AN ALGEBRAIC COMBINATION THEOREM FOR GRAPHS OF RELATIVELY HYPERBOLIC GROUPS

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Abstract. We prove a combination theorem for finite graphs of relatively hyperbolic groups, with both Farb's and Gromov's definitions.

1. Introduction

We prove in this paper a quite general theorem (termed *combination theorem*) giving a condition for the fundamental group of a graph of *relatively hyperbolic groups* being a relatively hyperbolic group (with both Farb's and Gromov's definitions - we refer the reader to [20, 10, 5, 29, 7] for the various definitions of relative hyperbolicity and their relationships). This is an extension, to the relative hyperbolicity setting, of a famous theorem due to Bestvina and Feighn [3] (see also [23]): the authors introduce there the notion of (finite) graphs of qi-embedded groups and, assuming the Gromov hyperbolicity of the vertex groups, give a sufficient condition for the Gromov hyperbolicity of the fundamental group of the given graph of groups. Since then different proofs have appeared, which treat the so-called 'acylindrical case': see, among others, [19, 26]. A graph of groups is acylindrical if the fixed set of the action of any element of its fundamental group on the universal covering has uniformly bounded diameter. The non-acylindrical case is less common: see [24] which relies on [3] but clarifies its consequences when dealing with a certain class of mapping-tori of injective, non surjective free group endomorphisms, or [11] which, by an approach similar to the one presented here, gives a new proof of [3] in the case of mapping-tori of free group endomorphisms.

First combination theorems in some particular (essentially acylindrical) cases have been given in the setting of relative hyperbolicity: [1], [8] or [28, 30]. One result [15] treats a particular non-acylindrical case, namely the relative hyperbolicity of one-ended hyperbolic by cyclic groups. In [27] the authors give a combination theorem dealing with more general non-acylindrical cases than [15]. This last paper heavily relies upon [3], which is used as a "black-box". The current paper is a sequel to [13] where we proved a dynamical and geometric combination theorem for trees of hyperbolic and relatively hyperbolic spaces. The paper [13] is independent of [3] and in fact presents a new proof of the geometric combination theorem of [3]. Our work here is to derive from [13] the algebraic consequences. We only prove here the sufficiency of the conditions we introduce (algebraic exponential-separation property and its strengthening) for the relative hyperbolicity of a graph of relatively hyperbolic groups. However they are also necessary conditions: in the absolute hyperbolicity case, Gersten was the first to give the converse to the combination theorem, using cohomological arguments [17] and we adapt his arguments in [14]. Bowditch exposed a more direct proof in [6] that [27] adapts in his setting.

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2. Main result

We start by recalling the definitions of weak and strong relative hyperbolicity. Both notions were defined in [10]. However the approach of strong relative hyperbolicity is taken from [5].

Definition 2.1. [10, 5] Let G be a group with finite generating set S and Cayley graph $\Gamma_S(G)$. Let Λ be a set and let $\mathcal{H} = \{H_i\}_{i\in\Lambda}$ be a a family of subgroups H_i of G.

The *$\mathcal{H}$-coned graph* $\Gamma_S^{\mathcal{H}}(G)$ is the graph obtained from $\Gamma_S(G)$ by:

- adding an *exceptional vertex* $v(gH_i)$ for each left H_i-coset,
- putting an *exceptional edge* of length $\frac{1}{2}$ between $v(gH_i)$ and each vertex of $\Gamma_S(G)$ associated to an element in the coset gH_i,

for each $H_i \in \mathcal{H}$.

Definition 2.2. [10, 5] With the notations of Definition 2.1:

The group G is *weakly hyperbolic relative to $\mathcal{H}$* if and only if $\Gamma_S^{\mathcal{H}}(G)$ is Gromov hyperbolic.

The group G is *strongly hyperbolic relative to $\mathcal{H}$* if and only if:

(a) The graph $\Gamma_S^{\mathcal{H}}(G)$ is Gromov hyperbolic;
(b) For any positive integer n, any edge in $\Gamma_S^{\mathcal{H}}(G)$ is contained in only finitely many embedded loops of length n.

The subgroups H_i in the family $\mathcal{H}$ are the *parabolic subgroups* of $(G, \mathcal{H})$.

There is a new "word-metric" on the group G which is naturally associated to the metric on a $\mathcal{H}$-coned graph.

Definition 2.3. Let G be a group with finite generating set S. Let Λ be a set and let $\mathcal{H} = \{H_i\}_{i\in\Lambda}$ be a family of subgroups of G.

The *$\mathcal{H}$-word metric* $|.|_{\mathcal{H}}$ is the word-metric for G equipped with the generating set $S_{\mathcal{H}} = S \cup \bigcup_{i\in\Lambda} H_i$.

As the subgroups H_i are usually infinite, so will be $S_{\mathcal{H}}$. It was observed in [10] that in the setting of strong relative hyperbolicity the family of parabolic subgroups is necessarily *almost malnormal*:

Definition 2.4. Let G be a group, let Λ be a set and let $\mathcal{H} = \{H_i\}_{i\in\Lambda}$ be a a family of subgroups of G. The family $\mathcal{H}$ is *almost malnormal* if and only if:

(a) any subgroup H_i is almost malnormal in G, i.e. $g^{-1}H_ig \cap H_i$ is finite for any $g \notin H_i$.
(b) for any two $H_i, H_j \in \mathcal{H}$ with $i \neq j$, $g^{-1}H_ig \cap H_j$ is finite for any $g \in G$.

Since the ultimate goal is a theorem about graphs of relatively hyperbolic groups, we introduce some notations for graphs and graphs of groups. If Γ is a graph, $V(\Gamma)$ (resp. $E(\Gamma)$) denotes its set of vertices (resp. of oriented edges). For $e \in E(\Gamma)$ we denote by e^{-1} the same edge with opposite orientation. The map $e \mapsto e^{-1}$ is a fixed-point free involution of $E(\Gamma)$. If p is an edge-path in Γ, in particular if p is an edge, $i(p)$ (resp. $t(p)$) denotes the initial (resp. terminal) vertex of p. An edge-path p is *reduced* if no edge e in p is followed by its opposite e^{-1}. In a tree, given any two vertices x, y, we denote by $[x, y]$ the unique reduced edge-path from x to y. In a metric graph Γ (i.e. a graph equipped with a positive length l_e on each edge e and an isometry from e to the real interval $(0, l_e)$ - for instance a Cayley graph or a $\mathcal{H}$-coned graph), $d_\Gamma(x, y)$ denotes the geodesic distance between two vertices x, y, $[x, y]$ any geodesic, i.e. length-minimizing, edge-path between x and y and $|p|_\Gamma$ the length of an edge-path p in Γ.

Let us now consider a graph of groups $\mathcal{G} = (\Gamma, \{G_e\}, \{G_v\}, \{\imath_e\})$ where

- Γ is a graph,
- for each oriented edge e, G_e denotes the group associated to the edge e (an *edge-group*) and $G_e = G_{e^{-1}}$,
- for each vertex v, G_v denotes the group associated to the vertex v (*a vertex-group*),
- for each oriented edge e, $\imath_e \colon G_e \to G_{t(e)}$ is a monomorphism.

Definition 2.5. Let G and G' be two groups and let $\mathcal{H}$ (resp. $\mathcal{H}'$) denote a family of subgroups of G (resp. of G'). A morphism $\alpha \colon G \to G'$ is a *relative morphism from* $(G, \mathcal{H})$ *to* $(G', \mathcal{H}')$ if and only if for each subgroup $H \in \mathcal{H}$ there is a subgroup $H' \in \mathcal{H}'$ such that $\alpha(H)$ is conjugated to a subgroup of H'.

Definition 2.6. A *graph of weakly relatively hyperbolic groups* (resp. *of strongly relatively hyperbolic groups*) $\mathcal{G} = (\Gamma, \{(G_v, \mathcal{H}_v)\}, \{(G_e, \mathcal{H}_e)\}, \{\imath_e\})$ is a finite graph of finitely generated groups $(\Gamma, \{G_e\}, \{G_v\}, \{\imath_e\})$ satisfying the following properties:

(a) Each edge-group G_e and each vertex-group G_v is weakly hyperbolic (resp. strongly hyperbolic) relative to a specified, possibly empty, finite family of infinite subgroups $\mathcal{H}_e = \mathcal{H}_{e^{-1}}$ and $\mathcal{H}_v$.

(b) There are $\mathfrak{a} \geq 1$, $\mathfrak{b} \geq 0$ such that for any oriented edge e of Γ, $\imath_e$ is a $(\mathfrak{a}, \mathfrak{b})$-quasi isometric embedding from $(G_e, |.|_{\mathcal{H}_e})$ to $(G_{t(e)}, |.|_{\mathcal{H}_{t(e)}})$.

(c) For each oriented edge e, $\imath_e$ is a relative endomorphism from $(G_e, \mathcal{H}_e)$ to $(G_{t(e)}, \mathcal{H}_{t(e)})$.

When considering a finite graph of groups $\mathcal{G} = (\Gamma, \{G_e\}, \{G_v\}, \{\imath_e\})$, the group we are interested in is its *fundamental group* $\pi(\mathcal{G})$, that we are now going to define. We borrow the following presentation from [32] (see also [25]). We refer the interested reader to [34] for a nice and quite complete introduction to Bass-Serre theory (another point of view is developed in [9]). A *(finite)* $\mathcal{G}$*-path* p from $v \in V(\Gamma)$ to $v' \in V(\Gamma)$ is a sequence $g_0, e_1, g_1, \cdots, e_k, g_k$ where $k \geq 0$ is an integer, $e_1 \cdots e_k$ is and edge-path in Γ from v to v', $g_0 \in G_v$, $g_k \in G_{v'}$ and $g_j \in G_{t(e_j)} = G_{i(e_{j+1})}$ for $j = 1, \cdots, k-1$. The integer k is the *length* of the $\mathcal{G}$-path, denoted by $|p|_{\mathcal{G}}$. If $l \geq 0$ is any integer smaller than the length of p, we denote by p_l the initial subpath $g_0, e_1, g_1, \cdots, e_l, g_l$ of p. The concatenation of $\mathcal{G}$-paths is defined in the obvious way. We write $p \sim q$ if and only if p and q are two $\mathcal{G}$-paths which are equivalent for the equivalence relation generated by the elementary equivalences $g, e, g' \sim g\imath_{e^{-1}}(h), e, \imath_e(h^{-1})g'$ and $g, e, 1, e^{-1}, g' \sim gg'$. We then denote by $[p]$ the $\sim$-equivalence class of a $\mathcal{G}$-path p. Once a base-vertex $v_0 \in V(\Gamma)$ has been chosen, the fundamental group $\pi(\mathcal{G}, v_0)$ is the group of $\sim$-equivalence classes of $\mathcal{G}$-paths from v_0 to v_0. The group operation is the concatenation: $[p][q] := [pq]$.

A *reduction* of a finite $\mathcal{G}$-path p consists of the substitution of a subsequence in p of the form $g, e, \imath_e(h), e^{-1}, g'$ by the sequence $g\imath_{e^{-1}}(h)g'$. Observe that, if q is the resulting $\mathcal{G}$-path, then $p \sim q$. A $\mathcal{G}$-path is *reduced* if no reduction is possible. Any $\mathcal{G}$-path is equivalent to a reduced one. More precisely, if $p = g_0, e_1, g_1, \cdots, e_k, g_k$ and $q = g_0', e_1', g_1', \cdots, e_{k'}', g_{k'}'$ are two equivalent reduced $\mathcal{G}$-paths then $k = k'$, $e_j' = e_j$ for $j = 1, \cdots, k$ and $p_j^{-1}q_j$ defines an element in $G_{t(e_j)}$ which belongs to $\imath_{e_{j+1}^{-1}}(G_{e_{j+1}})$ for $j = 0, \cdots, k-1$. Finally, if p is a reduced $\mathcal{G}$-path, we denote by $\overline{p}$ the set of all $\mathcal{G}$-paths equivalent to p modulo a right-multiplication by $g \in G_{t(p)}$.

In order to state the combination theorem for graphs of weakly relatively hyperbolic groups we are now going to introduce a new property termed *algebraic exponential-separation property.*

Definition 2.7. The graph of (weakly or strongly) relatively hyperbolic groups $\mathcal{G} = (\Gamma, \{(G_v, \mathcal{H}_v)\}, \{(G_e, \mathcal{H}_e)\}, \{\imath_e\})$ satisfies the *algebraic exponential-separation property* (*algebraic ESP* in short) if and only if there exist $\lambda > 1$ and integers $M, N \geq 1$ such that for any reduced $\mathcal{G}$-paths $p = g_0e_1g_1 \cdots e_Ng_N$ and $q = h_0f_1h_1 \cdots f_Nh_N$ with $i(e_1) = i(f_1) := v$ and satisfying that $p^{-1}q$ is a reduced edge-path, for any element $g \in G_v$ with $p^{-1}gp \in G_{t(e_N)}$, $q^{-1}gq \in G_{t(f_N)}$ and $|g|_{\mathcal{H}_v} \geq M$ we have

$$\lambda |g|_{\mathcal{H}_v} \leq \max(|p^{-1}gp|_{\mathcal{H}_{t(e_N)}}, |q^{-1}gq|_{\mathcal{H}_{t(f_N)}}).$$

The combination theorem for graphs of weakly relatively hyperbolic groups is then stated as follows (we recall that a parabolic subgroup in a relatively hyperbolic group $(G, \mathcal{H})$ is a subgroup in the family $\mathcal{H}$):

Theorem 2.8. *If a graph of weakly relatively hyperbolic groups satisfies the algebraic ESP then its fundamental group is weakly hyperbolic relative to the family composed of all the parabolic subgroups of the vertex-groups.*

Definition 2.9 below gives the strengthening of the algebraic ESP which is needed to deal with strong relative hyperbolicity. We adopt the convention that, given a graph of relatively hyperbolic groups $\mathcal{G} = (\Gamma, \{(G_v, \mathcal{H}_v)\}, \{(G_e, \mathcal{H}_e)\}, \{\imath_e\})$ and $h \in \pi(\mathcal{G})$, the length $|h|_{\mathcal{H}_v}$ of h measured with respect to the relative $\mathcal{H}_v$-metric of the vertex-group G_v is infinite if h does not belong to G_v.

Definition 2.9. A graph of relatively hyperbolic groups $\mathcal{G} = (\Gamma, \{(G_v, \mathcal{H}_v)\}, \{(G_e, \mathcal{H}_e)\}, \{\imath_e\})$ satisfies the *strong algebraic exponential-separation property* (*strong algebraic ESP* in short) if and only if there exist $\lambda > 1$ and integers $M, N \geq 1$ such that:

(a) For any reduced $\mathcal{G}$-paths $p = g_0e_1g_1 \cdots e_Ng_N$ and $q = h_0f_1h_1 \cdots f_Nh_N$ with $i(e_1) = i(f_1) := v$ and satisfying that $p^{-1}q$ is a reduced edge-path, for any element $g \in G_v$ with $p^{-1}gp \in G_{t(e_N)}$, $q^{-1}gq \in G_{t(f_N)}$ and $|g|_{\mathcal{H}_v} \geq M$ we have

$$\lambda |g|_{\mathcal{H}_v} \leq \max(|p^{-1}gp|_{\mathcal{H}_{t(e_N)}}, |q^{-1}gq|_{\mathcal{H}_{t(f_N)}}).$$

(b) For any element g in some vertex-group G_v with $|g|_{\mathcal{H}_v} < M$, for any parabolic subgroup $H \in \mathcal{H}_v$ with $g \notin H$ which admits some length N reduced $\mathcal{G}$-path p such that $[pH] \cap [Hp]$ is infinite, we have $|p^{-1}gp|_{\mathcal{H}_v} \geq M$.

Roughly speaking, item (b) of Definition 2.9 amounts to asking that any two distinct orbits of left H_i-cosets such that H_i admits a length N reduced $\mathcal{G}$-path p such that $[pH_i] \cap [Hp_i]$ is infinite separate exponentially. Indeed, by item (a) and with the notations of Definition 2.9, the length of $p^{-1}gp$ is necessarily exponentially dilated in all the directions with the exception of the one given by p.

Definition 2.10. Let $\mathcal{G} = (\Gamma, \{(G_v, \mathcal{H}_v)\}, \{(G_e, \mathcal{H}_e)\}, \{\imath_e\})$ be a graph of strongly relatively hyperbolic groups. The *induced graph of parabolic subgroups* is the graph of groups $\mathcal{G}_{\mathcal{P}} = (\Gamma_{\mathcal{P}}, \{J_v\}, \{J_e\}, i_e)$ defined by:

(a) There is a bijection σ_V (resp. σ_E) from the set of vertices (resp. edges) of $\Gamma_{\mathcal{P}}$ to the set of all the parabolic subgroups of the vertex-groups (resp. edge-groups) of $\mathcal{G}$, which are the vertex-groups J_v (resp. edge-groups J_e) of $\mathcal{G}_{\mathcal{P}}$.

(b) There is an oriented edge e with terminal vertex v in $\Gamma_{\mathcal{P}}$ if and only if $\imath_{\sigma_E(e)}(J_e) \subset J_v$. In this case i_e is the restriction of $\imath_{\sigma_E(e)}$ to J_e.

An *induced elementary graph of parabolic subgroups* is any connected component of the induced graph of parabolic subgroups.

Our main result in the setting of srong relative hyperbolicity is now stated as follows:

Theorem 2.11. *If a graph of strongly relatively hyperbolic groups satisfies the strong algebraic ESP, then its fundamental group is strongly hyperbolic relatively to the family composed of the fundamental groups of all the induced elementary graphs of parabolic spaces.*

3. The geometric combination theorems

This section is borrowed from [13].

3.1. **Relatively hyperbolic spaces.** If S is a set, the *cone with base* S is the space $S \times [0, \frac{1}{2}]$ with $S \times \{0\}$ collapsed to a point, termed the *vertex of the cone* or *cone-vertex.* This cone is considered as a metric space, with distance function $d_S((x,t),(y,t')) = t + t'$. Let (X, d) be a geodesic space. Putting a cone over a subset S of X consists of pasting to X a cone with base S by identifying $S \times \{1/2\}$ with $S \subset X$. The resulting metric space is denoted by $\widehat{X}$ and its subspace consisting of the cone over S by $\widehat{S}$. The space $\widehat{X}$ is such that all the points in S are now at distance $\frac{1}{2}$ from the cone-vertex and so at distance 1 one from each other.

Definition 3.1.
A *geodesic pair* $(X, \mathcal{P})$ is a geodesic space X equipped with a family of disjoint subspaces $\mathcal{P} = \{P_i\}_{i \in \Lambda}$, termed *parabolic subspaces.*

Definition 3.2. [10]
Let $(X, \mathcal{P})$ be a geodesic pair.

(a) The *coned-space* $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ is the metric space obtained from $(X, \mathcal{P})$ by putting a cone over each parabolic subspace in $\mathcal{P}$ and $d_{\mathcal{P}}$ is the *coned*, or *relative distance.*

(b) The space X is weakly hyperbolic relative to $\mathcal{P}$ if and only if the coned-space $(\widehat{X}, d_{\mathcal{P}})$ is Gromov hyperbolic.

Let $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ be a coned-space. We say that a path $\widehat{g}$ in $\widehat{X}$ *backtracks* if for the arc-length parametrization of $g \colon [0, l] \to \widehat{X}$ there exists a parabolic subspace P_i and times $0 \leq t_0 < t_1 < t_2 \leq l$ such that $g(t) \notin P_i$ if $t_0 - \epsilon < t < t_0$ and $t_1 < t < t_1 + \epsilon$ for $\epsilon > 0$ sufficiently small, $g([t_0, t_1]) \subset P_i$ and $g(t_2) \in P_i$. In other words a path backtracks if and only if it reenters a parabolic subspace that it left before. Let $\widehat{g}$ be a (u, v)-quasi geodesic path in $(\widehat{X}_{\mathcal{S}}, d_{\mathcal{S}})$ which does not backtrack. A *trace* g of $\widehat{g}$ is a subpath of X obtained by substituting each subpath of $\widehat{g}$ not in X by a subpath in some parabolic subspace P_i, which is a geodesic for the path-metric induced by X on P_i.

Definition 3.3. [10] Let $(X, \mathcal{P})$ be a geodesic pair.

The coned-space $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ satisfies the *Bounded-Parabolic Penetration property (BPP)* if and only if there exists $C(u,v) \geq 0$ such that, for any two (u,v)-quasi geodesics $\widehat{g}_0, \widehat{g}_1$ of $(\widehat{X}_{\mathcal{S}}, d_{\mathcal{S}})$ with traces g_0, g_1 in (X,d), which have the same initial point, which have terminal points at most 1-apart and which do not backtrack, the following two properties are satisfied:

(a) if both g_0 and g_1 intersects a parabolic subspace P_i then their first intersection points with S_i are $C(u,v)$-close in (X,d),

(b) if g_0 intersects a parabolic subspace P_i that g_1 does not, then the length in (X,d) of $g_0 \cap P_i$ is smaller than $C(u,v)$.

Definition 3.4. [10] Let $(X, \mathcal{P})$ be a geodesic pair.

The space X is *strongly hyperbolic relative to* $\mathcal{P}$ if and only if the coned-space $(\widehat{X}_{\mathcal{P}}, d_{\mathcal{P}})$ is Gromov hyperbolic and satisfies the BPP.

3.2. **Trees of spaces.**

Definition 3.5. (compare [3])

(a) A *tree of metric spaces* $\mathfrak{T} = (\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ is a metric tree $\mathcal{T}$ with length 1 edges, together with two collections of geodesic spaces, the collection of *edge-spaces* $\{X_e\}_{e \in E(\Gamma)}$ indexed over the oriented edges e of $\mathcal{T}$ which satisfy $X_e = X_{e^{-1}}$ and the collection of *vertex-spaces* $\{X_v\}_{v \in V(\Gamma)}$ indexed over the vertices v of $\mathcal{T}$, and a collection of maps $\jmath_e \colon X_e \to X_{t(e)}$ from the edge-spaces to the vertex-spaces.

(b) A *tree of qi-embedded metric spaces* is a tree of metric spaces $(\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ such that there exist two fixed real constants $\mathfrak{a} \geq 1$ and $\mathfrak{b} \geq 0$ such that the maps $\jmath_e \colon X_e \to X_{t(e)}$ from the edge-spaces X_e to the vertex-spaces X_v are $(\mathfrak{a}, \mathfrak{b})$-quasi isometric embeddings.

(c) A *tree of hyperbolic spaces* is a tree of qi-embedded metric spaces such that there is $\delta \geq 0$ for which each edge- and vertex-space is a δ-hyperbolic space.

Before defining trees of relatively hyperbolic spaces we need to introduce the notion of the *coned-extension* of a map between geodesic pairs.

Definition 3.6. Let $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$ be two geodesic pairs.

(a) A map $f \colon X \to Y$ is a *pair-map from* $(X, \mathcal{P})$ *to* $(Y, \mathcal{Q})$ if and only if for every parabolic subspace $P \in \mathcal{P}$ there is a parabolic subspace $Q \in \mathcal{Q}$ such that $f(P) \subset Q$.

(b) Let $f \colon (X, \mathcal{P}) \to (Y, \mathcal{Q})$ be a pair-map and let $\widehat{X}, \widehat{Y}$ be the coned-spaces associated respectively to $(X, \mathcal{P})$ and $(Y, \mathcal{Q})$. A map $\widehat{f} \colon \widehat{X} \to \widehat{Y}$ is a *coned-extension of* f if and only if it satisfies the following properties:

- Its restriction to X is equal to f.
- For any parabolic subspace $P \in \mathcal{P}$ with $f(P) \subset Q \in \widehat{Q}$, $\widehat{f}$ is a pair-map from $(\widehat{X}, \widehat{P} \setminus P)$ to $(\widehat{Y}, \widehat{Q} \setminus Q)$ which sends the cone-vertex of $\widehat{P}$ to the cone-vertex of $\widehat{Q}$.

Definition 3.7.

(a) A *tree of geodesic pairs* $(\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ is a tree of metric spaces $(\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ such that for each edge e and each vertex v, $(X_e, \mathcal{P}_e)$ and $(X_v, \mathcal{P}_v)$ are geodesic pairs, for each edge e, $\mathcal{P}_e = \mathcal{P}_{e^{-1}}$ and $\jmath_e \colon (X_e, \mathcal{P}_e) \to (X_{t(e)}, \mathcal{P}_{t(e)})$ is a pair-map.

(b) A *tree of weakly* (resp. *strongly*) relatively hyperbolic spaces is a tree of geodesic pairs $\mathfrak{T} = (\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ such that:
- For each edge e, the edge-space X_e is weakly (resp. strongly) hyperbolic relatively to the family of parabolic subspaces $\mathcal{P}_e$. For each vertex v the vertex-space X_v is weakly (resp. strongly) hyperbolic relative to the family of parabolic subspaces $\mathcal{P}_v$.
- If $\widehat{X}_e$ and $\widehat{X}_v$ denote the coned-spaces equipped with the relative metrics associated to the geodesic pairs $(X_e, \mathcal{P}_e)$ and $(X_v, \mathcal{P}_v)$ and $\widehat{\jmath}_e$ is a coned-extension of $\jmath_e$ then $\widehat{\mathfrak{T}} = (\mathcal{T}, \{\widehat{X}_e\}, \{\widehat{X}_v\}, \{\widehat{\jmath}_e\})$ is a tree of qi-embedded metric spaces.

Definition 3.8. Let $\mathfrak{T} = (\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ be a tree of metric spaces.

If $E^+(\mathcal{T})$ denotes the subset of $E(\mathcal{T})$ composed of exactly one representative in each pair (e, e^{-1}) then the space $\widetilde{X}$ obtained from

$$\bigsqcup_{e \in E^+(\mathcal{T})} (X_e \times [0,1]) \sqcup \bigsqcup_{v \in V(\mathcal{T})} X_v$$

by identifying $(x, 1) \in X_e \times [0, 1]$ with $\jmath_e(x) \in X_{t(e)}$ and $(x, 0) \in X_e \times [0, 1]$ with $\jmath_{e^{-1}}(x) \in X_{i(e)}$ for each $e \in E^+(\mathcal{T})$ is called the *geometric realization of* $\mathfrak{T}$.

Definition 3.9. Let $(\widetilde{X}, \pi, \mathcal{T})$ be the geometric realization of a tree of qi-embedded metric spaces. For any two points x, y in $\widetilde{X}$, let $\mathcal{P}(x, y)$ be the set of all the continuous paths from x to y which are the concatenation of horizontal paths and of non-trivial intervals.

The *tree of spaces-distance* between any two points x, y in $\widetilde{X}$, denoted by $d_{\widetilde{X}}(x, y)$, is the infimum of the lengths of the paths in $\mathcal{P}(x, y)$, measured as the sum of the horizontal and interval-lengths of their subpaths.

The following lemma is obvious:

Lemma 3.10. *With the notations of Definition 3.9, the space $\widetilde{X}$ equipped with the tree of spaces-distance $d_{\widetilde{X}}$ is a quasi geodesic metric space.*

Remark 3.11. Since the geometric realization of a tree of metric spaces actually is the space we will work on and is well-defined once given the tree, with a slight abuse of terminology we will often denote by $(\widetilde{X}, \pi, \mathcal{T})$ a tree of qi-embedded metric spaces and write "a tree of metric spaces ..." for "the geometric realization of a tree of metric spaces ...".

3.3. **The theorems.**

Definition 3.12. Let $(\widetilde{X}, \mathcal{T}, \pi)$ be the geometric realization of a tree of qi-embedded metric spaces.

For $v \geq 0$, a *v-vertical segment* in $\widetilde{X}$ is a section σ_ω of π over a geodesic ω of $\mathcal{T}$ which is a $(v+1, v)$-quasi isometric embedding of ω in $(\widetilde{X}, d_{(\widetilde{X}})$.

Definition 3.13. (compare [3])

A tree of qi-embedded spaces satisfies the *geometric exponential-separation property* (*geometric ESP* in short) if and only if for any $v \geq 0$ there exist $\lambda > 1$ and positive integers t_0, M such that, for any geodesic segment $[\beta, \gamma] \subset \mathcal{T}$ of length $2t_0$ and midpoint α, any two v-vertical segments s_0, s_1 over $[\beta, \gamma]$ with $d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha) \geq M$ satisfy:

$$\max(d_{hor}(s_0 \cap X_\beta, s_1 \cap X_\beta), d_{hor}(s_0 \cap X_\gamma, s_1 \cap X_\gamma)) \geq \lambda d_{hor}(s_0 \cap X_\alpha, s_1 \cap X_\alpha)$$

We will sometimes say that the v-vertical segments are *exponentially separated.*

Theorem 3.14. *Let $\mathfrak{X} = (\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ be a tree of weakly relatively hyperbolic spaces. If $\widehat{\mathfrak{X}}$ satisfies the geometric ESP then $\mathfrak{X}$ is weakly hyperbolic relative to the family composed of all the parabolic subspaces of the vertex-spaces.*

Definition 3.15. Let $(\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ be a tree of geodesic pairs.

The *induced forest of parabolic spaces* is the tree of spaces $(\mathcal{F}_{\mathcal{P}}, \{\mathfrak{P}_e\}, \{\mathfrak{P}_v\}, \{\imath_e\})$ defined as follows:

(a) There is a bijection σ_E (resp. σ_V) from the set of edges (resp. vertices) of $\mathcal{F}_{\mathcal{P}}$ to the set of all the parabolic subspaces of the edge-spaces (resp. vertex-spaces) of $\mathcal{T}$ which are the edge-spaces $\mathfrak{P}_e$ (resp. vertex-spaces $\mathfrak{P}_v$) of $\mathcal{F}_{\mathcal{P}}$.

(b) There is an oriented edge e with terminal vertex v in $\mathcal{F}_{\mathcal{P}}$ if and only if $\jmath_{\sigma_E(e)}(\mathfrak{P}_e) \subset \mathfrak{P}_v$. In this case $\imath_e$ is the restriction of $\jmath_{\sigma_E(e)}$ to $\mathfrak{P}_e$.

An *induced tree of parabolic spaces* is any connected component of the induced forest of parabolic spaces.

Remark 3.16. The geometric realization of the induced forest of parabolic spaces of a tree of geodesic pairs is naturally embedded in the geometric realization of the latter. So, assimilating this forest and the tree to their geometric realizations, it makes sense to speak about the "horizontal distance between two induced trees of parabolic spaces" or about the vertical diameter of some of their subsets.

Definition 3.17. A tree of strongly relatively hyperbolic spaces satisfies the *strong geometric ESP* if and only if it satisfies the geometric ESP and for any $l \geq 0$ there is $t \geq 0$ such that for any two distinct induced trees of parabolic spaces, the region where they are at horizontal distance smaller than l has vertical diameter smaller than t.

Theorem 3.18. *Let $\mathfrak{X} = (\mathcal{T}, \{(X_e, \mathcal{P}_e)\}, \{(X_v, \mathcal{P}_v)\}, \{\jmath_e\})$ be a tree of strongly relatively hyperbolic spaces. If $\widehat{\mathfrak{X}}$ satisfies the strong geometric ESP then $\mathfrak{X}$ is strongly hyperbolic relatively to the family composed of all the induced trees of parabolic spaces.*

3.4. **Technical lemma.** We will need the following two basic, technical lemma proven in [13].

Lemma 3.19. *Let $\delta \geq 0$ and let $(\mathcal{T}, \{X_e\}, \{X_v\}, \{\jmath_e\})$ be a tree of δ-hyperbolic spaces. There exists $C \geq 0$ such that if $v \geq C$, if e is an edge of $\mathcal{T}$ and if h is a horizontal geodesic in $X_{t(e)}$, then:*

- *If no v-vertical segment starting at h can be defined over e, then*

$$\mathrm{diam}_{X_{t(e)}}(P_h^{hor}(\jmath_e(X_e))) \leq 2\delta$$

- *If v-vertical segments can be defined over e starting at the initial and terminal points of h, then v-vertical segments can be defined over e starting at any point in h.*

The constants λ, M, t_0 appearing in Definition 3.13 will be referred to as the *constants of hyperbolicity.*

Lemma 3.20. *Let $(\widetilde{X}, \mathcal{T}, \pi)$ be a tree of hyperbolic spaces.*

If $v \geq C_{3.19}$ is such that the v-vertical segments of $\widetilde{X}$ are exponentially separated with constants of hyperbolicity $\lambda_v > 1$, $M_v, t_0 \geq 0$ then for any $w \geq 0$, the w-vertical segments are exponentially separated, with constants of hyperbolicity $\lambda_w > 1$, $M_w \geq 0$ and t_0.

4. From Geometry to Algebra

The goal of this section is to derive the algebraic combination theorems of Section 2 (Theorem 2.8 - weak relative hyperbolicity case - and Theorem 2.11 - strong relative hyperbolicity case) from the geometric combination theorems of Section 3 (Theorems 3.14 and 3.18).

4.1. **A geometric model for a tree of relatively hyperbolic groups.** Let G be a group with generating set S, let $\mathcal{H}$ be a finite family of subgroups of G and let $\Gamma_S^{\mathcal{H}}(G)$ be the $\mathcal{H}$-coned graph. We subdivide each exceptional edge at its middle point. The *augmented Cayley graph* $\Gamma_S^{aug}(G)$ is the closure of the unique connected component of $\Gamma_S^{\mathcal{H}}(G)\setminus\{$ middle points of exceptional edges $\}$ which contains no exceptional vertex. The edges in $\Gamma_S^{aug}(G)\setminus\Gamma_S(G)$ are the *auxiliary edges*. We denote by M_i each maximal set of non-exceptional vertices of auxiliary edges which are all connected to a same exceptional vertex, and by $\mathcal{M}$ the family composed of all the sets M_i. Observe that $\widehat{\Gamma}_S^{aug}(G)$, the coned-space obtained from $\Gamma_S^{aug}(G)$ by putting a cone over each set $M_i\in\mathcal{M}$, is equivalently defined as the metric space obtained by putting a length of 1 on each edge in the complement of $\Gamma_S^{aug}(G)$ in $\Gamma_S^{\mathcal{H}}(G)$.

Let $\alpha\colon(G,\mathcal{H})\to(G',\mathcal{H}')$ be a relative endomorphism (see Definition 2.5), where G and G' are two groups with respective generating sets S and S'. An *α-map* is a continuous PL-map $f_\alpha\colon\Gamma_S^{aug}(G)\to\Gamma_{S'}^{aug}(G')$ which satisfies the following properties:

- For each edge (g,gx_i) $(x_i\in S)$ of $\Gamma_S(G)$, $f_\alpha(x_i)$ is the edge-path reading $\alpha(x_i)$ between the vertex $\alpha(g)$ and the vertex $\alpha(gx_i)$.
- For each auxiliary edge e from $g\in\Gamma_S(G)$ to the midpoint of the edge $(g,v(gH))$ in $\Gamma_S^{\mathcal{H}}(G)$ (H is a parabolic subgroup in $\mathcal{H}$), $f_\alpha(e)$ starts at $\alpha(g)$, reads a conjugacy-element h such that $\alpha(H)=h^{-1}H_0h$ with H_0 a subgroup of some parabolic subgroup H' in $\mathcal{H}'$ and ends with the auxiliary edge of ${\Gamma'}_{S'}^{aug}(G')$ contained in the exceptional edge $(\alpha(g)h,v(\alpha(g)hH'))$ of ${\Gamma'}_{S'}^{\mathcal{H}'}(G')$.

If $(G,\mathcal{H})$ and $(G',\mathcal{H}')$ are strongly relatively hyperbolic groups, the conjugacy element h above is well-defined up to a right-multiplication by an element in H so that the map f_α is uniquely defined. Otherwise there is a choice of the conjugacy element.

Let $\mathcal{G}=(\Gamma,\{(G_v,\mathcal{H}_v)\},\{(G_e,\mathcal{H}_e)\},\{\imath_e\})$ be a graph of (weakly or strongly) relatively hyperbolic groups. For each edge- or vertex-group we consider the augmented Cayley graph defined above (we assume a generating set has been chosen for each one of these groups, but for simplifying the notations we will not indicate them). For each oriented edge e, we consider an $\imath_e$-map $f_{\imath_e}$ as defined above. Let $\mathcal{T}$ be the universal covering of $\mathcal{G}$, in other words the Bass-Serre tree of $\mathcal{G}$, we denote by T the corresponding combinatorial tree. To each vertex $\overline{v}$ of T corresponds a left-class xG_v of some vertex-group G_v of $\mathcal{G}$, to each edge $\overline{e}$ of T an edge-group G_e of $\mathcal{G}$. The attaching-maps $\overline{\imath}_{\overline{e}}$ are left-translates $x\imath_e$ of the attaching-morphisms of $\mathcal{G}$, where at a given vertex $G_{t(\overline{e})}$ all the $x\in G_{t(e)}$ belong to distinct left $\imath(G_e)$-classes. We denote by $\overline{f}_{\imath_e}$ the corresponding attaching-maps between the associated augmented Cayley graphs. The following lemma - definition is obvious:

Lemma 4.1. *With the notations above: if $\mathcal{G}=(\Gamma,\{(G_v,\mathcal{H}_v)\},\{(G_e,\mathcal{H}_e)\},\{\imath_e\})$ is a graph of weakly (resp. strongly) relatively hyperbolic groups then*

$$\mathcal{G}^{geom}=(T,\{(\Gamma^{aug}(G_v),\mathcal{M}_v)\},\{(\Gamma^{aug}(G_e),\mathcal{M}_e)\},\{\overline{f}_{\imath_e}\})$$

is a tree of weakly (resp. strongly) relatively hyperbolic spaces, termed a model-space of $\mathcal{G}$.

4.2. **From the algebraic ESP to the geometric ESP.** We prove here the following results:

Lemma 4.2. *With the notations above: if $\mathcal{G}$ satisfies the algebraic ESP (resp. strong algebraic ESP) then $\widehat{\mathcal{G}}^{geom}$ satisfies the geometric ESP (resp. strong geometric ESP).*

Proof. We first observe that the left-cosets of the parabolic subgroups of the vertex- and edge-groups of $\mathcal{G}$ exactly correspond to the parabolic subspaces of the vertex- and edge-spaces of the model-space $\mathcal{G}^{geom}$. Hence the metrics on the vertex- and edge-groups of $\widehat{\mathcal{G}}$ and on the vertex- and edge-spaces of $\widehat{\mathcal{G}}^{geom}$ are quasi isometric. The algebraic ESP then gives the exponential separation of any two v-vertical segments s, t which start at two distinct points x, y in a same stratum and *read a same reduced edge-path in the geometric realization $\widehat{X}$ of $\widehat{\mathcal{G}}^{geom}$*. We have to check that the exponential separation holds if this last property is not satisfied. Observe however that of course s and t project, under $\pi\colon \widehat{X} \to T$, to a same reduced edge-path ω in T: otherwise the horizontal distance between their endpoints is infinite since they lie in two distinct strata and they are exponentially separated. If $\omega = e_{i_1}^{\epsilon_1} \cdots e_{i_k}^{\epsilon_k}$ then any v-vertical segment over ω can be approximated by a sequence of intervals $x_i \times (0,1)$ over the e_i's, the Hausdorff distance between the v-vertical segment and these intervals only depending on v. Thus we can assume that the only differences between s and t are horizontal geodesics, denoted by $h_1 \in X_{\alpha^1}, \cdots, h_r \in X_{\alpha^r}$ for s and $l_1 \in X_{\alpha^1}, \cdots, l_r \in X_{\alpha^r}$ for t, which have horizontal length bounded above by a constant $C(v)$ only depending on v. These horizontal geodesics read words of the form $w_1 H_{i_1} w_2 \cdots H_{i_j} w_{j+1}$ (where w_j stands for a passage of the geodesic in the Cayley graph of a vertex-group whereas H_{i_j} stands for a passage of the geodesic in a left-coset for H_{i_j}). In each stratum X_{α^k} we consider a horizontal geodesic g_k between the initial points $i(h_k)$ and $i(l_k)$ of h_k and l_k and a horizontal geodesic g'_k between the terminal points $t(h_k)$ and $t(l_k)$ of h_k and l_k. Assuming that the geometric ESP does not hold, by Lemma 3.20 we can fix v as large as we wish. Moreover, once v has been fixed, we can assume that the horizontal geodesics g_k and g'_k are as large as we wish. Let $\delta \geq 0$ be such that the strata are δ-hyperbolic. The geodesic rectangles with sides g_k, g'_k, h_k, l_k are 2δ-thin. Let $[x_k, y_k] \subset g_k$ and $[u_k, v_k] \subset g'_k$ be the largest subpaths with $d_{hor}(x_k, u_k) \leq 2\delta$ and $d_{hor}(y_k, v_k) \leq 2\delta$, where $d_{hor}(.,.)$ denotes the horizontal distance. The finiteness of each family of parabolic subgroups and the finite generation of the vertex-groups imply together that there are only finitely many geodesic words of a given form which have relative length smaller than a given constant: let X be the number of edge-paths of length less than 2δ reading distinct words. At bounded distance from $[x_k, u_k]$ on the one hand, $[y_k, v_k]$ on the other hand, the bound only depending on the data, there are two edge-paths reading the same words: we substitute x_k, u_k on the one hand and y_k, v_k on the other hand by the endpoints of these edge-paths. Thus, as soon as v has been chosen sufficiently large, there exist v-vertical segments s', t' over ω passing through x_k and u_k for s' and through y_k, v_k for t'. The horizontal lengths of $[x_k, y_k] \subset g_k$ and $[u_k, v_k] \subset g'_k$ tend toward infinity with the horizontal lengths of g_k and g'_k since the horizontal lengths of h_k and l_k are bounded above by $C(v)$ and the geodesic rectangles with sides g_k, g'_k, h_k, l_k are 2δ-thin. Once chosen sufficiently large, the exponential separation of s' and t' would imply the exponential separation of s and t. Since we assumed on the contrary that s and t are not exponentially separated, neither are s' and t'. We so get a contradiction with the fact that the algebraic ESP is satisfied.

For the strong version of the ESP, just observe that the additional properties in one case as in the other are exactly equivalent with the construction given for $\mathcal{G}^{geom}$. □

4.3. **Conclusions.** By Lemma 4.2, Theorem 3.14 implies Theorem 2.8 and Theorem 3.18 implies Theorem 2.11.

5. Nice graphs of strongly relatively hyperbolic groups

The restriction we put for a graph $\mathcal{G}$ of strongly relatively hyperbolic groups to be *nice* allows one to get a clearer description of the parabolic subgroups of$\pi(\mathcal{G})$. We hope that this restriction is a not too bad compromise between clarity and generality.

Definition 5.1. A graph of strongly relatively hyperbolic groups $(\mathcal{G}, \mathcal{H}_v, \mathcal{H}_e)$ is *fine* if and only if (recall that $\mathcal{G} = (\Gamma, \{G_e\}, \{G_v\}, \{\imath_e\})$):

(a) If a parabolic subgroup H of a vertex-group admits a non-trivial reduced $\mathcal{G}$-path p such that $[Hp] \cap [pH]$ is infinite then $[Hp] = [pH]$.
Such a parabolic subgroup H is termed *periodic*.

(b) If H is a periodic parabolic subgroup and H' is any other parabolic subgroup of a vertex-group such that $[qH'] \cap [Hq]$ is infinite for some non-trivial reduced $\mathcal{G}$-path q then $[qH'] = [Hq]$ and H' is a periodic parabolic subgroup.

The *normalizer* $N_G(H)$ of a subgroup H in a group G is the set of all elements $g \in G$ which satisfy $g^{-1}Hg = H$. This is the largest subgroup of G which contains H as a normal subgroup.

Theorem 5.2. *Let $\mathcal{G}$ be a fine graph of strongly relatively hyperbolic groups. Let $\{N_{\pi(\mathcal{G})}(H_i)\}_{\substack{H_i \in \mathcal{H}_v \\ v \in V(\mathcal{G})}}$ be the family of all the normalizers of the parabolic subgroups of the vertex groups of $\mathcal{G}$. Let $\mathcal{N} \subset \{N_{\pi(\mathcal{G})}(H_i)\}_{\substack{H_i \in \mathcal{H}_v \\ v \in V(\mathcal{G})}}$ be the subfamily, unique up to conjugacy in $\pi(\mathcal{G})$, composed of exactly one representative of each conjugacy-class. If $\mathcal{G}$ satisfies the strong algebraic ESP then its fundamental group $\pi(\mathcal{G})$ is strongly hyperbolic relative to $\mathcal{N}$.*

Remark 5.3. There is another way of describing "the" family of parabolic subgroups for $\pi(\mathcal{G})$, in a more explicit but less concise way. If H, H' are two parabolic subgroups of some vertex-groups, we write $H \simeq H'$ if and only if there is a reduced $\mathcal{G}$-path p such that $[pH] \cap [H'p]$ is infinite. The $\simeq$-equivalence class of a parabolic subgroup is called its *orbit*. An orbit is *periodic* if and only if it contains a periodic parabolic subgroup. Theorem 5.2 then tells us that, if $\mathcal{G}$ satisfies the strong algebraic ESP, $\pi(\mathcal{G})$ is strongly hyperbolic relative to "the" family composed of exactly one parabolic subgroup in each orbit which is not periodic, and for exactly one periodic parabolic subgroup H in each periodic orbit, the subgroup composed of H and all the elements associated to the non-trivial reduced $\mathcal{G}$-paths p such that $[pH] = [Hp]$.

6. About extensions over relatively hyperbolic automorphisms

We begin with a result about free extensions of relatively hyperbolic groups. The *relatively hyperbolic automorphims* we define below first appeared in [12] where we announced a (weak) version of the results of the present paper. They generalize Gromov hyperbolic automorphisms [3].

Definition 6.1. Let G be a group and let $\mathcal{H} = \{H_i\}_{i \in \Lambda}$ be a finite family of subgroups of G. A *relative automorphism of* $(G, \mathcal{H})$ is an automorphism α of G such that there is a permutation $\sigma(\alpha) \in \mathrm{Sym}(\Lambda)$ and $g_i \in G$ for all $i \in \Lambda$ such that $\alpha(H_i) = g_i^{-1} H_{\sigma(\alpha)(i)} g_i$.

A relative automorphism α of $(G, \mathcal{H})$ *fixes* $\mathcal{H}$, or *fixes each* H_i, *up to conjugacy* if and only if $\sigma(\alpha)$ is the identity.

We denote by $\mathrm{Aut}(G, \mathcal{H})$ the set of all relative automorphisms of $(G, \mathcal{H})$: this is a subgroup of $\mathrm{Aut}(G)$, the group of automorphisms of G. The map

$$\begin{array}{rccc} \sigma\colon & \mathrm{Aut}(G, \mathcal{H}) & \to & \mathrm{Sym}(\Lambda) \\ & \alpha & \mapsto & \sigma(\alpha) \end{array}$$

is clearly a homomorphism.

Observe that the definition of a relative automorphism is not a simple rewriting of the similar definition for endomorphisms (Definition 2.5) but is much more restrictive.

Definition 6.2. With the notations of Definitions 2.3 and 6.1: A *relatively hyperbolic automorphism of* $(G, \mathcal{H})$ is a relative automorphism $\alpha \in \mathrm{Aut}(G, \mathcal{H})$ satisfying the following property:
There exist $\lambda > 1$ and $M, N \geq 1$ such that for any $w \in G$ with $|w|_{\mathcal{H}} \geq M$:

$$\lambda |w|_{\mathcal{H}} \leq \max(|\alpha^N(w)|_{\mathcal{H}}, |\alpha^{-N}(w)|_{\mathcal{H}}).$$

We also say in this case that α *is hyperbolic relative to* $\mathcal{H}$.

This definition is slightly more general than the definition given in [12].

Lemma 6.3. *Definition 6.2 is invariant:*

(a) Under conjugacy of α *in* $\mathrm{Aut}(G, \mathcal{H})$.
More generally, if $\beta \in \mathrm{Aut}(G)$ *and* $\alpha \in \mathrm{Aut}(G, \mathcal{H})$ *is relatively hyperbolic then* $\beta^{-1} \circ \alpha \circ \beta \in \mathrm{Aut}(G, \beta^{-1}(\mathcal{H}))$ *is relatively hyperbolic.*

(b) Under the substitution of any subgroup H_i *in* $\mathcal{H}$ *by a conjugate* $g^{-1}H_i g$ *with* $g \in G$.

Proof. Let us prove item (a). Any $\beta \in \mathrm{Aut}(G, \mathcal{H})$ acts as a bi-lipschitz map on $(G, \mathcal{H})$, i.e. there is $a \geq 1$ such that for any $w \in G$

$$\frac{1}{a}|w|_{\mathcal{H}} \leq |\beta(w)|_{\mathcal{H}} \leq a|w|_{\mathcal{H}}.$$

Hence by choosing $M' \geq aM$ we get for any $w \in G$ with $|w|_{\mathcal{H}} \geq M'$ either $|(\beta^{-1} \circ \alpha \circ \beta)^{jN}(w)|_{\mathcal{H}} \geq \lambda^j |w|_{\mathcal{H}}$ for any $j \geq 1$ or $|(\beta^{-1} \circ \alpha \circ \beta)^{-jN}(w)|_{\mathcal{H}} \geq \lambda^j |w|_{\mathcal{H}}$ for any $j \geq 1$. By choosing j such that $\lambda^j \geq a^2\lambda$ we get $N' = jN$ such that for any $w \in G$ with $|w|_{\mathcal{H}} \geq M'$ we have

$$\lambda |w|_{\mathcal{H}} \leq \max(|\alpha^{N'}(w)|_{\mathcal{H}}, |\alpha^{-N'}(w)|_{\mathcal{H}}).$$

For the generalization, just observe that β acts as a bi-lipschitz map from $(G, \beta^{-1}(\mathcal{H}))$ to $(G, \mathcal{H})$ and applies the same computation as above.

For item (b), if $\mathcal{H}'$ denotes the new family of subgroups, the metric spaces $(G, \mathcal{H})$ and $(G, \mathcal{H}')$ are bi-lipschitz equivalent with Lipschitz constant $a = 2\max(|g|_{\mathcal{H}}, |g|_{\mathcal{H}'}) + 1$. Then choose M' and N' as above. ☐

Definition 6.4. Let G be a finitely generated group and let $\mathcal{H}$ be a finite family of subgroups of G. Let $\imath\colon \mathfrak{A} \hookrightarrow \mathrm{Aut}(G)$ be a monomorphism from a finitely generated group $\mathfrak{A}$ into the group of automorphisms of G.

The pair $(\imath, \mathfrak{A})$ *defines a group of uniformly relatively hyperbolic automorphisms of* $(G, \mathcal{H})$ if and only if $\imath(\mathfrak{A}) < \mathrm{Aut}(G, \mathcal{H})$ and for any finite generating set $\mathcal{A}$ of $\mathfrak{A}$ there exist $\lambda > 1$ and $M, N \geq 1$ such that for any element $w \in G$ with $|w|_{\mathcal{H}} \geq M$, for any $a_1, a_2 \in \mathfrak{A}$ with $|a_1|_{\mathcal{A}} = |a_2|_{\mathcal{A}} = N$ and $d_{\mathcal{A}}(a_1, a_2) = 2N$:

$$\lambda |w|_{\mathcal{H}} \leq \max(|\imath(a_1)(w)|_{\mathcal{H}}, |\imath(a_2)(w)|_{\mathcal{H}}).$$

In Definition 6.4 the existence of the constants λ, M, N holds for *any* finite generating set $\mathcal{A}$ if and only if it holds for *some* such generating set. However λ, M, N depend on the choice of $\mathcal{A}$.

With the notations of Definitions 2.3 and 6.1, let r be a positive integer, let $\mathbb{F}_r$ be the free group of rank r and let $\imath\colon \mathbb{F}_r \hookrightarrow \mathrm{Aut}(G, \mathcal{H})$ be a monomorphism. For each $i \in \Lambda$ let $K_i := (\sigma \circ \imath)^{-1}(\mathrm{Stab}_{\mathrm{Sym}(\Lambda)}(i))$. Since a subgroup of a free group is free (Schreier's theorem) each K_i admits a free basis $\mathcal{B}_i = \{u_{i,j}\}_{j=1,2,\dots}$. Since the subgroups $\mathrm{Stab}_{\mathrm{Sym}(\Lambda)}(i)$ are finite, each K_i is of finite index in $\mathbb{F}_r$ and so is a finitely generated free group, the rank of which is denoted by r_i.

Definition 6.5. With the notations of Definitions 2.3 and 6.1, assume that $\mathcal{H}$ is a finite family of almost malnormal, infinite subgroups of G. Let r be a positive integer, let $\mathbb{F}_r$ be the free group of rank r and let $\imath\colon \mathbb{F}_r \hookrightarrow \mathrm{Aut}(G, \mathcal{H})$ be a monomorphism. For each $i \in \Lambda$ let $K_i := (\sigma \circ \imath)^{-1}(\mathrm{Stab}_{\mathrm{Sym}(\Lambda)}(i))$ and let $\mathcal{B}_i = \{u_{i,1}, \cdots, u_{i,r_i}\}$ be a free basis of K_i. For each $u_{i,j} \in \mathcal{B}_i$ let $g_{i,j} \in G$ such that $\imath(u_{i,j})(H_i) = g_{i,j}^{-1} H_i g_{i,j}$.

Then the subgroup of $G \rtimes_\imath \mathbb{F}_r$ generated by H_i and by the elements $u_{i,j} g_{i,j}^{-1}$ for $j \in \{1, \cdots, r_i\}$ is the *$(\mathbb{F}_r, \imath)$-extension of H_i*. It is denoted by $H_i^\imath$ (see Remark 6.6 below).

Let $\mathcal{K} \subset \{K_i\}_{i \in \Lambda}$ be a subfamily whose associated set of indices in Λ contains exactly one representative of each $(\sigma \circ \imath)(\mathbb{F}_r)$-orbit. Then the family $\{H_i^\imath\}_{K_i \in \mathcal{K}}$ is a *$(\mathbb{F}_r, \imath)$-extension of $\mathcal{H}$ over $\mathcal{K}$* and is denoted by $\mathcal{H}_{\imath, \mathcal{K}}$.

When $r = 1$ in Definition 6.5, i.e. when $\mathbb{F}_r = \langle t \rangle$, a $\mathbb{F}_r$-extension of $\mathcal{H}$ is preferably called *mapping-torus of $\mathcal{H}$ under* $\alpha \in \mathrm{Aut}(G, \mathcal{H})$, where $\alpha = \imath(t)$. Lemma 6.6 below is a straightforward consequence of the fact that the subgroups in $\mathcal{H}$ are almost malnormal and infinite.

Lemma 6.6. *With the assumptions and notations of Definition 6.5: the element $g_{i,j}$ such that $\imath(u_{i,j})(H_i) = g_{i,j}^{-1} H_i g_{i,j}$ is unique up to left-multiplication by an element in H_i. Moreover $\mathcal{H}_{\imath, \mathcal{K}}$ is unique up to substitution of some of its subgroups by conjugates in $G \rtimes_\imath \mathbb{F}_r$.*

The first assertion of Lemma 6.6 allows one to speak of *the* $(\mathbb{F}_r, \imath)$-extension of H_i. The second assertion allows one to make a slight abuse of language and write *the* $(\mathbb{F}_r, \imath)$-*extension of $\mathcal{H}$*.

Theorem 6.7. *Let G be a finitely generated group and let $\mathcal{H}$ be a finite family of infinite subgroups of G. Let $(\mathbb{F}_r, \imath)$ define a free group of uniformly relatively hyperbolic automorphisms of $(G, \mathcal{H})$. If G is weakly hyperbolic relative to $\mathcal{H}$, then $G \rtimes_\imath \mathbb{F}_r$ is weakly hyperbolic relative to $\mathcal{H}$. If G is strongly hyperbolic relative to $\mathcal{H}$, then $G \rtimes_\imath \mathbb{F}_r$ is strongly hyperbolic relative to the $(\mathbb{F}_r, \imath)$-extension of $\mathcal{H}$.*

When $r = 1$ in the above theorem, that is when the considered free group is just $\mathbb{Z}$, we get the classical "mapping-torus" case, that is the case of semi-direct products $G \rtimes \mathbb{Z}$ with G a relatively hyperbolic group.

Proof. With a slight abuse of terminology, we consider $\mathbb{F}_r$ as a subgroup of $\mathrm{Aut}(G, \mathcal{H})$ generated by the automorphisms α_i's. The group $G \rtimes \mathbb{F}_r$ is the fundamental group of the graph of groups which has G as unique vertex group G_v and G as the r edge-groups G_{e_i} (the e_i's are loops with initial and terminal vertex v). The attaching endomorphisms $\imath_{e_i}\colon G_{e_i} \hookrightarrow G_v$ are the automorphisms α_i whereas the $\imath_{e_i^{-1}}\colon G_{e_i} \hookrightarrow G_v$, are the identity. Since the α_i's are relative automorphisms of $(G, \mathcal{H})$, each one induces a quasi isometry from $(G_{e_i}, \mathcal{H})$ to $(G_v, \mathcal{H})$. We so got a graph of relatively hyperbolic groups. Since $\mathbb{F}_r$ is a uniform free group of relatively hyperbolic automorphisms, it satisfies the algebraic ESP.

Hence the weak relative hyperbolicity case of Theorem 6.7 is then a corollary of Theorem 2.8.

For the strong relative hyperbolicity case, it suffices to check that the definition of a uniform free group of relatively hyperbolic automorphisms implies the *strong* algebraic ESP. It suffices to prove that *any two* induced trees of parabolic groups separate exponentially one from each other. Assume that this is not satisfied. Then, there is $M \geq 0$ such that for any $N \geq 1$, there is $\alpha_w \in \mathbb{F}_r$ with $|w| \geq N$, s.t. there is a geodesic word u in $(G, |.|_{\mathcal{H}})$ of the form $h_1 H_{i_1} h_2 \cdots H_{i_k} h_{k+1}$ (where h_j stands for a passage of the geodesic in the Cayley graph of G whereas H_{i_j} stands for a passage of the geodesic in a left-coset for H_{i_j}) satisfying the following properties:

(a) $|u|_{\mathcal{H}} \leq M$,

(b) the image under α_w of any element with geodesic word HuH' has the form $rHuH's$, where H, H' stand for passages through left-cosets for the corresponding parabolic subgroups, and where the relative lengths of r and s only depend on the length of w.

Here H and H' are the parabolic subgroups of G corresponding to the left-cosets associated to the two induced trees of parabolic subgroups which violate, for the considered w, the strong algebraic ESP. The existence of u above comes from the finiteness of the family $\mathcal{H}$ and from the finite generation of G: they imply together that there are only finitely many geodesic words of a given form which have relative length smaller than M.

Since G is strongly hyperbolic relative to $\mathcal{H}$, $\mathcal{H}$ is almost malnormal in G. This readily implies, by choosing elements in H and H' which are sufficiently long in $(G, |.|_S)$, that there is an element g of the form $HuH'.H'u^{-1}H = HuH'u^{-1}H$ which is not conjugate to an element of a parabolic subgroup. Furthermore g can be chosen not to be a torsion element. From Corollary 4.20 of [29], $\lim_{n \to +\infty} |g^n|_{\mathcal{H}} = +\infty$. However $\alpha_w(g)$ has the form $rHuH'ss^{-1}H'u^{-1}Hr^{-1} = rHuH'u^{-1}Hr^{-1}$. Thus $|\alpha_w(g^n)|_{\mathcal{H}} \leq |g^n|_{\mathcal{H}} + 2|r|_{\mathcal{H}}$. Since $|r|_{\mathcal{H}}$ is a constant only depending on $|w|_{\mathcal{H}}$, by choosing n sufficiently large we get a contradiction with the uniform hyperbolicity of $\mathbb{F}_r$. □

We now give a corollary for this case. From [18], a hyperbolic group is weakly hyperbolic relative to any finite family of quasi convex subgroups. From [5] or [31], a hyperbolic group G is strongly hyperbolic relative to any almost malnormal finite family of quasi convex subgroups. We so get:

Corollary 6.8. *Let G be a hyperbolic group, let $\mathcal{H}$ be a finite family of infinite subgroups of G and let $\alpha \in \mathrm{Aut}(G, \mathcal{H})$ be hyperbolic relative to $\mathcal{H}$. If $\mathcal{H}$ is quasi convex in G then the mapping-torus group $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ is weakly hyperbolic relative to $\mathcal{H}$. If $\mathcal{H}$ is quasi convex and almost malnormal in G then the mapping-torus group $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ is strongly hyperbolic relative to the mapping-torus of $\mathcal{H}$.*

This corollary may be specialized to torsion free one-ended hyperbolic groups, and so in particular to fundamental groups of surfaces. We so re-prove the result of [15]. Since there we gave only an idea for the statement and the proof in the Gromov's (i.e. strong) relative hyperbolicity case, we include here the full statement of this result:

Corollary 6.9. *Let G be a torsion free one-ended hyperbolic group and let α be an automorphism of G. Let $\mathcal{H}$ be a family of maximal subgroups of G which consist entirely of elements on which α acts up to conjugacy periodically or with linear growth and such that each infinite-order element on which α acts up to conjugacy periodically or with linear growth is conjugate to an element in a subgroup in $\mathcal{H}$. Then $G_\alpha = G \rtimes_\alpha \mathbb{Z}$ is* *weakly hyperbolic relative to $\mathcal{H}$, and strongly hyperbolic relative to the mapping-torus of $\mathcal{H}$.*

If G is the fundamental group of a compact surface S (possibly with boundary) with negative Euler characteristic and if h is a homeomorphism of S inducing α on $\pi_1(S)$ (up to inner automorphism), then the subgroups in $\mathcal{H}$ are:

(i) the cyclic subgroups generated by the boundary curves,

(ii) the subgroups associated to the maximal subsurfaces which are unions of components on which h acts periodically, pasted together along reduction curves of the Nielsen-Thurston decomposition,

(iii) the cyclic subgroups generated by the reduction curves not contained in the previous subsurfaces.

Proof. From Theorem 6.7 and Corollary 6.8, we only have to prove that the considered automorphism α of G is hyperbolic relative to the given family of subgroups (indeed these families are quasi convex and almost malnormal - see [15] and [16]). The passage from the surface case to the torsion free one-ended hyperbolic group case is done thanks to the JSJ-decomposition theorems of [4]. We refer the reader to [15] for more precisions and concentrate on the surface case. The fundamental group of S is the fundamental group of a graph of groups $\mathcal{G}$ such that:

- the edge groups are cyclic subgroups associated to the reduction curves and boundary components,
- the vertex groups are the subgroups associated to the pseudo-Anosov components (type I vertices) and to the maximal subsurfaces with no pseudo-Anosov components (type II vertices),
- the (outer) automorphism α induced by the homeomorphism preserves the graph of groups structure.

We consider the universal covering of $\mathcal{G}$ and the associated tree of spaces. We measure the length of a geodesic in this tree of spaces as follows:

- we count zero for the passages through the edge-spaces and through the type II vertex-spaces,
- we measure the length of the pieces through the type I vertex-spaces by integrating against the stable and unstable measures of the invariant foliations (a boundary-component is considered to belong to both invariant foliations and so the contribution of a path in such a leaf amounts to zero).

There is $N \geq 1$ such that, when the total stable (resp. unstable) length of a geodesic in a type I-vertex space is two times its unstable (resp. stable) length, then it is dilated by a factor $\lambda > 1$ under N iterations of α^{-1} (resp. of α). In the other cases, we find $N \geq 1$ such that the total length is dilated under N iterations of both α and α^{-1}. Similar computations have been presented in [15]. The conclusion of the relative hyperbolicity of α now comes easily since pieces with positive length, dilated either under α^N or under α^{-N}, and pieces with zero length alternate. □

7. Ascending HNN-extensions and 3-manifold groups

The corollaries of Theorems 2.8 and 5.2 given up to now treat the case of the semi-direct product of a finite rank free group $\mathbb{F}_r$ with a finite type relatively hyperbolic group $(G, \mathcal{H})$. However a semi-direct product is only a particular case of HNN-extension. Alibegovic in [1], Dahmani in [8] or Osin in [30] treat acylindrical HNN-extensions and amalgated products. Let us now give a corollary about non-acylindrical HNN-extensions. Corollary

7.3 below deals with injective, not necessarily surjective, endomorphisms of relatively hyperbolic groups.

Definition 7.1. Let G be a group and let $\mathcal{H} = \{H_1, \cdots, H_k\}$ be a finite family of subgroups of G. A subgroup H' of G is *almost malnormal relative to* $\mathcal{H}$ if and only if there is an upper-bound on the $\mathcal{H}$-word length of the elements in the set $\{w \in H' \ ; \exists g \in G \setminus H' \text{ with } w \in g^{-1}H'g\}$.

If $\mathcal{H}$ is empty in the definition above, we get the usual notion of almost malnormality of a subgroup. If in addition there is no torsion, we get the notion of malnormality.

Whereas the definitions of a relative automorphism and of a mapping-torus of a family of subgroups given in Definition 2.3 remain valid for injective endomorphisms, the definition of relative hyperbolicity for automorphisms is easily adapted to the more general case of injective endomorphisms:

Definition 7.2. Let G be a finitely generated group and let $\mathcal{H}$ be a finite family of infinite subgroups of G. An injective endomorphism α of G is *hyperbolic relative to* $\mathcal{H}$ if and only if α is a relative endomorphism of $(G, \mathcal{H})$ and there exist $\lambda > 1$ and $M, N \geq 1$ such that, for any $w \in \mathrm{Im}(\alpha^N)$ with $|w|_{\mathcal{H}} \geq M$, if $|\alpha^N(w)|_{\mathcal{H}} \geq \lambda |w|_{\mathcal{H}}$ does not hold then $w = \alpha^N(w')$ with $|w'|_{\mathcal{H}} \geq \lambda |w|_{\mathcal{H}}$.

Corollary 7.3. *Let G be a finitely generated group, let α be an injective endomorphism of G and let G_α be the associated mapping-torus group, i.e. the associated ascending HNN-extension. Let $\mathcal{H}$ be a finite family of infinite subgroups of G such that α is hyperbolic relative to $\mathcal{H}$. Assume that* $\mathrm{Im}(\alpha)$ *is almost malnormal relative to $\mathcal{H}$. Then, if G is strongly hyperbolic relative to $\mathcal{H}$, G_α is weakly hyperbolic relative to $\mathcal{H}$ and strongly hyperbolic relative to the mapping-torus of $\mathcal{H}$.*

The reader will notice at once that the above theorem does not treat the extension of weakly relatively hyperbolic groups. The reason is that the condition of relative almost malnormality does not imply in this case the strong algebraic ESP.

Proof of Corollary 7.3. Before stating a first lemma let us recall that if h is a geodesic in a Gromov hyperbolic space then $P_h(.)$ denotes a quasi projection on h.

Lemma 7.4. *Let $G = \langle S \rangle$ be a finitely generated group which is strongly hyperbolic relative to a finite family of subgroups $\mathcal{H}$. There exists $C > 0$ such that if K is a finitely generated subgroup of G satisfying the following properties:*

- *it is almost malnormal relative to $\mathcal{H}$,*
- *it is strongly hyperbolic relative to a (possibly empty) finite family $\mathcal{H}'$ the subgroups of which are conjugated to subgroups in $\mathcal{H}$,*
- *$(K, |.|_{\mathcal{H}'})$ is quasi isometrically embedded in $(G, |.|_{\mathcal{H}})$,*

and if x, y (resp. z, t) are any two vertices in a same left-coset gK (resp. hK) with $g \neq h$ then $d_{\Gamma^{\mathcal{H}}_S(G)}(P_{[z,t]}(x), P_{[z,t]}(y)) \leq C$.

Proof. In order to simplify the notations we write $d_{\mathcal{H}}(.,.)$ for $d_{\Gamma^{\mathcal{H}}_S(G)}(.,.)$. Since $\Gamma^{\mathcal{H}}_S(G)$ is hyperbolic, there is a constant $\delta \geq 0$ such that the geodesic triangles of are δ-thin, and geodesic rectangles are 2δ-thin. This implies the existence of a quadruple of vertices x_0, y_0, z_0, t_0 with $x_0, y_0 \in [x, y]$, $z_0, t_0 \in [z, t]$ and $d_{\mathcal{H}}(x_0, z_0) \leq 2\delta + 1$, $d_{\mathcal{H}}(y_0, t_0) \leq 2\delta + 1$. Since $(K, |.|_{\mathcal{H}'})$ is (λ, μ)-quasi isometrically embedded in $(G, |.|_{\mathcal{H}})$, and $\Gamma^{\mathcal{H}}_S(G)$ is δ-hyperbolic, there exist $c_0(\lambda, \mu, \delta)$ and x_1, y_1, z_1, t_1 such that $g^{-1}x_1, g^{-1}y_1 \in K$, $h^{-1}z_1, h^{-1}t_1 \in K$ and $d_{\mathcal{H}}(x_0, x_1) \leq c_0(\lambda, \mu, \delta)$, $d_{\mathcal{H}}(y_0, y_1) \leq c_0(\lambda, \mu, \delta)$, $d_{\mathcal{H}}(z_0, z_1) \leq c_0(\lambda, \mu, \delta)$, $d_{\mathcal{H}}(t_0, t_1) \leq c_0(\lambda, \mu, \delta)$. We choose x_1, y_1, z_1, t_1 to minimize the distance *in* $\Gamma_S(G)$ (that is the distance associated to the given finite set of generators S of G) respectively to x_0, y_0, z_0, t_0. We denote by $[x_1, y_1]_K$ (resp. $[z_1, t_1]_K$) the images, under the embedding of K in G, of geodesics between the pre-images of x_1, y_1 (resp. z_1, t_1) in K. Both $[x_1, y_1]_K$ and $[z_1, t_1]_K$ are (λ, μ)-quasi geodesics. Moreover $[x_1, z_1][z_1, t_1]_K[t_1, y_1]$ is a $(\lambda, 4\delta + 2 + 4c_0(\lambda, \mu, \delta) + \mu)$-quasi geodesic between x_1 and y_1. Since G is strongly hyperbolic relative to $\mathcal{H}$, $\Gamma_S^{\mathcal{H}}(G)$ satisfies the BCP property with respect to $\mathcal{H}$. This gives a constant $c_1(\lambda, \mu, \delta)$ such that the $\mathcal{H}$-cosets $[x_1, z_1]$ and $[t_1, y_1]$ go through correspond to geodesics in $\Gamma_S(G)$ with length smaller than $c_1(\lambda, \mu, \delta)$: indeed, since x_1, y_1, z_1, t_1 were chosen to minimize the distances in $\Gamma_S(G)$ with respect to x_0, y_0, z_0, t_0, the $\mathcal{H}$-cosets crossed by $[x_1, z_1]$ and $[t_1, y_1]$ are not crossed by $[x_1, y_1]_K$. Therefore the distance in (G, S) between x_1 and z_1 on the one hand, and between y_1 and t_1 on the other hand is less or equal to $(2\delta + 1 + 2c_0(\lambda, \mu, \delta))c_1(\lambda, \mu, \delta)$. There are a finite number of elements in G with such an upper-bound on the length, measured with a word-metric associated to a finite set of generators. Whence, by the almost normality of K relative to $\mathcal{H}$, an upper-bound on the length between x_1 and y_1, and so also between x_0 and y_0. Lemma 7.4 is proved. □

From Lemma 7.4, the overlapping of two distinct left $\mathrm{Im}(\alpha)$-cosets is bounded above by a constant. Together with the fact that α is a relatively hyperbolic endomorphism, this implies the algebraic ESP. Getting the strong version of this property is done as in the proof of the strong relative hyperbolicity case of Theorem 6.7. Corollary 7.3 now follows from Theorem 2.11. □

The next result is about fundamental groups of 3-manifolds, which generalizes the case of 3-manifolds fibering over $\mathbb{S}^1$ treated by Corollary 6.9. We refer the reader to [21] or [22] for the basis about Seifert fibered spaces, graph-manifolds and the decomposition of 3-manifolds, and to [33] for a nice review about the geometries of 3-manifolds.

Corollary 7.5. *Let M^3 be a closed (i.e. compact, without boundary), irreducible, orientable 3-manifold. Then the fundamental group of M^3 is strongly hyperbolic relative to the family composed of*

- *(a) the subgroups G_i corresponding to conjugates of the fundamental groups of the maximal graph-submanifolds $\mathcal{GM}_1, \cdots, \mathcal{GM}_r$ in M^3,*
- *(b) the $\mathbb{Z} \oplus \mathbb{Z}$-subgroups corresponding to the incompressible tori in $M^3 \setminus \bigcup_{i=1}^r \mathcal{GM}_i$.*

Proof. The fundamental group of M^3 is the fundamental group of a graph of groups $\mathcal{G}$ satisfying the following properties:

- the vertex groups are of two kinds: there are the subgroups associated to the maximal graph-submanifolds, denoted by G_i, and the subgroups associated to the finite volume hyperbolic 3-submanifolds with cusps, denoted by H_j;
- the edge groups are $\mathbb{Z} \oplus \mathbb{Z}$-subgroups;
- two vertex groups of the first kind, G_i and G_j with $i \neq j$, are not adjacent.

The graph $\mathcal{G}$ becomes a graph of strongly relatively hyperbolic groups when considering each edge-group and each vertex-group G_i strongly hyperbolic relative to itself, whereas each vertex-group H_j is considered as a group strongly hyperbolic relative to the $\mathbb{Z} \oplus \mathbb{Z}$-subgroups of the cusps [10] (i.e. boundary components). Since the cusp subgroups are malnormal we have the following assertion: if $\mathcal{T}$ is the universal covering of $\mathcal{G}$ then there is a uniform bound M on the length of the $\mathcal{G}$-paths which conjugate a horizontal element to another one. It follows from Theorem 2.11 that the fundamental group of $\mathcal{G}$ is strongly hyperbolic relative to a family of subgroups as given by Corollary 7.5 since the strong algebraic ESP is vacuously satisfied. □

Since this case is an acylindrical case, an alternative to the proof proposed here is obtained by combining [10] (any hyperbolic 3-manifold with boundary tori is strongly hyperbolic relative to the boundary subgroups) and the combination theorem of [8].

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