group $\mathbb{Z}$. Set $\Lambda = \{1,2\}$, $H_i = \langle h_i\rangle$, $i = 1,2$. As in Section 1, let $\mathcal{H}$ be the disjoint union of $\widetilde{H}_1 \setminus \{1\}$ with $\widetilde{H}_2 \setminus \{1\}$. Obviously, $\mathbb{Z}$ is weakly hyperbolic relative to $\mathfrak{H} = (H_\lambda)_{\lambda\in\Lambda}$ since $\Gamma(G,\mathcal{H})$ is a bounded metric space. The presentation $\mathcal{P} = \langle H_1, H_2 \mid h_1 h_2 = 1\rangle$ is a finite presentation of $\mathbb{Z}$ relative to $\mathfrak{H}$: X is empty and $\mathcal{R}$ is just $h_1 h_2 = 1$. We construct the 2-complex $K_{\mathcal{P}}$ as above. The complex L_i is homeomorphic to the circle S^1 with one 0-cell e^0_i and one 1-cell $e^1_{h_i}$. There is no 1-cell e^1_x since X is empty so that $K_{\mathcal{P}}$ is a cylinder: the attaching map of the single 2-cell $e^2 = e^2_{h_1h_2}$ is given by the edge path $e^1_1 e^1_{h_1} \overline{e^1_1} e^1_2 e^1_{h_2} \overline{e^1_2}$. Here and in the sequel $\overline{e^1_j}$ denotes the 1-cell e^1_j with its opposite orientation. The subcomplex L consists of two disjoint loops and K is aspherical (see figure 1).

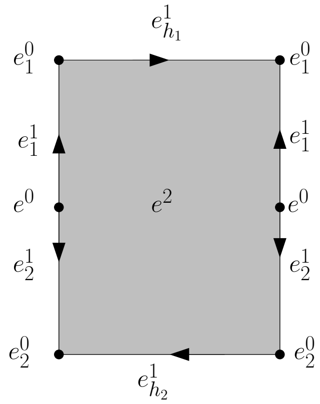


FIGURE 1. The complex K.

The universal covering $\pi\colon \widetilde{K} \to K$ is homeomorphic to the strip $\mathbf{R} \times [1,2]$ with two boundary componens $\overline{L_i} = \mathbf{R} \times \{i\}$, $i = 1,2$. We fix a lift of each cell of K such that $\tilde{e}^0 \in \pi^{-1}(e^0)$, $\tilde{e}^0_i \in \pi^{-1}(e^0_i)$, $\tilde{e}^1_{h_i} \in \pi^{-1}(e^1_{h_i})$, $i = 1,2$, and $\tilde{e}^2 \in \pi^{-1}(e^2)$ such that

$$\partial\tilde{e}^1_i = \tilde{e}^0_i - \tilde{e}^0, \quad \partial\tilde{e}^1_{h_1} = h_1\tilde{e}^0_1 - \tilde{e}^0_1, \quad \partial\tilde{e}^1_{h_2} = \tilde{e}^0_2 - h_1\tilde{e}^0_2,$$

and

$$\partial\tilde{e}^2 = \tilde{e}^1_1 + \tilde{e}^1_{h_1} - h_1\tilde{e}^1_1 + h_1\tilde{e}^1_2 + \tilde{e}^1_{h_2} - \tilde{e}^1_2\,.$$

We define a bounded relative 2-cocycle f by setting $f(h_1^k\tilde{e}^2) = 1$ for each $k \in \mathbb{Z}$. Assume now $f = \delta m$ for some relative 1-cochain m. The equalities $\langle f, h_1^k\tilde{e}^2\rangle = \langle m, h_1^k\partial\tilde{e}^2\rangle$ and $m(h_1^k\tilde{e}^1_{h_i}) = 0$ give

$$1 = m(h_1^k\tilde{e}^1_1) - m(h_1^{k+1}\tilde{e}^1_1) + m(h_1^{k+1}\tilde{e}^1_2) - m(h_1^k\tilde{e}^1_2) \text{ for each } k \in \mathbb{Z}.$$

By summing from $k = 0$ to $k = n-1$, we get: $n = m(\tilde{e}^1_1) - m(h_1^n\tilde{e}^1_1) + m(h_1^n\tilde{e}^1_2) - m(\tilde{e}^1_2)$. This implies that the difference $|m(h_1^n\tilde{e}^1_2) - m(h_1^n\tilde{e}^1_1)|$ tends toward infinity with $n \to +\infty$. Thus m is not bounded, so that f is not a bounded coboundary. Therefore $H^2_{(\infty)}(\mathbb{Z},\mathfrak{H})$ does not vanish.

4. Strong vanishing of $H^2_{(\infty)}(G, \mathfrak{H})$ for strongly relatively hyperbolic groups

Let G be a group which is strongly hyperbolic relative to a family of subgroups $\mathfrak{H} = (H_\lambda)_{\lambda\in\Lambda}$. As in the previous section we consider a canonical pair (K, L) associated to a finite presentation $\langle X, \mathcal{H} \mid \mathcal{S}, \mathcal{R}\rangle$ of G relative to $\mathcal{H}$. We suppose that L_λ is the canonical $K(H_\lambda, 1)$ i.e. L_λ has one 0-cell, one 1-cell e^1_h for each $h \in H_\lambda \setminus \{1\}$ and one 2-cell e^2_S for each relation $S \in \mathcal{S}_\lambda$.

Let $\pi\colon \widetilde{K} \to K$ be the universal covering as above. We equip $\widetilde{K}^{(1)}$ with the following pseudo–metric, termed the *L-relative metric*: each 1-cell of $\overline{L}$ has length zero, each 1-cell in $\pi^{-1}(e^1_\lambda)$, $\lambda \in \Lambda$, has length $1/2$, each 1-cell in $\pi^{-1}(e^1_x)$, $x \in X$, has length 1.

With the notations above:

Definition 4.1. The length of an edge-path p in $\widetilde{K}^1$ with respect to the L-relative metric is termed *relative length* and is denoted by $l_{rel}(p)$.

Each cycle c in $\widetilde{K}^1$ can be filled by a singular disc diagram $D \to \widetilde{K}^2$. The *relative area* $A_{rel}(D)$ of D is the number of 2-cells in the domain in D which correspond to 2-cells e^2_R, $R \in \mathcal{R}$. The *relative area $A_{rel}(c)$ of a cycle c in $\widetilde{K}^1$* is the minimal relative area of all diagrams D filling c.

An immediate consequence of the definition of strong relative hyperbolicity is:

Lemma 4.2. *Suppose that G is strongly hyperbolic relative to $\mathfrak{H}$. Then there exists a constant $C \geq 1$ such that, for any cycle c in $\widetilde{K}^1$, $A_{rel}(c) \leq C\, l_{rel}(c)$ holds.*

Proof. Note that $\Gamma(G, X \cup \mathcal{H})$ is obtained from $\widetilde{K}^1$ by contracting each edge from $\pi^{-1}(e^1_\lambda)$, $\lambda \in \Lambda$. Let $\kappa\colon \widetilde{K}^1 \to \Gamma(G, X \cup \mathcal{H})$ denote the corresponding surjection.

Now by the very definition we have for each cycle c in $\widetilde{K}^1$:

$$l_{rel}(c) = l_{X\cup\mathcal{H}}(\kappa(c)) \text{ and } A_{rel}(c) = Area^{\mathcal{R}}(\kappa(c))\,.$$

The Lemma follows from the existence of a linear relative Dehn-function. □

Let $z \in Z^2_{(\infty)}(\widetilde{K}, \overline{L})$ be a ℓ_∞-relative 2-cocycle. Let $m \in C^1(\widetilde{K}, \overline{L})$ be a relative 1-cochain with $z = \delta m$. Note that $H^2(\widetilde{K}, \overline{L}) = 0$ since $H^2(\widetilde{K}) = 0$ and $H^1(\overline{L}) = 0$. From now on, z and m are fixed. We want to prove the existence of $k \in C^{(1)}_{(\infty)}(\widetilde{K}, \overline{L})$ such that $z = \delta k$.

Let $P, P' \in \widetilde{K}^0$. Following [11], we define a *maximizing path* $w(P, P')$ to be a path from P to P' which maximizes the integer valued function $\nu(\gamma) = \langle m, \gamma\rangle - C||z||_\infty l_{rel}(\gamma)$. Here the maximum is taken among all paths γ from P to P'. The existence of a maximizing path follows as in [11, Sec. 5]:

Lemma 4.3. [11] *The function ν always attains its maximum, i.e. this maximum is finite and maximizing paths always exist.*

Proof. Let w, γ be two paths in $\widetilde{K}^1$ from P to P' and let D be a minimal filling disk of $\gamma^{-1}w$. Then

$$\langle z, D\rangle = \langle \delta m, D\rangle = \langle m, w\rangle - \langle m, \gamma\rangle$$

and

$$\langle z, D\rangle \leq \|z\|_\infty A_{rel}(D) \leq C\|z\|_\infty \big(l_{rel}(w) + l_{rel}(\gamma)\big)$$

which implies: $\nu(w) = \langle m, w\rangle - C||z||_\infty l_{rel}(w) \leq \langle m, \gamma\rangle + C\|z\|_\infty l_{rel}(\gamma)$. □

In the sequel we fix some cells in $\widetilde{K}^0$ as follows: Let $\tilde{e}^0 \in \pi^{-1}(e^0)$, $\tilde{e}^0_\lambda \in \pi^{-1}(e^0_\lambda)$, $\tilde{e}^1_\lambda \in \pi^{-1}(e^1_\lambda)$, $\lambda \in \Lambda$, $\tilde{e}^1_h \in \pi^{-1}(e^1_h)$, $h \in \mathcal{H}$, and $\tilde{e}^1_x \in \pi^{-1}(e^1_x)$, $x \in X$, such that

$$\partial\tilde{e}^1_\lambda = \tilde{e}^0_\lambda - \tilde{e}^0, \quad \partial\tilde{e}^1_x = x\tilde{e}^0 - \tilde{e}^0 \text{ and } \partial\tilde{e}^1_h = h\tilde{e}^0_\lambda - \tilde{e}^0_\lambda, \text{ if } h \in \widetilde{H}_\lambda \smallsetminus \{1\}.$$

Let $P \in \widetilde{K}^0$. We denote by $w(P)$ a maximizing path $w(\tilde{e}^0, P)$ in $\widetilde{K}^1$ with initial point $\tilde{e}^0$ and terminal endpoint P.

From now on we fix for each $\lambda \in \Lambda$ a system of representatives $\{g^\lambda_i\}_{i\in I_\lambda}$, of G/H_λ. For a given $g \in G$, we set $\overline{g}^\lambda := g^\lambda_i$ if $gH_\lambda = g^\lambda_i H_\lambda$. Hence we have $\overline{g_1}^\lambda = \overline{g_2}^\lambda$ if and only if $g_1^{-1}g_2 \in H_\lambda$.

Lemma 4.4. *Let $d\colon C_0(\widetilde{X}) \to \mathbb{Z}$ be the 0-cochain defined by $d(g\tilde{e}^0) = \nu(w(g\tilde{e}^0))$ and $d(g\tilde{e}^0_\lambda) = \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda))$. Then $\delta d \in C^1(\widetilde{K}^1, \overline{L})$.*

Proof. We check that δd vanishes on $C_1(\overline{L})$. Let $h \in \widetilde{H}_\lambda \smallsetminus \{1\}$ then

$$(\delta d)(g\tilde{e}^1_h) = d(g\partial\tilde{e}^1_h) = d(gh\tilde{e}^0_\lambda) - d(g\tilde{e}^0_\lambda) = 0$$

because $\overline{gh}^\lambda = \overline{g}^\lambda$ if $h \in H_\lambda$. □

Proposition 4.5. *Set $k = -m + \delta d$. Then $k \in C^1_{(\infty)}(\widetilde{K}, \overline{L})$.*

Proof. Let w and w' be two edge-paths in $\widetilde{K}^1$. If the terminal vertex of w coincides with the initial vertex of w' we will simply denote by ww' the composition of the two paths. The inverse of an edge $g\tilde{e}^1$ will be denoted by $\overline{g\tilde{e}^1}$.

For all $x \in X$ and all $g \in G$ there exist constants $H_1, H_2 \geq 0$ such that:

$$\begin{cases} \nu(w(g\tilde{e}^0)g\tilde{e}^1_x) + H_1 & = \nu(w(gx\tilde{e}^0)) \\ \nu(w(gx\tilde{e}^0)\overline{g\tilde{e}^1_x}) + H_2 & = \nu(w(g\tilde{e}^0))\,. \end{cases}$$

This is equivalent to:

$$\begin{cases} \nu(w(g\tilde{e}^0)) + \langle m, g\tilde{e}^1_x\rangle - C\|z\|_\infty + H_1 & = \nu(w(gx\tilde{e}^0)) \\ \nu(w(gx\tilde{e}^0)) - \langle m, g\tilde{e}^1_x\rangle - C\|z\|_\infty + H_2 & = \nu(w(g\tilde{e}^0)) \end{cases}$$

and hence $H_1 + H_2 = 2C\|z\|_\infty$ and

$$\begin{aligned}0 &\le \nu(w(gx\tilde{e}^0)) - \nu(w(g\tilde{e}^0)g\tilde{e}^1_x) \\ &= \nu(w(gx\tilde{e}^0)) - \nu(w(g\tilde{e}^0)) - \langle m, g\tilde{e}^1_x\rangle + C\|z\|_\infty \\ &= H_1 \le H_1 + H_2 = 2C\|z\|_\infty\,.\end{aligned}$$

This implies:

$$\begin{aligned}0 \le k(g\tilde{e}^1_x) + C\|z\|_\infty &= \nu(w(gx\tilde{e}^0)) - \nu(w(g\tilde{e}^0)) - \nu(g\tilde{e}^1_x) \\ &= \nu(w(gx\tilde{e}^0)) - \nu(w(g\tilde{e}^0)g\tilde{e}^1_x) \le 2C\|z\|_\infty\end{aligned}$$

and therefore $|k(g\tilde{e}^1_x)| \le C\|z\|_\infty$.

Note that $w(\overline{g}^\lambda\tilde{e}^0_\lambda)$ is an edge-path from $\tilde{e}^0$ to $\overline{g}^\lambda\tilde{e}^0_\lambda$ and $g\tilde{e}^1_\lambda$ is an edge from $g\tilde{e}^0$ to $g\tilde{e}^0_\lambda$. Moreover, $h = (\overline{g}^\lambda)^{-1}g \in H_\lambda$ and that $\overline{g}^\lambda\tilde{e}^1_h$ is an edge from $\overline{g}^\lambda\tilde{e}^0_\lambda$ to $g\tilde{e}^0_\lambda$. With this notation we obtain that $w(\overline{g}^\lambda\tilde{e}^0_\lambda)(\overline{g}^\lambda\tilde{e}^1_h)(\overline{g\tilde{e}^1_\lambda})$ is a path from $\tilde{e}^0$ to $g\tilde{e}^0$ and $w(g\tilde{e}^0)(g\tilde{e}^1_\lambda)(\overline{\overline{g}^\lambda\tilde{e}^1_h})$ is a path from $\tilde{e}^0$ to $\overline{g}^\lambda\tilde{e}^0_\lambda$. Therefore there exist constants $H_3, H_4 \ge 0$ such that:

$$\begin{cases}\nu(w(g\tilde{e}^0)) &= H_3 + \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)(\overline{g}^\lambda\tilde{e}^1_h)(\overline{g\tilde{e}^1_\lambda})) \\ \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)) &= H_4 + \nu(w(g\tilde{e}^0)(g\tilde{e}^1_\lambda)(\overline{\overline{g}^\lambda\tilde{e}^1_h}))\,.\end{cases}$$

This is equivalent to:

$$\begin{cases}\nu(w(g\tilde{e}^0)) &= H_3 + \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)) - \langle m, g\tilde{e}^1_\lambda\rangle - 2C\|z\|_\infty \\ \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)) &= H_4 + \nu(w(g\tilde{e}^0)) + \langle m, g\tilde{e}^1_\lambda\rangle - 2C\|z\|_\infty\,.\end{cases}$$

As above we obtain

$$0 \le \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)) - \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)(\overline{g}^\lambda\tilde{e}^1_h)(\overline{g\tilde{e}^1_\lambda})) = H_3 \le H_3 + H_4 \le 4C\|z\|_\infty$$

and therefore

$$k(g\tilde{e}^1_\lambda) + 2C\|Z\|_\infty = \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)) - \nu(w(\overline{g}^\lambda\tilde{e}^0_\lambda)(\overline{g}^\lambda\tilde{e}^1_h)(\overline{g\tilde{e}^1_\lambda})) \le 4C\|z\|_\infty\,.$$

Here we have used that $\langle m, g\tilde{e}^1_h\rangle = 0$ for all $g \in G$ and all $h \in \mathcal{H}$. □

Proof of Theorem 1. By definition, $H^2_{(\infty)}(G, \mathfrak{H}) = H^2_{(\infty)}(\widetilde{K}, \overline{L})$. Let z be a bounded relative 2-cocycle of $(\widetilde{K}, \overline{L})$. Since $H^2(\widetilde{K}, \overline{L}) = 0$, z is a relative 2-coboundary, $z = \delta m$. From Proposition 4.5, $k = -m + \delta d$ is a bounded relative 1-cochain. But $\delta(-k) = \delta m = z$. Therefore z is a bounded relative 2-coboundary. Whence the vanishing of $H^2_{(\infty)}(G, \mathfrak{H})$. For the strong vanishing, just defining $\sigma(z) = -k$ yields the announced section since $\|\sigma(z)\|_\infty = \|k\|_\infty \le 4C\|z\|_\infty$. □

5. A converse to the combination theorem

5.1. **Definitions and statement of theorem.** Let G be a group and let $\mathfrak{H} = (H_\lambda)_{\lambda\in\Lambda}$ be a family of subgroups of G.

Assumption 5.1. We shall suppose in the sequel that H_λ and $H_{\lambda'}$ are not conjugated for $\lambda \neq \lambda'$. Moreover the H_λ's are infinite subgroups.

Definition 5.2. A *relative automorphism of* $(G, \mathfrak{H})$ is an automorphism α of G which preserves $\mathfrak{H}$ up to conjugacy. More precisely, there is a permutation $\sigma \in \mathrm{Sym}(\Lambda)$ such that for any $\lambda \in \Lambda$ there is $g_\lambda \in G$ such that $\alpha(H_\lambda) = g_\lambda^{-1} H_{\sigma(\lambda)} g_\lambda$ i.e. we have $i_{g_\alpha} \circ \alpha(H_\lambda) = H_{\sigma(\lambda)}$ where i_{g_α} is an inner automorphism of G. We call σ the permutation *associated* to α. If σ is the identity we will say that α *fixes* $\mathfrak{H}$ *up to conjugacy*. The group of relative automorphisms will be denoted by $\mathrm{Aut}(G, \mathfrak{H})$ and the subgroup of relative automorphisms which fix $\mathfrak{H}$ up to conjugacy by $\mathrm{Aut}_0(G, \mathfrak{H})$.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a finite set and let $\mathbb{F}_n = \langle \mathcal{A} \rangle$ be the free group with basis $\mathcal{A}$. All the free groups considered are finitely generated free groups. We will denote by $|w|_{\mathbb{F}_n}$ or $|w|_{\mathcal{A}}$ the word-length of an element of $\mathbb{F}_n$, depending on whether the basis $\mathcal{A}$ has been specified or not. Our convention is that the distance between g and h in the free group is given by $|g^{-1}h|_{\mathbb{F}_n}$.

We suppose that there is an injective homomorphism $\alpha\colon \mathbb{F}_n \to \mathrm{Aut}(G, \mathfrak{H})$. We will denote $\alpha_i := \alpha(a_i) \in \mathrm{Aut}(G, \mathfrak{H})$ and more generally $\alpha_a := \alpha(a)$ i.e. for all $a, a' \in \mathbb{F}_n$ we have $\alpha_{aa'} = \alpha_a \circ \alpha_{a'}$.

We will define a new pair $(G_{\mathcal{A}}, \mathfrak{H}_{\mathcal{A}})$ in the following way: we let $G_{\mathcal{A}}$ denote the semidirect product $G_{\mathcal{A}} = G \rtimes_\alpha \mathbb{F}_n$ i.e. $ga \cdot g'a' = g\alpha_a(g')\, aa'$. For each $\lambda \in \Lambda$ we denote

$$\mathcal{H}_\lambda := \{ga \in G_{\mathcal{A}} \mid \alpha_a(H_\lambda) = g^{-1} H_\lambda g\}\,.$$

It is easy to see that $\mathcal{H}_\lambda \subset G_{\mathcal{A}}$ is a subgroup and that $H_\lambda < \mathcal{H}_\lambda$.

Remark 5.3. If G is strongly hyperbolic relative to $\mathcal{H}$, then $G \cap \mathcal{H}_\lambda = H_\lambda$. Indeed, by definition, if $g \in \mathcal{H}_\lambda$ then $H_\lambda = g^{-1} H_\lambda g$, which is forbidden by the strong relative hyperbolicity (parabolic subgroups are almost malnormal) if $g \notin H_\lambda$.

Remark 5.4. If there exists $g_b b \in G_{\mathcal{A}}$ such that $\alpha_b(H_\lambda) = g_b^{-1} H_{\sigma(\lambda)} g_b$ then the subgroups $\mathcal{H}_\lambda$ and $\mathcal{H}_{\sigma(\lambda)}$ are conjugate.

In order to obtain a family of non conjugated subgroups $\mathfrak{H}_{\mathcal{A}}$ of $G_{\mathcal{A}}$ we are proceeding as follows:

Definition 5.5 ($\mathbb{F}_n$-extension of $\mathfrak{H}$)**.** For each $i = 1, \ldots, n$, we let $\sigma_i \in \mathrm{Sym}(\Lambda)$ denote the permutation associated to α_i and we let $U = \langle \sigma_1, \ldots, \sigma_n \rangle < \mathrm{Sym}$ denote the subgroup generated by the σ_i. Let

$L \subset \Lambda$ be a system of orbit representatives i.e.

$$\Lambda = \bigsqcup_{\lambda \in L} U \cdot \lambda \,.$$

We now define a $\mathbb{F}_n$-*extension* $\mathfrak{H}_{\mathcal{A}}$ by $\mathfrak{H}_{\mathcal{A}} = (\mathcal{H}_\lambda)_{\lambda \in L}$.

Note that the groups $\mathcal{H}_\lambda$, $\lambda \in L$, are uniquely defined up to conjugacy in the group $G_{\mathcal{A}}$.

Let us now recall the definition of a "uniform free group of relatively hyperbolic automorphisms":

Definition 5.6. [9] Let $\mathcal{P} = \langle X, \mathcal{H}\,;\mathcal{S}, \mathcal{R}\rangle$ be a finite relative presentation of a group G. A *uniform free group of relatively hyperbolic automorphisms of* $(G, \mathfrak{H})$ is a free group $\mathbb{F}_n$ together with a monomorphism $\alpha\colon \mathbb{F}_n \to \mathrm{Aut}(G, \mathfrak{H})$ for which there exists $\lambda > 1$, $N, M \geq 1$ such that, for any $g \in G$ with $l_{X\cup\mathcal{H}}(g) \geq M$, any pair of $a, b \in \mathbb{F}_n$ with $|a|_{\mathbb{F}_n} = |b|_{\mathbb{F}_n} = N$ and $|a^{-1}b|_{\mathbb{F}_n} = 2N$ satisfies:

$$\lambda l_{X\cup\mathcal{H}}(g) \leq \max(l_{X\cup\mathcal{H}}(\alpha_a(g)), l_{X\cup\mathcal{H}}(\alpha_b(g))).$$

With these definitions in mind, the reader can now go back to Theorem 2, which is the theorem we are going to prove. We will however adopt the following:

Assumption 5.7. We will assume for the moment that $\alpha\colon \mathbb{F}_n \to \mathrm{Aut}_0(G, \mathfrak{H})$.

As we shall see, the general case is a straightforward implication of this particular case.

As in Theorem 2, we will assume that G is a group which is strongly hyperbolic relative to a finite family $\mathfrak{H}$ of finitely generated subgroups $(H_i)_{i=1}^k$. Under the finiteness hypothesis of $\mathfrak{H}$, the automorphisms considered induce quasi isometries on the group G equipped with the relative metric.

Assume that a finite relative presentation $\mathcal{P} = \langle X, \mathcal{H}\,;\mathcal{S}, \mathcal{R}\rangle$ has been chosen. Let (K, L) be a canonical CW-pair for $\mathcal{P}$ where each connected component L_λ of L is a $K(H_\lambda, 1)$, and such that the 1-cells in L are in bijection with the generators of the subgroups in $\mathfrak{H}$. For each automorphism α_i we choose a cellular map $f_i\colon (K, L) \to (K, L)$ with $(f_i)_\# = \alpha_i$ which fixes the base-point e^0 and such that $f_i(L_\lambda) \subset L_\lambda$ for each connected component L_λ of L. Let $K(G_{\mathcal{A}})$ be the graph of spaces defined as follows:

- the associated combinatorial graph Γ is the rose with n petals i.e. the one point union $\vee_{i=1}^n S^1$ labelled by the a_i's;
- the edge and vertex spaces are copies of the complex K;
  Let us recall that, over each open edge of Γ, $\mathcal{G}$ is homeomorphic to $K \times (0, 1)$.

- the space $K \times \{0\}$ (resp. $K \times \{1\}$) associated to the edge with label a_i is glued along the vertex space K by the map f_i (resp. by the identity-map).

It is easily checked that $L_\lambda \subset K$ gives rise to a subcomplex $L_\lambda(G_\mathcal{A})$ and hence a CW-pair, denoted by $(K(G_\mathcal{A}), L(G_\mathcal{A}))$. Observe in particular that, by construction, each connected component $L_\lambda(G_\mathcal{A})$ is a $K(\mathcal{H}_\lambda, 1)$.

Since G is strongly hyperbolic relative to $\mathfrak{H}$, G admits a finite presentation relative to $\mathfrak{H}$. Let $\langle X, \mathcal{H} \mid \mathcal{R}, \mathcal{S}\rangle$ be such a finite relative presentation. By definition of $\mathcal{A}$, for each $a_i \in \mathcal{A}$, for each $H_\lambda \in \mathfrak{H}$, there is $g_{i,\lambda} \in G$ such that $\alpha_i(H_\lambda) = g_{i,\lambda}^{-1} H_\lambda g_{i,\lambda}$. We denote by $S_{i,\lambda}$ such a relation. Let $\mathcal{S}'$ be the union of the relations in $\mathcal{S}$ with the relations $S_{i,\lambda}$. Let $\mathcal{R}'$ be the union of the relations in $\mathcal{R}$ with the relations $\alpha_i(x_j) = a_i x_j a_i^{-1}$. Then $G_\mathcal{A}$ admits $\langle X, \mathcal{A}, \mathfrak{H}_\mathcal{A} \mid \mathcal{R}', \mathcal{S}'\rangle$ as finite relative presentation.

Remark 5.8. Constructing a finite relative presentation for $(G_\mathcal{A}, \mathfrak{H}_\mathcal{A})$ as above is not so hard when $\alpha(\mathbb{F}_n) \subset \mathrm{Aut}_0(G, \mathfrak{H})$. However, if the automorphisms α_i only preserve $\mathfrak{H}$ up to conjugacy, such a finite relative presentation does not come so easily without the finite generation of G (or of the H_λ's).

Assumption 5.9. In what follows, the CW-pairs (K, L) and $(K(G_\mathcal{A}), L(G_\mathcal{A}))$ are graph of spaces as detailed above.

With this assumption, (K, L) is canonically embedded in $(K(G_\mathcal{A}), L(G_\mathcal{A}))$ since Γ has a unique vertex. We denote by $j\colon (K, L) \to (K(G_\mathcal{A}), L(G_\mathcal{A}))$ this embedding. As suggested by the notation, it satisfies $j(L) \subset L(G_\mathcal{A})$ (j is the embedding which induces the canonical injection of G in $G_\mathcal{A}$). The situation is similar for the universal coverings, which we denote by $\pi\colon \widetilde{K} \to K$ and $\pi_\mathcal{A}\colon \widetilde{K}(G_\mathcal{A}) \to K(G_\mathcal{A})$ ($\pi_\mathcal{A}^{-1}(j(K))$ consists of an infinite number of copies of $\widetilde{K} = \pi^{-1}(K)$).

Definition 5.10. A *horizontal edge-path* in $\widetilde{K}(G_\mathcal{A})$ is an edge-path γ between two lifts of the base-point e^0 which is contained in a connected component $\widetilde{K}$ of $\pi_\mathcal{A}^{-1}(j(K))$ (the lift, under $\pi_\mathcal{A}$, of the complex K canonically embedded in $K(G_\mathcal{A})$).

A *horizontal geodesic* is a horizontal edge-path which defines a geodesic of $\widetilde{K}$ equipped with the $\overline{L}$-relative metric.

A *corridor* $\mathcal{C}_g$ in $\widetilde{K}(G_\mathcal{A})$ is a union of horizontal geodesics which contains, for a given $g \in G$ and for any $a \in \mathbb{F}_n$, exactly one horizontal geodesic, denoted by $\gamma_g(a)$, from $a\tilde{e}^0$ to $a\alpha_{a^{-1}}(g)\tilde{e}^0$.

Remark 5.11. A horizontal edge-path in $\widetilde{K}(G_\mathcal{A})$ projects under $\pi_\mathcal{A}$ to a closed path representing an element of $G < G_\mathcal{A}$.

A horizontal geodesic is not necessarily (and most often won't be) a geodesic for $(\widetilde{K}(G_\mathcal{A}), \overline{L}(G_\mathcal{A}))$ equipped with the $\overline{L}(G_\mathcal{A})$-relative metric.

The fibers $\widetilde{K}$ in $\pi_{\mathcal{A}}^{-1}(j(K))$ are indeed distorted (from a geometrical point of view) in the total space.

Definition 5.12. A corridor $\mathcal{C}_g$ is (λ, N, M)-*separated*, with $\lambda > 1$, $N, M \geq 1$, if for any horizontal geodesic $\gamma_g(w) \in \mathcal{C}_g$ with $l_{rel}(\gamma_g(w)) \geq M$, any pair of elements $u, v \in \mathbb{F}_n$ with $|w^{-1}u|_{\mathbb{F}_n} = |w^{-1}v|_{\mathbb{F}_n} = N$ and $|u^{-1}v|_{\mathbb{F}_n} = 2N$ satisfies:

$$\lambda l_{rel}(\gamma_g(w)) \leq \max(l_{rel}(\gamma_g(wu)), l_{rel}(\gamma_g(wv))).$$

Remark 5.13. If there exist $\lambda > 1$, $M, N \geq 1$ such that all corridors of $(\widetilde{K}(G_{\mathcal{A}}), \overline{L}(G_{\mathcal{A}}))$ are (λ, M, N)-separated then $\mathbb{F}_n$ is a uniform free group of relative automorphisms of G.

The theorem we want to prove is:

Theorem 5.14. *Let G be a group which is strongly hyperbolic relative to a finite family $\mathfrak{H}$ of finitely generated subgroups. Let $\alpha\colon \mathbb{F}_n \to \mathrm{Aut}_0(G, \mathfrak{H})$ be a monomorphism, and let $\mathcal{A}$ be a basis of $\mathbb{F}_n$. If $H^2_{(\infty)}(G_{\mathcal{A}}, \mathfrak{H}_{\mathcal{A}})$ strongly vanishes, then there exists $\lambda > 1$, $N, M \geq 1$ such that the corridors of $(\widetilde{K}(G_{\mathcal{A}}), \overline{L}(G_{\mathcal{A}}))$ are (λ, N, M)-separated.*

Remark 5.15. The strong exponential separation property of [9] involves another condition, which is the exponential separation of any two vertices representing elements in distinct right $\mathcal{H}$-classes, even if the (relative) distance between these vertices is smaller than the constant M. This condition is obviously necessary, this is most easily seen with Farb's approach [8]: not satisfying this property contradicts the BCP, and has nothing to do with ℓ_∞-cohomology. This is why we do not evoke it in Theorem 5.14 above.

5.2. **From Theorem 5.14 to Theorem 2.** The full statement of Theorem 2, i.e. when $\alpha(\mathbb{F}_n)$ is not necessarily contained in $\mathrm{Aut}_0(G, \mathfrak{H})$, is deduced from Theorem 5.14 thanks to the following four lemmas. In order to proceed, we fix a monomorphism $\alpha\colon \mathbb{F}_n \to \mathrm{Aut}(G, \mathfrak{H})$. We denote by α_0 the restriction of α to $\mathbb{F}_0 := \alpha^{-1}(\mathrm{Aut}_0(G, \mathfrak{H}))$. We set $\mathcal{A}$ and $\mathcal{A}_0$ two basis respectively of $\mathbb{F}_n$ and $\mathbb{F}_0$.

Lemma 5.16. *With the above notations: the subgroup $\mathbb{F}_0$ is a finitely generated subgroup of $\mathbb{F}_n$ and its natural embedding in $\mathbb{F}_n$ defines a quasi isometry between $\mathbb{F}_0$ and $\mathbb{F}_n$.*

Proof. Since the family $\mathfrak{H}$ is finite, $\mathbb{F}_0$ is of finite index in $\mathbb{F}_n$. [12, Proposition 3.19] gives the lemma. □

Lemma 5.17. *Let Γ be a finitely generated group which admits a finite presentation relative to a finite family of finitely generated infinite subgroups, denoted by $\mathfrak{H} = (H_j)_{j=1}^k$. Let $\Gamma_0 < \Gamma$ be a subgroup of finite index $p \in \mathbb{N}$.*

We fix a finite system $\{g_{ij} \mid j = 1, \ldots, k,\ i = 1, \ldots, p_j\}$ of representatives for the double cosets $\Gamma_0/\Gamma\backslash H_j$ and we define a finite family *of subgroups of* Γ_0 *by* $H_{ij} := (g_{ij} H_j g_{ij}^{-1}) \cap \Gamma_0$. *Let* $\mathfrak{H}_0$ *be the family* $(H_{ij})_{i,j=1}^{p_j,k}$.

Then $H^2_{(\infty)}(\Gamma_0, \mathfrak{H}_0)$ *strongly vanishes if and only if* $H^2_{(\infty)}(\Gamma, \mathfrak{H})$ *strongly vanishes.*

Proof. We consider a canonical CW-pair (K, L) for $(\Gamma, \mathfrak{H})$. Since $\Gamma_0 < \Gamma$ is of finite index, a finite covering of (K, L) gives a (not necessarily canonical) CW-pair (K_0, L_0) for $(\Gamma_0, \mathfrak{H}_0)$ (see [18, 3.12, 3.13]). Since the covering is finite, one can pass from (K_0, L_0) to a canonical pair (K_0', L_0') for $(\Gamma_0, \mathfrak{H}_0)$ by a finite sequence of elementary expansions and collapses. Moreover one can choose such a finite sequence so that:

(a) one collapses a maximal tree in each connected component $L_\lambda \subset L_0$ and we let denote $L'_\lambda \subset L_0'$ the resulting subcomplex with base vertex e'_λ,

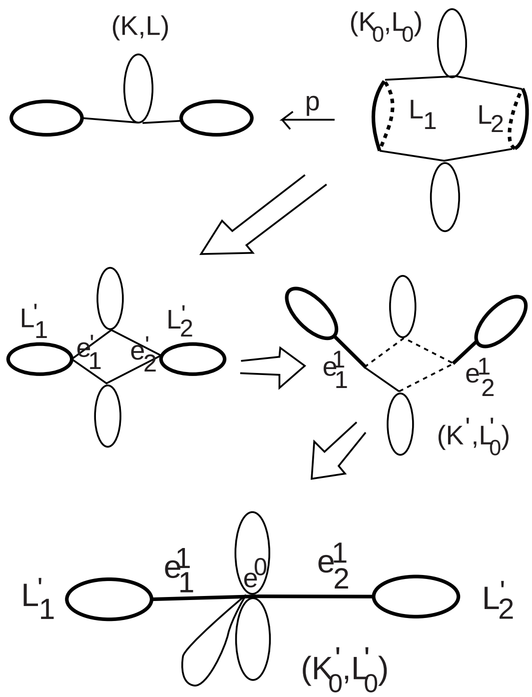
(b) one expands at each e'_λ to get an edge e^1_λ and we let denote K' the resulting complex, $L_0' \subset K'$ and $e^1_\lambda \in K' \setminus L_0'$.

(c) one collapses to a 0-cell denoted by e^0 a maximal tree in the complement of $L_0' \cup (\cup_{\lambda \in \Lambda} \{e^1_\lambda\})$. We let denote K_0' the resulting complex.

Let X_0 be the closed complement of L_0 in K_0. Let X_0' be the image of X_0 in K_0' under the above collapses. Then the relative 1-skeleton of K_0' (that is the 1-skeleton outside L_0') consists of the 1-skeleton of X_0', the 0-cell e^0 and the finitely many 1-cells $\{e^1_\lambda\}$ (see an example just after this proof).

By following the procedure described above one ensures that the associated elementary expansions and collapses from K_0 to K_0' do not modify the second relative l_∞-cohomology group, that is $H^2_\infty(\Gamma_0, \mathfrak{H}_0) = H^2_\infty(\widetilde{K_0}, \overline{L_0})$ (see Section 2). Moreover, since (K_0, L_0) is a covering of (K, L), the pairs (K, L) and (K_0, L_0) have same universal covering $(\widetilde{K}, \overline{L}) = (\widetilde{K_0}, \overline{L_0})$ thus in particular $H^2_\infty(\widetilde{K_0}, \overline{L_0})$ strongly vanishes if and only if $H^2_\infty(\widetilde{K}, \overline{L}) = H^2_\infty(G, \mathfrak{H})$ does. The lemma follows. $\square$

Example: Let (K, L) be a graph which is a canonical pair for the rank 3 free group Γ with the finite relative presentation $\Gamma = \langle X, H_1, H_2 \rangle$ (no relations) where $H_i = \langle h_i \rangle$. Let $p \colon E \to \mathbb{Z}/2\mathbb{Z}$ be the epimorphism defined by $p(X) = 0$ and $p(h_i) = 1$ and let $\Gamma_0 = \mathrm{Ker}(p)$. The figure above illustrates what happens when taking the finite covering (K_0, L_0) of (K, L) associated to Γ_0, and then applying the elemantary collapses and expansions described in the proof of Lemma 5.17 to get a canonical pair (K_0', L_0') for Γ_0. The associated finite relative presentation is $\Gamma_0 = \langle X^0_1, X^0_2, X^0_3, H^0_1, H^0_2 \rangle$ where $p(X^0_1) = X, p(X^0_2) = h_1 X h_1^{-1}, p(X^0_3) = h_1 h_2$ and $H^0_i := \Gamma_0 \cap H_i = \langle h_i^2 \rangle$.



We define $G_{\mathcal{A}} := G \rtimes_{\alpha} \mathbb{F}_n$ and $G_{\mathcal{A}_0} := G \rtimes_{\alpha_0} \mathbb{F}_0$. As in Definition 5.5 let $\mathfrak{H}_{\mathcal{A}}$ be the $\mathbb{F}_n$-extension of $\mathfrak{H}$ and let $\mathfrak{H}_{\mathcal{A}_0}$ be the $\mathbb{F}_0$-extension of $\mathfrak{H}$ i.e.

$$\mathfrak{H}_{\mathcal{A}} = (\mathcal{H}_\lambda)_{\lambda \in L} \text{ and } \mathfrak{H}_{\mathcal{A}_0} = (\mathcal{H}^0_\lambda)_{\lambda \in \Lambda}$$

where $\mathcal{H}^0_\lambda = \{ga \in G_{\mathcal{A}_0} \mid \alpha_a(H_\lambda) = g^{-1}H_\lambda g\}$.

Lemma 5.18. *$H^2_{(\infty)}(G_{\mathcal{A}}, \mathfrak{H}_{\mathcal{A}})$ strongly vanishes if and only if $H^2_{(\infty)}(G_{\mathcal{A}_0}, \mathfrak{H}_{\mathcal{A}_0})$ does.*

Proof. Since the parabolic subgroups in $\mathfrak{H}$ are finitely generated and in finite number, and since G admits a finite presentation relative to $\mathfrak{H}$, G is finitely generated. Therefore $G_{\mathcal{A}}$ is also finitely generated. Since $\mathbb{F}_0$ is a finite index subgroup of $\mathbb{F}_n$, $G_{\mathcal{A}_0}$ is a finite index subgroup of $G_{\mathcal{A}}$. Moreover we have for every $\lambda \in L$ that $\mathcal{H}^0_\lambda = \mathcal{H}_\lambda \cap G_{\mathcal{A}_0}$. If $\mu \in \Lambda \setminus L$ then there exist $\lambda \in L$ and $ga \in G_{\mathcal{A}}$ such that $ga\mathcal{H}_\lambda a^{-1}g^{-1} = \mathcal{H}_\mu$ (see Remark 5.4). Hence

$$ga\mathcal{H}_\lambda a^{-1}g^{-1} \cap G_{\mathcal{A}_0} = \mathcal{H}_\mu \cap G_{\mathcal{A}_0} = \mathcal{H}^0_\mu\,.$$

The family $\mathfrak{H}_{\mathcal{A}_0} = (\mathcal{H}^0_\lambda)_{\lambda\in\Lambda}$ is clearly a maximal family of non conjugated subgroups, i.e. is a family as defined in Lemma 5.17. We can thus apply this lemma, which gives the announced conclusion. □

Lemma 5.19. *Assume that $\mathbb{F}_0$ is a uniform free group of relatively hyperbolic automorphisms. Then $\mathbb{F}_n$ is a uniform free group of relatively hyperbolic automorphisms.*

Proof. We begin by the following:

Claim 1. Assume that $\mathbb{F}_n$ is not a uniform free group of relatively hyperbolic automorphisms. Then for any sufficiently large $M > 0$ and $N \geq 1$, for any $\lambda > 1$, there exist three elements $g, \alpha_{w_0}(g), \alpha_{w_1}(g)$ in G satisfying the following properties:

(a) $w_i \in \mathbb{F}_0$, $|w_i|_{\mathbb{F}_0} \geq N$, $|w_1^{-1}w_2|_{\mathbb{F}_0} = |w_0|_{\mathbb{F}_0} + |w_1|_{\mathbb{F}_0}$,
(b) $l_{rel}(g) \geq M$ and $l_{rel}(\alpha_{w_i}(g)) < \lambda l_{rel}(g)$.

Proof. If $\mathbb{F}_n$ is not a uniform free group of relatively hyperbolic automorphisms then:

For any $M > 0$, $N \geq 1$ and $\lambda > 1$, there exist three elements $g, \alpha_{w_0}(g), \alpha_{w_1}(g)$ in G, with $l_{rel}(g) \geq M$, $|w_i|_{\mathcal{A}} = N$, $|w_1^{-1}w_2|_{\mathcal{A}} = 2N$ and $l_{rel}(\alpha_{w_i}(g)) < \lambda l_{rel}(g)$.

By Lemma 5.16, $\mathbb{F}_0$ is quasi-isometrically embedded in $\mathbb{F}_n$. Thus for every finite basis $\mathcal{A}_0$ of $\mathbb{F}_0$ there exists $\mu \geq 1$ such that, for any $w \in \mathbb{F}_0$:

$$\frac{1}{\mu}|w|_{\mathcal{A}} \leq |w|_{\mathcal{A}_0}.$$

Still by Lemma 5.16, there is $C > 0$ such that each w_i is C-close, for some positive constant C, to an element $w_i' \in \mathbb{F}_0$. If N is strictly greater than C, we can assume $|w_0'^{-1}w_1'|_{\mathcal{A}_0} = |w_0'|_{\mathcal{A}_0} + |w_1'|_{\mathcal{A}_0}$.

From the above two observations, we get $\mu \geq 1$, $C \geq 0$ and two elements $w_i' \in \mathbb{F}_0$ with $|w_0'^{-1}w_1'|_{\mathcal{A}_0} = |w_0'|_{\mathcal{A}_0} + |w_1'|_{\mathcal{A}_0}$ such that:

$$\frac{1}{\mu}|w|_{\mathcal{A}} - C \leq |w_i'|_{\mathcal{A}_0}.$$

The automorphisms act by quasi isometry on G equipped with the $\mathfrak{H}$-relative metric. Since the distance in $\mathbb{F}_n$ from w_i' to w_i is bounded above by C, we get a constant $D \geq 1$ such that

$$l_{rel}(\alpha_{w_i'}(g)) \leq D l_{rel}(\alpha_{w_i}(g)).$$

The proof of the claim readily follows from the above observations. $\square$

As an easy consequence of the definition of a uniform free group of relatively hyperbolic automorphisms (and using the fact that automorphisms act by quasi isometries - this is needed for the existence of C below) we have:

Claim 2. Assume that $\mathbb{F}_0$ is a uniform free group of relatively hyperbolic automorphisms. Then there are $M, N \geq 1$, $\lambda > 1$, $C > 0$ such that, for any $g \in G$ with $l_{rel}(g) \geq M$, for any integer $j \geq 1$, for any $u, v \in \mathbb{F}_0$ with $|u|_{\mathbb{F}_0} \geq jN$, $|v|_{\mathbb{F}_0} \geq jN$ and $|u^{-1}v|_{\mathbb{F}_0} = |u|_{\mathbb{F}_0} + |v|_{\mathbb{F}_0}$,

$$C\lambda^j l_{rel}(g) \leq \max(l_{rel}(\alpha_u(g)), l_{rel}(\alpha_v(g))).$$

Claims 1 and 2 are obviously in contradiction. We so get the lemma. $\square$

Proof of Theorem 2 assuming Theorem 5.14: The implication $(a) \Rightarrow (b)$ comes from Theorem 1. Assume now that $H^2_{(\infty)}(G_{\mathcal{A}}, \mathfrak{H}_{\mathcal{A}})$ strongly vanishes. By Lemma 5.18, $H^2_{(\infty)}(G_{\mathcal{A}_0}, \mathfrak{H}_{\mathcal{A}_0})$ strongly vanishes. By Theorem 5.14 and Remark 5.13, $\mathbb{F}_0 = \langle \mathcal{A}_0 \rangle$ is a uniform free group of relatively hyperbolic automorphisms. By Lemma 5.19, $\mathbb{F}_n = \langle \mathcal{A} \rangle$ is a uniform free group of relatively hyperbolic automorphisms. We so proved $(b) \Rightarrow (c)$. The implication $(c) \Rightarrow (a)$ is the content of [9]. $\square$

Remark 5.20. The essential difference between the general case of Theorem 2 and the particular case where the automorphisms fix each subgroup of $\mathfrak{H}$ up to conjugacy lies in the fact that in the first case G, equipped with the "$\mathfrak{H}_{\mathcal{A}} \cap G$-relative metric", *is not* quasi isometrically embedded in itself by the automorphisms of $\langle \mathcal{A} \rangle$. This quasi isometric embedding is used in a crucial way when working with corridors in the next section.

6. Proof of Theorem 5.14

We follow the strategy of [11]. In Lemma 6.1 below, we give a key inequality which was proven there for the usual hyperbolic setting and whose generalization to the relative setting is straightforward:

Lemma 6.1. *Assume that G is strongly hyperbolic relative to $\mathfrak{H}$ and let (K, L) be a canonical CW-pair for $(G, \mathfrak{H})$. Let $z = \delta h \in B^2_{(\infty)}(\widetilde{K}, \overline{L})$. Let D be a filling of a closed edge-path w in $\widetilde{K}^1$. Then $\langle z, D \rangle \leq ||h||_\infty l_{rel}(w)$. In particular, in the case of strong vanishing of the bounded relative 2-cohomology, there exists $C > 0$ such that, for any filling D in $\widetilde{K}^2$, $\langle z, D \rangle \leq C l_{rel}(\partial D)$.*

Remark 6.2. By definition of a corridor $\mathcal{C} := \mathcal{C}_g$ ($g \in G$ is fixed) in $\widetilde{K}(G_{\mathcal{A}})$, a unique horizontal geodesic $\gamma_g(a)$ in $\mathcal{C}$ is associated to each element $a \in \mathbb{F}_n$. Two geodesics $\gamma := \gamma_g(a)$ and $\gamma' := \gamma_g(a')$ of $\mathcal{C}$ are called *consecutive* if $|a^{-1}a'|_{\mathcal{A}} = 1$. We suppose for the moment that $a' = aa_i^\epsilon$. The horizontal path $\gamma'' = a_i^\epsilon \cdot \gamma$ has the same endpoints as the horizontal geodesic γ'. Hence there is a horizontal filling $D' := D'_{\gamma', \gamma''}$ of the loop $\gamma'(\gamma'')^{-1}$. There is a loop given by $a_i^\epsilon \gamma'' a_i^{-\epsilon} (\gamma')^{-1}$. Let D'' denote a filling of this loop. A *filling* $D_{\gamma, \gamma'}$ of two consecutive geodesics γ, γ' in $\mathcal{C}$ is defined by concatenating D' and D''. By concatenating the fillings $D_{\gamma, \gamma'}$ so defined for each pair of consecutive horizontal geodesics in $\mathcal{C}$, we get a "filling" of $\mathcal{C}$.

The following lemma is a straightforward generalization, to the relative setting, of [11, Proposition 3.1].

Lemma 6.3. *Let $\mathcal{C} := \mathcal{C}_g$ be a corridor in $(\widetilde{K}(G_{\mathcal{A}}), \overline{L}(G_{\mathcal{A}}))$. Let u, v be two elements of $\mathbb{F}_n$ and let $w_1, \cdots, w_n$, $w_1 = u$, $w_n = v$ be the elements of $\mathbb{F}_n$ in the geodesic, in $\Gamma(\mathbb{F}_n, \mathcal{A})$, from w_1 to w_n. Let $\mathcal{C}_{u,v}$ be the union of the horizontal geodesics $\gamma_i := \gamma_g(w_i)$ in $\mathcal{C}$.*

There is a filling D of $\mathcal{C}_{u,v}$ and $z \in Z^2_{(\infty)}(\widetilde{K}(G_{\mathcal{A}}), \overline{L}(G_{\mathcal{A}}))$ such that

$$\langle z, D\rangle = \sum_{i=1}^{n-1} l_{rel}(\gamma_i)\,.$$

Proof. We follow the proof of [11]. We first define a bounded relative 1-cocycle of $(\widetilde{K}, \overline{L})$ by setting, for any horizontal edge $\tilde{e}$: $f(\tilde{e}) = d_{rel}(\tilde{e}^0, t(\tilde{e})) - d_{rel}(\tilde{e}^0, i(\tilde{e}))$, where d_{rel} denotes the relative distance in $(\widetilde{K}, \overline{L})$. Obviously, when applied to a geodesic horizontal edge-path γ in $\widetilde{K}(G_{\mathcal{A}})$, we get $\langle f, \gamma\rangle = l_{rel}(\gamma) = d_{rel}(\tilde{e}^0, t(\gamma))$. We define a 2-cochain z by:

- $\langle z, c\rangle = 1$ if the bottom of c is a 1-cell $\tilde{e}^1_x$ in $\mathcal{C}_{u,v}$, where the $\tilde{e}^1_x$ are the lifts, under $\pi_{\mathcal{A}}$, of the 1-cells e^1_x in $j(K)$ (the image of K in $K(G_{\mathcal{A}})$ under its canonical embedding), i.e. the 1-cells associated to the finite set X in the finite relative presentation of G;
- $\langle z, c\rangle = 0$ if the bottom of c is a 1-cell in $\mathcal{C}_{u,v} \cap \pi^{-1}_{\mathcal{A}}(j(L))$, i.e. the lift of a 1-cell coming from the relative part of G;
- $\langle z, c\rangle = \frac{1}{2}$ if the bottom of c is a 1-cell $\tilde{e}^1_\lambda$ in $\mathcal{C}_{u,v}$, i.e. the lift under $\pi_{\mathcal{A}}$ of a 1-cell between the base point of the complex and the base point of some L_λ;
- $\langle z, c\rangle = 0$ if c is any other 2-cell, in particular if c is a horizontal 2-cell.

The *key-observation* is the following one: if h is a horizontal edge-path, the value of z on the sum of 2-cells which have a 1-cell of h as bottom is equal to the value of f on h. Since f is a 1-cocycle of $(\widetilde{K}, \overline{L})$, the product-structure of $K(G_{\mathcal{A}})$ then implies that z vanishes on the 2-boundaries of $(\widetilde{K}(G_{\mathcal{A}}), \overline{L}(G_{\mathcal{A}}))$, i.e. z is a 2-cocycle. By construction, z vanishes on $\overline{L}(G_{\mathcal{A}})$, i.e. is a *relative* 2-cocycle. When applying the key-observation above to the filling D between two consecutive geodesics γ, γ' in $\mathcal{C}_{u,v}$ as described in Remark 6.2, we get $\langle z, D\rangle = \langle f, \gamma\rangle = l_{rel}(\gamma)$. We so get the announced equality. □

Given a corridor $\mathcal{C}_g$ and a horizontal geodesic $\gamma_g(w)$ in $\mathcal{C}_g$, we say that two horizontal geodesics $\gamma_g(w_0)$ and $\gamma_g(w_1)$ in $\mathcal{C}_g$ are in a same side of $\gamma_g(w)$ if and only if the geodesic in $\Gamma(\mathbb{F}_n, \mathcal{A})$ from w_0 to w_1 does not contain w. A *side of* $\gamma_g(w)$ in $\mathcal{C}_g$ is then a maximal union of horizontal geodesics in $\mathcal{C}_g$ which are all in a same side of $\gamma_g(w)$.

Lemma 6.4. *There exists $\lambda_+ > 1$ and $M \geq 1$ such that, if γ is a horizontal geodesic in a corridor $\mathcal{C}$ with $l_{rel}(\gamma) \geq M$ then there is at least one side of γ in $\mathcal{C}$ such that any horizontal geodesic γ' in this side satisfies $l_{rel}(\gamma') \geq \frac{1}{\lambda_+} l_{rel}(\gamma)$.*

Proof. As was already observed, the finiteness of $\mathfrak{H}$ implies that the relative automorphisms α_i associated to the a_i's generating $\mathbb{F}_n$ act by quasi isometries on G equipped with the $\mathfrak{H}$-relative metric. Thus, there is $\mu > 1$ such that, if γ_0 is a horizontal geodesic in $\mathcal{C}$ consecutive to γ, then $l_{rel}(\gamma_0) \geq \frac{1}{\mu} l_{rel}(\gamma)$.

The strong vanishing of $H^2_{(\infty)}(G_{\mathcal{A}}, \mathfrak{H}_{\mathcal{A}})$ gives a positive constant C such that $\langle z, D\rangle \leq C l_{rel}(\partial D)$ (C is the supremum of $||\sigma(z)||_\infty$ where σ is the bounded section given by the strong vanishing).

Assume the existence, in $\mathcal{C}$, of a horizontal geodesic γ such that there exist two horizontal geodesics $\gamma_0 := \gamma_g(w_0), \gamma_1 := \gamma_g(w_1)$ in two distinct sides of γ in $\mathcal{C}$, satisfying the following properties for some integer $j \geq 1$ (we want to prove that j cannot be chosen arbitrarily large):

(a) $l_{rel}(\gamma_i) < \frac{1}{\mu^j} l_{rel}(\gamma)$,
(b) no horizontal geodesic in $\mathcal{C}$ between γ and γ_i satisfies the above inequality,
(c) $\frac{1}{\mu^j} l_{rel}(\gamma) \geq 3C$.

We consider a filling D of the subset of $\mathcal{C}$ between γ_0 and γ_1, and a cocycle $z \in Z^2_{(\infty)}(\widetilde{K}(G_{\mathcal{A}}), \overline{L}(G_{\mathcal{A}}))$ as given by Lemma 6.3. We want to find a minoration of $\frac{\langle z,D\rangle}{l_{rel}(\partial D)}$ which tends toward infinity with j.

Claim 3. $\frac{\langle z,D\rangle}{l_{rel}(\partial D)}$ is minimal when the $|w_i|_{\mathcal{A}}$'s are minimal.

Proof. Let A_{min} and L_{min} be equal to the values respectively of $\langle z, D\rangle$ and $l_{rel}(\partial D)$ when the $|w_i|_{\mathcal{A}}$'s are minimal. From (c) above, if $l_{rel}(\partial D) = L_{min} + x$, then $\langle z, D\rangle \geq A_{min} + Cx$. Thus $\frac{\langle z,D\rangle}{l_{rel}(\partial D)} \geq \frac{A_{min}}{L_{min}}$ if and only if $C \geq \frac{A_{min}}{L_{min}}$. As was observed before, this last assertion is true thanks to the strong vanishing of $H^2_{(\infty)}(G_{\mathcal{A}}, \mathfrak{H}_{\mathcal{A}})$, which proves the claim. □

From our starting observation, Item (a) implies $|w_i|_{\mathcal{A}} \geq j+1$. By the claim, we can assume $|w_i| = j+1$. Then $l_{rel}(\partial D) \leq 4(j+1) + \frac{2}{\lambda^j} l_{rel}(\gamma)$, whereas $\langle z, D\rangle \geq (\frac{2}{\lambda} + \cdots + \frac{2}{\lambda^{j-1}} + \frac{1}{\lambda^j} + 1) l_{rel}(\gamma)$. It follows that the quotient $\frac{\langle z,D\rangle}{l_{rel}(\partial D)}$ always tends toward infinity with j. As was observed before, this gives a contradiction with the strong vanishing of the second relative ℓ_∞-cohomology group and Lemma 6.4 follows. □

Proof of Theorem 5.14. We argue by contradiction. By a translation in $G_{\mathcal{A}}$, in order to simplify the notations we can take $w = 1_{\mathbb{F}_n}$ in Definition 5.12. Thus, we assume that, for any $\lambda > 1$, for any $N, M \geq 1$, there exist $\gamma := \gamma_g(1_{\mathbb{F}_n}), \gamma_0 := \gamma_g(w_0), \gamma_1 := \gamma_g(w_1)$ in a corridor $\mathcal{C} := \mathcal{C}_g$ such that $l_{rel}(\gamma) \geq M$, $l_{rel}(\gamma_i) < \lambda l_{rel}(\gamma)$, $|w_0|_{\mathcal{A}} = |w_1|_{\mathcal{A}} = N$ and $|w_0^{-1} w_1|_{\mathcal{A}} = 2N$. From Lemma 6.4, we can assume that between γ_0 and γ all horizontal geodesics of $\mathcal{C}$ have relative length at least $\frac{1}{\lambda_+} l_{rel}(\gamma)$. We once again appeal to Lemma 6.3 for the subset of $\mathcal{C}$ between γ_0 and γ. We get a bounded relative 2-cocycle z and a filling D with $\langle z, D\rangle \geq N \frac{1}{\lambda_+} l_{rel}(\gamma)$. But $l_{rel}(\partial D) \leq 2N + (1 + \lambda) l_{rel}(\gamma)$. As soon as $N, l_{rel}(\gamma)$ are sufficiently large enough and $\lambda > 1$ sufficiently small enough, we get a contradiction with the inequality $\langle z, D\rangle \leq Cl_{rel}(\partial D)$ given by the strong vanishing of $H^2_{(\infty)}(G_{\mathcal{A}}, \mathfrak{H}_{\mathcal{A}})$, whence Theorem 5.14. $\square$

Remark 6.5. The finiteness of the family $\mathfrak{H}$ is essential for Theorem 2. However the condition of finite generation for the parabolic subgroups is not necessary if one is only interested in the implication $(a) \Rightarrow (c)$ of Theorem 2. This hypothesis is only essential when appealing to [9] because this last paper uses Farb's approach to relative hyperbolicity and so needs the finite generation of the group (which is here implied by the finite generation of the parabolic subgroups).

For the implication $(a) \Rightarrow (c)$, we could have required only the finite generation of $G_{\mathcal{A}} = G \rtimes_\alpha \mathbb{F}_n$. Even this last assumption would have been unnecessary if working with a slightly restricted kind of $\mathbb{F}_n$-extensions, namely those which fix (up to conjugacy) each subgroup of $\mathfrak{H}$.

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