Eigenvalue pinching on convex domains in space forms

E. Aubry*, J. Bertrand, B. Colbois

Abstract. In this paper, we show that the convex domains of $\mathbb{H}^n$ which are almost extremal for the Faber-Krahn or the Payne-Polya-Weinberger inequalities are close to geodesic balls. Our proof is also valid in other space forms and allows us to recover known results in $\mathbb{R}^n$ and $\mathbb{S}^n$.

1. Introduction

This paper aims to study some optimal inequalities involving the first eigenvalues of the Dirichlet spectrum of convex domains in space forms, and to ask how stable they are. The paper essentially deals with the most intricate case of the hyperblic space.

The inequalities we are interested in are the Faber-Krahn inequality and the Payne-Polya-Weinberger inequality. The Faber-Krahn inequality asserts that among all bounded domains with the same volume in a given space form, the geodesic ball has the smallest first Dirichlet eigenvalue. Moreover, the geodesic ball is the unique minimizer (up to an isometry) among smooth domains. In this setting, such an inequality is stable if a bounded domain Ω whose $\lambda_1(\Omega)$ is close to $\lambda_1(B)$ (B is a geodesic ball with the same volume as Ω), is close for the Hausdorff distance to B (up to an isometry). This general statement does not hold true, because it is possible to attach very long and thin tentacles to a ball without affecting significantly the volume and the spectrum. In fact, for Euclidean domains, weaker forms of stability have been established. One form is to prove that a domain whose first Dirichlet eigenvalue is close to the first eigenvalue

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of a suitable ball, ressembles a ball up to sets of small volume (see [15] for a precise statement). The other form is to consider only convex bodies, in this case the Faber-Krahn inequality is stable [14]. The stability of the Faber-Krahn inequality has also been established for convex domains in $\mathbb{H}^2$ and $\mathbb{S}^2$ [3].

The first result of this paper is to prove the stability of the Faber-Krahn inequality for convex domains in a space form of arbitrary dimension and arbitrary curvature. In the sequel, we will denote by $X^1 = (\mathbb{S}^n, can)$, $X^0 = (\mathbb{R}^n, can)$ and $X^{-1} = (\mathbb{H}^n, can)$ the space forms of curvature 1, 0 and -1 respectively.

Theorem 1.1. *Let $V_0 > 0$. Let $\lambda_1^*(V_0)$ be the first Dirichlet eigenvalue of a geodesic ball B of volume V_0 in X^δ. For any $\epsilon > 0$, there exists $\eta > 0$ such that, if Ω is a convex domain of volume V_0 in X^δ and if $\lambda_1(\Omega) \leq \lambda_1^*(V_0) + \eta$ then, up to an isometry,*

$$d_H(\Omega, B) \leq \epsilon,$$

where d_H denotes the Hausdorff distance. In the case $\delta = 0$ we have $\eta = \eta'(\epsilon) V_0^{-2/n}$.

Remark 1.1. We do not assume that the convex domains are bounded.

The method developed is the same whatever the space form. Nevertheless, the case $\delta = -1$ is considerably harder. The primary difficulty is that the hyperbolic space comprises unbounded convex domains with finite volume therefore, have a discrete Dirichlet spectrum. This is contrary to the case of $\mathbb{R}^n$, where an upper bound of the type $\mathrm{Diam}\Omega \leq C(\mathrm{Vol}\,\Omega, \lambda_1(\Omega), n)$ holds. To deal with this difficulty, we need to prove the thus for unsolved Faber-Krahn inequality for unbounded convex domains.

Proposition 1.1 (Faber-Krahn Inequality). *Let Ω be a convex set in X^δ of finite volume V_0. The first Dirichlet eigenvalue of Ω satisfies*

$$\lambda_1(\Omega) \geq \lambda_1^*(V_0)$$

where $\lambda_1^*(V_0)$ denotes the first Dirichlet eigenvalue of a geodesic ball with volume V_0. Moreover, the equality $\lambda_1(\Omega) = \lambda_1^*(V_0)$ implies that Ω is isometric to a geodesic ball.

Remark 1.2. The difficulty is in proving the case of equality.

The second result of this paper concerns the stability of the Payne-Polya-Weinberger inequality (PPW inequality for short). This famous conjecture has been proved by M.S. Ashbaugh and R.D. Benguria [1].

Theorem 1.2 ([1]). *Let Ω be a smooth bounded domain in Euclidean space (resp. a smooth domain included in an hemisphere in $\mathbb{S}^n$). Then, the following inequality holds*

$$\frac{\lambda_2}{\lambda_1}(\Omega) \leq \frac{\lambda_2}{\lambda_1}(B),$$

where B is an arbitrary Euclidean ball (resp. a spherical ball such that $\operatorname{Vol} B = \operatorname{Vol} \Omega$). Moreover the equality is achieved if and only if Ω is isometric to a geodesic ball.

Let us remark that the ratio $\frac{\lambda_2(B)}{\lambda_1(B)}$ is scale-invariant in Euclidean space and that M. Ashbaugh and R. Benguria also showed in [2], that the ratio of the two first eigenvalues of a geodesic ball in $\mathbb{S}^n$ is an increasing function of the radius r (if $r \leq \pi/2$). Consequently, the PPW inequality follows directly from the following theorem.

Theorem 1.3 ([1,2,4]). *Let Ω be a smooth bounded domain in X^δ ($\delta \in \{-1,0,1\}$) and such that Ω is included in an hemisphere if $\delta = 1$. The second Dirichlet eigenvalue of Ω satisfies*

$$\lambda_2(\Omega) \leq \lambda_2(B)$$

where B is a geodesic ball such that $\lambda_1(B) = \lambda_1(\Omega)$. Moreover, the equality holds if and only if Ω is isometric to B.

It is shown in [4] that λ_1/λ_2 is a decreasing function of the radius of hyperbolic balls and that the PPW is false in $\mathbb{H}^n$. This justifies the denomination "generalized PPW inequality" for the previous theorem.

We prove the following stability results.

Theorem 1.4. *Let Ω be a convex domain of $\mathbb{R}^n$ or $\mathbb{S}^n$ whose the volume is equal to V_0. For any $\epsilon > 0$ there exists $\eta > 0$ such that for all Ω as above, the assumption $\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \geq \frac{\lambda_2(B)}{\lambda_1(B)} - \eta$ implies*

$$d_H(\Omega, B) \leq \varepsilon,$$

where B is a (well centered) geodesic ball of volume V_0.

Theorem 1.5. *Let Ω be a convex domain of X^δ ($\delta \in \{-1,0,1\}$) with $\lambda_1(\Omega) = \lambda$ ($\lambda > \frac{(n-1)^2}{4}$ if $\delta = -1$). For any $\varepsilon > 0$, there exits η such that for all Ω as above, the assumption $\lambda_2(\Omega) \geq \lambda_2^*(\lambda) - \eta$ implies*

$$d_H(\Omega, B) \leq \varepsilon,$$

where $\lambda_2^*(\lambda)$ is the second Dirichlet eigenvalue of a (well centered) geodesic ball B of X^δ with $\lambda_1(B) = \lambda$.

Remark 1.3. We make no hypothesis about the volume of the convex domains we consider, not even the finiteness. This represents the main difference between this latter theorem and Theorems 1.4 and 1.1.

As for the Faber-Krahn inequality, it is necessary to generalize the PPW inequality to a more general setting, above all the characterization of the case of equality, in order to prove Theorems 1.4 and 1.5 (see Theorems 4.1 and 4.2 for precise statements).

The method developed to solve these stability questions is a rather general method, and based on the following abstract stability lemma. The proof is straightforward, therefore omitted .

Let X be a topological space and $f : X \to \mathbb{R}$ be a function. We say that f is *coercive* if there exists a compact subset K of X such that $\inf_{X\setminus K} f > \inf_X f$ (we set $\inf_\emptyset f = +\infty$).

Lemma 1.1. *Let X be a topological space. If $f : X \to \mathbb{R}$ is coercive and lower semi-continuous then f is bounded below, reaches its minimal value and the set $M_f = f^{-1}\{\inf f\}$ of its minima satisfies to the following stability property: for any neighborhood U of M_f, there exists $\eta > 0$ such that*

$$f^{-1}\big(]-\infty, \inf f + \eta]\big) \subset U.$$

This lemma is very close to the so-called *lower semi-continuity and compacity method*. This is typically used in calculus of variations to deal with the problem of minimisers existence (see [18, Chapter 1]). It can be applied to a wide variety of problems (as large as the lower semi-continuity and compacity method). It does not, however, give an explicit η (note in Proposition 4.2 for example, we do not assume that the boundary of the domain is of class C^2).

Our proof also shows that the infimum of the functional λ_1 (resp. $\frac{\lambda_1}{\lambda_2}$) on unbounded convex domains of $\mathbb{H}^n$ with a given volume (resp. with a given λ_1) is strictly larger than those on bounded domains. To our knowledge, this is also a new result.

The paper is organized as follows:

In section 2 we define a metric on the space $\mathcal{C}$ of convex, bounded domains in X^δ.

In section 3 we show that the eigenvalues and volume functions are continuous on $\mathcal{C}$.

In section 4 we extend the classical Faber-Krahn and Payne-Polya-Weinberger (as its generalized version) inequalites to the set of convex unbounded domains. This level of generality is required in our proof even if this set is restricted to bounded convex domains for the proof of the coercivity.

Finally, we reduce the proof of the stability theorems to the proof of the coercivity of the functionals λ_1 and λ_1/λ_2 on the set of bounded convex domains of given volume (resp. given λ_1) and prove the coercivity of these functionals in sections 5 and 6. In that purpose we prove several new qualitative result of the spectrum and eigenfunction of domains in space form. For instance we prove that a convex Euclidean domain with spectral gap is bounded (hence has a discrete spectrum) and that its diameter is bounded from above by $C(n)\left(\frac{1+\lambda_1}{\lambda_2-\lambda_1}\right)^{3/2}$ where $C(n)$ is a universal (explicitable) constant.

2. A distance on convex domains

In the following, we set $s_1(t) = \sin t$, $s_0(t) = t$, $s_{-1}(t) = \operatorname{sh} t$ *and* $c_\delta = s'_\delta$. *Let* x_0 *denote a fix point of* X^δ.

Definition 2.1. *Let* $\mathcal{C}$ *be the set of convex, bounded and open subsets* Ω *strictly included in* X^δ *which contain the point* x_0.

Remark 2.1. The isometry group of X^δ acts transitively on X^δ.

Remark 2.2. Each proper, convex set of the sphere is included in a hemisphere. Hence, up to the sphere itself, all the convex domains in $\mathbb{S}^n$ satisfy volume $\leq \operatorname{Vol}\mathbb{S}^n/2$ or fundamental tone $\geq n$.

So Theorems 1.1, 1.4 and 1.5 are obvious in the case $\Omega = \mathbb{S}^n$ and we just have to prove them for domains $\Omega \in \mathcal{C}$.

In the remaining part of this section we define a (proper) metric on $\mathcal{C}$. We choosed to work with a metric which has a better behaviour than the usual Hausdorff metric with respect to the volume and the Dirichlet spectrum. To define our metric, we need some facts on support functions.

2.1. Support functions

For any $\Omega \in \mathcal{C}$, the following function will be called *the support of* Ω:

$$\rho_\Omega : v \in S_{x_0} \mapsto \sup\{t \in \mathbb{R}_+ / \exp_{x_0}(sv) \in \Omega \text{ for all } s \in [0,t]\} \in \mathbb{R}_+$$

where S_{x_0} and $\exp_{x_0}$ are respectively the set of unit tangent vectors and the exponential map of X^δ at x_0. Note that, even on $\mathbb{S}^n$, we have $\rho_\Omega \leq R$ as soon as $\Omega \subset B(x_0, R)$.

The properties of ρ_Ω needed subsequently are summarized in the following lemma.

Lemma 2.1. ρ_Ω *is a Lipschitz map and if* $B(x_0,r) \subset \Omega \subset B(x_0,R)$ *then its Lipschitz constant is bounded above by* $s_\delta(R)\sqrt{\left(\frac{s_\delta(R)}{s_\delta(r)}\right)^2 - 1}$ *if* $\delta \neq 1$ *and by* $\operatorname{cotg} r$ *otherwise.*

Moreover we have

$\overline{\Omega} = \exp_{x_0}\{t.v/\, v \in S_{x_0},\, 0 \le t \le \rho_\Omega(v)\}$,
$\Omega = \exp_{x_0}\{t.v/\, v \in S_{x_0},\, 0 \le t < \rho_\Omega(v)\}$,
$\partial\Omega = \exp_{x_0}\{\rho_\Omega(v).v/\, v \in S_{x_0}\}$.

Proof. Fix $y_0 = \exp_{x_0}(\rho_\Omega(u_0)u_0) \in \partial\Omega$ and consider the geodesic double cone centered at y_0 and tangent to the ball $B(x_0, r)$. We claim that for each $v \in S_{x_0} \setminus \{u_0\}$ close enough to u_0, the geodesic $\gamma_v(t) = \exp_{x_0}(tv)$ meets the cone in exactly two points $Z(v), Z'(v)$ and that we have $l(v) \le \rho_\Omega(v) \le L(v)$, where

$$l\big(d_{S_{x_0}}(v,u_0)\big) = \min\{d(x_0, Z(v)), d(x_0, Z'(v))\}$$

and

$$L\big(d_{S_{x_0}}(v,u_0)\big) = \max\{d(x_0, Z(v)), d(x_0, Z'(v))\}.$$

From elementary trigonometric calculus (see appendix A for more details) we get

$$\liminf_{v\to u_0} \frac{\rho_\Omega(v) - \rho_\Omega(u_0)}{d_{S_{x_0}}(v,u_0)} \ge \liminf_{v\to u_0} \frac{l\big(d_{S_{x_0}}(v,u_0)\big) - l(0)}{d_{S_{x_0}}(v,u_0)}$$
$$= l'(0) = -s_\delta\big(d(x_0,y_0)\big)\sqrt{\Big(\frac{s_\delta\big(d(x_0,y_0)\big)}{s_\delta(r)}\Big)^2 - 1}$$

and

$$\limsup_{v\to u_0} \frac{\rho_\Omega(v) - \rho_\Omega(u_0)}{d_{S_{x_0}}(v,u_0)} \le \limsup_{v\to u_0} \frac{L\big(d_{S_{x_0}}(v,u_0)\big) - L(0)}{d_{S_{x_0}}(v,u_0)}$$
$$= L'(0) = s_\delta\big(d(x_0,y_0)\big)\sqrt{\Big(\frac{s_\delta\big(d(x_0,y_0)\big)}{s_\delta(r)}\Big)^2 - 1},$$

which implies that ρ_Ω is Lipschitzian and gives the bound on the constant by monotony properties of s_δ.

The last three equalities of the statement follow easily from the continuity of ρ_Ω and standard properties of the exponential map.

2.2. A distance on convex bounded domains

Definition 2.2. *Our distance on* $\mathcal{C}$ *is* $d(\Omega_1, \Omega_2) = \|\ln\Big(\frac{\rho_{\Omega_1}}{\rho_{\Omega_2}}\Big)\|_\infty$.

Proposition 2.1. $\big(\mathcal{C}, d\big)$ *is a proper metric space (i.e. every closed and bounded subset of* X *is a compact set).*

Proof. Let $(\Omega_i)_{i\in\mathbb{N}}$ be a bounded sequence in $\mathcal{C}$. Since there exist r and R such that $B(x_0,r)\subset\Omega_i\subset B(x_0,R)$ for every $i\in\mathbb{N}$, the functions $\rho_{\Omega_i}:S_{x_0}\to[r,R]$ are equicontinuous (by the Lemma 2.1), and so the sequence $(\rho_{\Omega_i})_{i\in\mathbb{N}}$ converges uniformly on S_{x_0} to a function ρ_∞, up to an extraction. Since $r\le\rho_\infty\le R$, we also have that $\lim\limits_{i\to\infty}\|\ln\big(\rho_{\Omega_i}/\rho_\infty\big)\|_\infty=0$. We set $\Omega_\infty=\{\exp_{x_0}\big(t.v\big)/\ v\in\mathbb{S}^{n-1}_{x_0},\ t\in[0,\rho_\infty(v)[\}$, which is a bounded, star-shaped domain and $\rho_{\Omega_\infty}=\rho_\infty$ because ρ_∞ is continuous and $\exp_{x_0}$ is a diffeomorphism of a neighbourhood of $B(0,R)$ onto a neighbourhood of $B(x_0,R)$. It remains to prove that Ω_∞ is convex.

Let y_1 and y_2 be any pair of points in Ω_∞. There exists only one minimizing geodesic γ from y_1 to y_2 in X^δ (for Ω_∞ is an open set of a hemisphere in the case δ=1). Since y_1 and y_2 are in Ω_j for all j large enough, we easily infer that $\gamma\subset\overline{\Omega}_\infty=\{\exp_{x_0}\big(t.v\big)/\ v\in\mathbb{S}^{n-1}_{x_0},\ t\in[0,\rho_\infty(v)]\}$. So for any $r>0$ small enough, the union of the minimizing geodesic from y_1 (resp. from y_2) to a point of $B(y_2,r)$ (resp. of $B(y_1,r)$) is contained in $\overline{\Omega}_\infty$. Since y_1 (resp. y_2) is in the injectivity domain of y_2 (resp. y_1), the union of this two sets is an open neighbourhood of γ contained in $\overline{\Omega}_\infty$, which implies the convexity of Ω_∞, the interior of $\overline{\Omega}_\infty$.

Corollary 2.1. *For any $R\ge r>0$ the set of the Ω in $\mathcal{C}$ which satisfy $B(x_0,r)\subset\Omega\subset B(x_0,R)$ is compact.*

3. Continuity of the volume and the eigenvalues

As proved in [9] the Dirichlet eigenfunctions of any convex (in fact Lipschitzian) domain Ω belong to $C^\infty(\Omega)\cap H^1_0(\Omega)\cap\mathcal{C}^0(\overline{\Omega})$ and are equal to 0 on $\partial\Omega$. Moreover (see [17, Corollary 10.10]) the Dirichlet spectrum of any open subset Ω of finite volume in X^δ is discrete. Enventually, for any integer $k\in\mathbb{N}^*$, we have a min-max principle

$$\lambda_k(\Omega)=\inf\{m(E)/\ E\text{ subspace of }\mathcal{C}^\infty_c(\Omega),\dim E=k\}$$
$$\text{where } m(E)=\sup_{f\in E}\frac{\int_\Omega|\nabla f|^2}{\int_\Omega f^2}.$$

We will say that Ω *has a spectral gap* if we have $\lambda_1(\Omega)<\lambda_2(\Omega)$, with the above definition of λ_1 and λ_2. This imply that $\lambda_1(\Omega)$ is an eigenvalue of the Dirichlet problem and always occure in finite volume.

Proposition 3.1. *For any $k\ge1$, we have*

$$\Big|\ln\Big(\frac{\lambda_k(\Omega_1)}{\lambda_k(\Omega_2)}\Big)\Big|\le\Lambda_\delta\big[d(\Omega_1,\Omega_2),R\big]$$

and

$$\Big|\ln\Big(\frac{\mathrm{Vol}\,\Omega_1}{\mathrm{Vol}\,\Omega_2}\Big)\Big| \leq \Lambda'_\delta\big[d(\Omega_1,\Omega_2),R\big],$$

where

$$\Omega_1\cup\Omega_2\subset B(x_0,R),\quad \Lambda_\delta(s,t)=\ln\Big[e^{2s}\Big(\frac{e^{2s}s_\delta(te^{-2s})}{s_\delta(t)}\Big)^{\delta(n-1)}\Big],$$

$$\Lambda'_1(s,t)=\Lambda'_0(s,t)=ns \quad \textit{and} \quad \Lambda'_{-1}=\ln\Big[e^{ns}\Big(\frac{e^{-s}\sinh(t)}{\sinh(e^{-s}t)}\Big)^{n-1}\Big].$$

Proof. In the case $\delta=1$, we denote by y_0 the antipodal point of x_0 in $\mathbb{S}^n$. For $\lambda\in]0,1]$, we define the map

$$\begin{aligned} H_\lambda : X^\delta\ (\textit{resp. } X^1\setminus\{y_0\}) &\to X^\delta \\ \exp_{x_0}(tv) &\mapsto \exp_{x_0}(\lambda tv) \end{aligned} \tag{1}$$

Set $d=d(\Omega_1,\Omega_2)$. Since $H_{e^{-d}}(\Omega_1)\subset\Omega_2$ we just have to bound the quotient $\lambda_k\big(H_\lambda(\Omega_1)\big)/\lambda_k(\Omega_1)$ for $\lambda=e^{-d}$.

For that purpose, we define a linear injective map $\Phi_\lambda:\mathcal{C}_c^\infty(\Omega)\to\mathcal{C}_c^\infty\big(H_\lambda(\Omega)\big)$ by $\Phi_\lambda(f)=f\circ H_{1/\lambda}$. Easy computations involving Jacobi fields give

$$\lambda\Big(\inf_{t\in]0,R]}\frac{s_\delta(\lambda t)}{s_\delta(t)}\Big)^{n-1}\|f\|_1\leq\|\Phi_\lambda(f)\|_1\leq\lambda\Big(\sup_{t\in]0,R]}\frac{s_\delta(\lambda t)}{s_\delta(t)}\Big)^{n-1}\|f\|_1$$

$$\frac{\big|d\big(\Phi_\lambda(f)\big)\big|^2(x)}{\Phi_\lambda\big(|df|^2\big)(x)}\leq\max\Big(\frac{1}{\lambda^2},\frac{s_\delta^2(d(x_0,x)/\lambda)}{s_\delta^2(d(x_0,x))}\Big)$$

The first inequality applied to $f\equiv1$ gives the volume estimate. The two inequalities imply

$$\frac{\|d\big(\Phi_\lambda(f)\big)\|_2^2}{\|\Phi_\lambda(f)\|_2^2}\leq e^{\Lambda_\delta(d,R)}\frac{\|df\|_2^2}{\|f\|_2^2}.$$

Which by the min-max principle implies

$$\lambda_k(\Omega_2)\leq\lambda_k\big(H_{e^{-d}}(\Omega_1)\big)\leq e^{\Lambda_\delta(d,R)}\lambda_k(\Omega_1).$$

4. Extremal convex domains

4.1. Schwarz symmetrization on noncompact domains

The aim of this paragraph is to recall some basic properties of the Schwarz symmetrization. However we will not assume as usual, that the domain to be symmetrized is bounded. To replace this property, some additional assumptions on the functions to be symmetrized are sometimes needed.

Definition 4.1 (Schwarz symmetrization). *Let f be a nonnegative function defined on an open set Ω in the space form X^δ. Let μ_f be the distribution function defined for $s \geq 0$, by $\mu_f(s) = vol(\{f > s\})$ and let $V : r \mapsto Vol(B(r))$ ($r \geq 0$). The nonincreasing Schwarz symmetrization of f is*

$$f^* = \mu_f^\sharp \circ V \circ d_{x_0},$$

where $d_{x_0}(x) = d(x_0, x)$ and $.^\sharp$ refers to the right inverse function of a nonincreasing function (i.e. $u^\#(s) = \inf\{t \geq 0/u(t) < s\}$). If the volume of Ω is finite, the Schwarz nondecreasing symmetrization of f is defined by

$$f_* = \mu_f^\sharp \circ H \circ d_{x_0},$$

where $H : r \mapsto \ vol\ (\Omega) - V(r)$.

The symmetrized functions satisfy

$$\mu_{f^*} = \mu_{f_*} = \mu_f. \tag{2}$$

Remark 4.1. For more details on symmetrization, we refer to [8,12,5].

Proposition 4.1. *Let Ω be an open set of finite volume in the space form X^δ ($\delta \in \{-1, 0, 1\}$).*

If u is in $L^2(\Omega)$ then u^* is in $L^2(\Omega^*)$ and

$$||u||_{L^2(\Omega)} = ||u^*||_{L^2(\Omega^*)}. \tag{3}$$

In addition, the following inequalities hold

$$\int_{\Omega^*} f_* g^* \leq \int_\Omega fg \leq \int_{\Omega^*} f^* g^* \tag{4}$$

for every nonnegative measurable functions f, g on Ω.

If u is now in $H_0^1(\Omega)$ then $u^* \in H_0^1(\Omega^*)$ and

$$\int_{\Omega^*} |\nabla u^*|^2 \leq \int_\Omega |\nabla u|^2. \tag{5}$$

Proof. The proof of the statement (3) is an immediate consequence of (2), the inequality (4) is easy to check for simple functions and the general case follows by density [12]. The proof of (5) also relies on an approximation argument, a suitable dense subset is introduced in the lemma below. The assumption on the volume is then used to conclude, using Rellich's Theorem on the symmetrized ball and the following inequality which is a direct consequence of (3) and (4).

$$||u^* - v^*||_{L^2(\Omega^*)} \leq ||u - v||_{L^2(\Omega)}.$$

Lemma 4.1. *Let f be a smooth nonnegative function in $H_0^1(\Omega)$, which is zero on $\partial\Omega$ and in $C^0(\overline{\Omega})$, where Ω is an open set of finite volume in X^δ. Suppose that the level sets of f are compact sets (except maybe $\{f = 0\}$) of measure zero. Under these assumptions, $\mu_f^\sharp$ is absolutely continous, the symmetrized function f^* is in $H_0^1(\Omega^*)$ and satisfies (5).*

Moreover, in case of equality in (5), the open set $\{f > 0\}$ is a ball.

Proof. Let $Reg(f)$ be the set of regular points of f which are included in $\{x \in \Omega; f > 0\}$. By assumption, $Reg(f)$ is an open set of full measure in $\{x \in \Omega; f > 0\}$. As a consequence, we deduce that f^* is continuously differentiable on an open set of full measure of $\{f^* > 0\}$ and satisfies the inequality (5) thanks to the coarea formula and the isoperimetric inequality (we refer to [6] for more details). We conclude that $\{f > 0\}$ is a ball using a decreasing sequence of regular values which goes to 0 and the case of equality in the isoperimetric inequality.

Remark 4.2. The set of functions which satisfy the assumptions of the lemma above contains the smooth functions with compact support and only nondegenerate critical points, therefore it is dense in $H_0^1(\Omega)$ (see [6] and references herein).

Remark 4.3. In the sequel, we will use the Schwarz symmetrization on convex domains of $\mathbb{H}^n$ whose the volume is not assumed to be finite. A priori, the nondecreasing Schwarz symmetrization cannot be defined in this setting, however the inequality

$$\int_{\Omega^*} f_* g^* \leq \int_\Omega fg$$

remains true for a function $f = F \circ d_{x_0}$, where F is a nonnegative and nondecreasing bounded function such that F is constant outside a compact set, if we define f_* as $f_*(x) = \begin{cases} (f|_{\Omega \cap B(x_0,r)})_* & \text{if } |x| < r \\ ||f||_\infty & \text{otherwise} \end{cases}$ for r large enough.

4.2. Faber-Krahn Inequality

In this section, we extend the Faber-Krahn inequality from the setting of smooth bounded domains to the setting of convex sets of finite volume. The main interest of the result below is the characterization of the case of equality.

Proposition 4.2 (Faber-Krahn Inequality). *Let Ω be a convex set in X^δ of finite volume V_0. The first Dirichlet eigenvalue of Ω satisfies*

$$\lambda_1(\Omega) \geq \lambda_1^*(V_0)$$

where $\lambda_1^*(V_0)$ denotes the first Dirichlet eigenvalue of a geodesic ball with volume V_0. Moreover, the equality $\lambda_1(\Omega) = \lambda_1^*(V_0)$ implies that Ω is isometric to a geodesic ball.

Remark 4.4. The characterization of the case of equality without assuming that Ω is bounded, is crucial in our proof of Theorem 1.1, when $\delta = -1$. Even when the domain is bounded, some regularity on the boundary is needed to deduce the case of equality. Indeed each ball with closed sets of capacity zero removed, satisfies the case of equality.

Proof. The proof of the inequality follows from Proposition 4.1 and does not rely on the convexity of Ω. As the volume of Ω is assumed to be finite, the Dirichlet spectrum of Ω is discrete [17, Corollary 10.10] and the eigenfunctions belong to $C^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ [9, Corollary 8.11 and theorem 8.29]. To prove the case of equality, it is enough to prove that the first eigenfunction (denoted by f_1) satisfies the assumptions of Lemma 4.1, which is a consequence of the lemma below. Indeed, thanks to this lemma and Sard's Theorem, the set of singular values of f_1 is a closed set of measure zero. Then, thanks to the fact that the function $\Delta f_1 = \lambda_1 f_1$ is positive on Ω, we deduce that each level set of the first eigenfunction is of measure zero.

Remark 4.5. Let us remark that the assumption on the finiteness of the volume is used only to prove that the bottom of the spectrum is an eigenvalue. It is also true for the lemma below; we will use this fact in paragraph 4.3.

Lemma 4.2. *Under the assumptions of the Proposition 4.2 the first Dirichlet eigenfunction f_1 on Ω can be assumed to be positive and proper: for all $s > 0$, the set $f_1^{-1}([s, +\infty[)$ is compact.*

Proof. By the maximum principle, we can suppose f_1 to be positive. To prove the second assertion, set y_0 be a fixed point of Ω, $R \geq 1$ and $x_0 \in \Omega \setminus B(y_0, 2R)$. Recall (see for instance [7]) that there exists a constant $C(n)$ such that:

$$\forall x_0 \in X^\delta, \quad \forall v \in H_0^1\big(B(x_0,1)\big), \quad \|v\|^2_{\frac{2n}{n-2}} \leq C(n)\|dv\|_2^2. \quad (*)$$

Note that in dimension $n = 2$, this inequality can be replaced by $\|v\|_4^2 \leq C\|dv\|_2^2$ in what follows. A standard Moser's iteration gives then us

$$f_1^2(x_0) \leq A(n)\big(1+\lambda_1\big)^{\gamma(n)} \int_{B(x_0,1)} f_1^2 \tag{6}$$

(where $A(n)$ and $\gamma(n)$ are constant that depend only on the dimension n, see appendix B for a proof), and from which we infer that

$$\sup_{\Omega\setminus B(y_0,2R)} f_1^2 \leq A(n)\big(1+\lambda_1\big)^{\gamma(n)} \int_{\Omega\setminus B(y_0,R)} f_1^2$$

which gives the compactness of the sets $f_1^{-1}\big([s,+\infty[\big)$ for all $s > 0$ since $\int_{\Omega\setminus B(y_0,R)} f_1^2 \to 0$ when $R \to \infty$ and f_1 is continuous on the convex set $\overline{\Omega}$ and is equal to 0 on $\partial\Omega$.

4.3. Payne-Polya-Weinberger Inequality

M.S. Ashbaugh and R.D. Benguria proved the Payne-Polya-Weinberger conjecture for smooth bounded sets of the Euclidean space and smooth sets included in an hemisphere of the sphere [1,2]. We need to extend this inequality to (possibly non smooth) convex sets in the space form X^δ ($\delta \in \{0,1\}$).

Theorem 4.1 (Payne-Polya-Weinberger Inequality). *Let Ω be a convex set of finite volume V_0 in X^δ ($\delta \in \{0,1\}$). Under these assumptions, the following inequality is satisfied,*

$$\frac{\lambda_2}{\lambda_1}(\Omega) \leq \frac{\lambda_2^*}{\lambda_1^*}(V_0).$$

Moreover, the equality is achieved only if Ω is isometric to a geodesic ball.

Remark 4.6. Actually, as in [1,2], the monotony properties of the ratio $\lambda_1(B)/\lambda_2(B)$ with respect to the radius of the geodesic ball B of X^δ make this theorem a direct corollary of the following result

Theorem 4.2. *Let Ω be a convex proper set in X^δ. Then the spectral gap of Ω is smaller or equal to $\lambda_2(B)-\lambda_1(B)$ (where B is a geodesic ball such that $\lambda_1(B)=\lambda_1(\Omega)$). If the spectral gap is equal to $\lambda_2(B)-\lambda_1(B)$, Ω is isometric to a geodesic ball.*

Let us remark that contrary to the cases $\delta \in \{0,1\}$, the assumptions in Theorem 4.2 do not imply an upper bound on the volume of Ω in $\mathbb{H}^n$.

We will prove Theorems 4.1 and 4.2 simultaneously. The scheme of the proof is the same as in [1,2,4], so we will mainly focuse on the extra arguments needed in our setting. The fisrt step of the proof is the following proposition

Proposition 4.3. *Let Ω be an open subset of X^δ (included in an hemisphere if $\delta = 1$) with spectral gap, u_1 an eigenfunction of Ω for the first eigenvalue and g be a positive, $\mathcal{C}^1$ by pieces function on $[0, \infty[$ (and with $\liminf_{+\infty} g > 0$ if Ω is not bounded. Then there exists a point $x_m \in X^\delta$ such that*

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_\Omega b\big(d(x_m, y)\big) u_1^2(y)\, dy}{\int_\Omega g^2\big(d(x_m, y)\big) u_1^2(y)\, dy}$$

where $b = g'^2 + \frac{n-1}{s_\delta^2} g^2$.

Note that for the proof of theorem 4.2, we can suppose that the spectral gap is non zero.

Proof. The min-max principle implies that

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_\Omega |\nabla P|^2 u_1^2}{\int_\Omega P^2 u_1^2},$$

for every non-zero function P such that Pu_1 is in $H_0^1(\Omega)$ and $\int_\Omega P u_1^2 = 0$.

The next step consists in choosing n suitable test functions. In that purpose we need the following lemma wich extend a result of [1, 2, 4] (the proof is postponed to appendix C)

Lemma 4.3. *For any $u \in L^2(X^\delta)$ (with support in an hemisphere if $\delta = 1$) and any $g : \mathbb{R}^+ \to \mathbb{R}^+$ continuous (bounded and with $\liminf_{+\infty} g > 0$ if u has not compact support), there is $x \in X^\delta$ such that*

$$\int_{X^\delta} g\big(d(x, y)\big) \frac{\exp_x^{-1}(y)}{d(x, y)} u^2(y)\, dy = 0_{T_x X^\delta}$$

To construct the test functions we apply this lemma to $u = u_1.1_\Omega$ and to g a nonnegative, increasing and bounded function (g will be specified later). The function $P_i = g(r) X_i$, where (r, X_i) are the geodesic coordinates at the point x_m given by lemma 4.3, satisfy $\int_\Omega P_i u_1^2 = 0$ for every i. To conclude the proof of proposition 4.3 we just have to sum the n inequations given by the min-max principle applied to the P_i and remark that $\sum_i P_i^2 = g^2$ and $\sum_i |\nabla P_i|^2 = b$.

Now, we choose for g a radial function such that we have the equality

$$\lambda_2(B) - \lambda_1(B) = \frac{\int_B b z^2}{\int_B g^2 z^2}$$

where z is a positive first eigenfunction of B. It is shown in [1, 2, 4] that we can choose g positive, nondecreasing and constant outside B and such that b is radial, positive and nonincreasing. We recall that B is such that $\lambda_1(B) = \lambda_1(\Omega)$. It remains to compare the spectral gaps. For that purpose, we first use properties of the Schwarz symmetrization (Proposition 4.1). We get

$$\int_\Omega b\, u_1^2 \leq \int_{\Omega^*} b^* u_1^{*2} \leq \int_{\Omega^*} b\, u_1^{*2}$$
$$\int_\Omega g\, u_1^2 \geq \int_{\Omega^*} g_* u_1^{*2} \geq \int_{\Omega^*} g\, u_1^{*2}.$$

The inequality involving the nonincreasing Schwarz symmetrization is valid without any assumption on the volume, thanks to remark 4.3. We conclude using the Chiti comparison result, which allows to compare z with u_1^* on B. This comparison result is valid as soon as the first eigenfunction u_1 satisfies the assumptions of Lemma 4.1 (this has been established in the proof of Proposition 4.2 and does not rely on any assumption on the volume), we refer to [1, pages 21-24] **[4]?** for more details. Using Chiti comparison result, we get [1, page 607]

$$\int_{\Omega^*} g\, u_1^{*2} \geq \int_B g\, z^2 \text{ and } \int_{\Omega^*} b\, u_1^{*2} \leq \int_B b\, z^2$$

and this conclude the proof of the inequality. The case of equality follows from the characterization of the equality in the Chiti comparison result.

5. Coercivity

We show that λ_1 (resp. λ_1/λ_2) is coercive on appropriate subsets of $\mathcal{C}$. We first need a control of the in-radii of the elements of $\mathcal{C}$.

5.1. In-radius estimate in $\mathcal{C}$

For any bounded domain Ω in X^δ, let Inrad Ω be the maximum radius of a geodesic ball included in $\overline{\Omega}$.

Proposition 5.1. *Let Ω be a bounded convex set in X^δ. Then*

$$\operatorname{Inrad} \Omega \geq \frac{\pi}{2\sqrt{\lambda_1(\Omega) + (n-1)}}.$$

This proposition has been proved by P. Li and S.T. Yau [13] for smooth domains of nonnegative mean curvature (see appendix D for a proof). It can be readily extended to any (non smooth) convex domains: indeed, for any $\epsilon > 0$ small enough, there exists a smooth convex domain Ω_ϵ such that $H_{1-\epsilon}(\Omega) \subset \Omega_\epsilon \subset H_{1+\epsilon}(\Omega)$, where H is the map defined by (1), p. 8 (see [10, Lemma 2.3.2] for the Euclidean case and use the Klein projective models of the hyperbolic space and the open hemisphere to infer this property in $\mathbb{H}^n$ and $\mathbb{S}^n$). The continuity of λ_1 on $\mathcal{C}$ allows to conclude.

5.2. Coercivity of λ_1

Subsequently we denote by $\mathcal{C}_{V_0}$ the set of convex bounded domains Ω of X^δ with $\mathrm{Vol}\,\Omega = V_0$ and $B(x_0, \mathrm{Inrad}\,\Omega) \subset \overline{\Omega}$ (remark that $\mathcal{C}_{V_0}$ contains, up to isometry, all the convex bounded domains of X^δ with volume V_0).

Combining Corollary 2.1 and Proposition 5.1, we get

Corollary 5.1. *For any $M > 0$, the set of elements Ω of $\mathcal{C}$ (resp. $\mathcal{C}_{V_0}$) with $\Omega \subset B(x_0, M)$ and $\lambda_1(\Omega) \le M$ is compact.*

5.2.1. case δ=1. Corollary 5.1 shows the compactness of the set $\{\Omega \in \mathcal{C}_{V_0} / \lambda_1(\Omega) \le M\}$. This implies that λ_1 is coercive. Actually, $\mathcal{C}_{V_0}$ itself is compact (see section 6).

5.2.2. Case δ=0. In this case, $\{\Omega \in \mathcal{C}_{V_0} / \lambda_1(\Omega) \le M\}$ is also a compact set and so λ_1 is coercive. Indeed by Proposition 5.1, a convex domain Ω in this set contains the ball $B(x_0, \frac{\pi}{2\sqrt{M+n-1}})$. Set $y \in \Omega$ such that $d(x_0, y) = \mathrm{diam}\,\Omega/4$. Since Ω is convex, it contains the convex hull of $B(x_0, \frac{\pi}{2\sqrt{M}}) \cup \{y\}$ whose volume must be bounded from above by V_0. We deduce that $\mathrm{diam}\,\Omega$ is bounded from above by a function of M and V_0. We conclude by Corollary 5.1.

5.2.3. Case δ=−1. We cannot argue as easily as in the previous case because in $\mathbb{H}^n$ the volume of the convex hull of $B(x_0, \frac{\pi}{2\sqrt{M+n-1}}) \cup \{y\}$ does not tend to ∞ with $d(x_0, y)$. We will prove simultaneously the coercivity of λ_1 and the property

$$\inf_{\mathcal{C}'} \lambda_1(\Omega) > \lambda_1^*(V_0), \tag{7}$$

where $\mathcal{C}' = \{\Omega \text{ unbounded convex sets}; \, vol(\Omega) = V_0\}$.

These two facts prove Theorem 1.1. First, we need to establish some lemmata.

Lemma 5.1. *Let Ω be a domain of a complete Riemannian manifold (M^n, g). Then for any $R \ge 1$, any α, $\gamma \in]0,1[$ and any $y_0 \in M$, we have*

$$\min\Big[\lambda_1\big(\Omega \cap B(y_0, R)\big), \lambda_1\big(\Omega \setminus B(y_0, \gamma R)\big)\Big] \le \frac{1}{(1 - R^{-\alpha})^2}\Big[\lambda_1(\Omega) + \frac{8}{(1-\gamma)^2 R^{2(1-\alpha)}}\Big]$$

where λ_1 stands for the bottom of the spectrum, Ω can be of infinite volume and we have set $\lambda_1(\emptyset) = \infty$.

Proof. The proof relies on the variational characterization of the first eigenvalue.

We set $N = E(R^\alpha) + 1$, $B_r = B(y_0, r)$, $A_{r,r'} = \Omega \cap (B_r \setminus B_{r'})$ and $r_k = \gamma R + (1-\gamma)R\frac{k}{N}$ for any integer $0 \leq k \leq N$. The for any $u \in H^1_0(\Omega)$, we have

$$\int_\Omega u^2 \geq \sum_{k=0}^{N-1} \int_{A_{r_{k+1},r_k}} u^2 \geq N \int_{A_{r_{k_0+1},r_{k_0}}} u^2$$

for at least one integer $0 \leq k_0 \leq N-1$. Let $r_0 = (1+\frac{k_0}{N})\frac{R}{2}$ and ϕ and ψ be the two functions defined on $\mathbb{R}^+$ by:

$-\phi$ is non-decreasing, $\phi = 0$ on $[0, \frac{r_{k_0}+r_{k_0+1}}{2}]$, $\phi = 1$ on $[r_{k_0+1}, \infty[$ and $\|\nabla\phi\|_\infty \leq \frac{2N}{(1-\gamma)R}$,

ψ is non-increasing, $\psi = 1$ on $[0, r_{k_0}]$, $\psi = 0$ on $[\frac{r_{k_0}+r_{k_0+1}}{2}, \infty[$ and $\|\nabla\psi\|_\infty \leq \frac{2N}{(1-\gamma)R}$,

For $g(x) = \psi\big(d(y_0,x)\big)u(x)$ and $h(x) = \phi\big(d(y_0,x)\big)u(x)$ we have

$$\begin{aligned}\int_{\Omega\cap B_{\frac{r_{k_0}+r_{k_0+1}}{2}}} g^2 + \int_{\Omega\setminus B_{\frac{r_{k_0}+r_{k_0+1}}{2}}} h^2 &= \int_\Omega (g+h)^2 \\ &\geq \int_\Omega u^2 - \int_{A_{r_{k_0+1},r_{k_0}}} u^2 \geq \frac{N-1}{N}\int_\Omega u^2\end{aligned}$$

Since $|dg+dh|^2 = |(\psi+\phi)du + ud(\psi+\phi)|^2$ we have

$$\begin{aligned}\int_{\Omega\cap B_{\frac{r_{k_0}+r_{k_0+1}}{2}}} |dg|^2 + \int_{\Omega\setminus B_{\frac{r_{k_0}+r_{k_0+1}}{2}}} |dh|^2 &= \int_\Omega |dg+dh|^2 \\ &\leq (1+R^{-\alpha})\int_\Omega (\phi+\psi)^2|du|^2 + (1+R^\alpha)\int_\Omega u^2|d\phi+d\psi|^2 \\ &\leq (1+R^{-\alpha})\int_\Omega |du|^2 + (1+R^\alpha)\int_{A_{r_{k_0+1},r_{k_0}}} u^2\frac{4N^2}{(1-\gamma)^2R^2} \\ &\leq (1+R^{-\alpha})\int_\Omega |du|^2 + (1+R^\alpha)\int_\Omega u^2\frac{4N}{(1-\gamma)^2R^2}\end{aligned}$$

We infer

$$\begin{aligned}
&\min\Big[\lambda_1\big(\Omega\cap B(y_0,R)\big),\lambda_1(\Omega\setminus B(y_0,\gamma R))\Big]\\
&\qquad\le \min\Big(\frac{\int_{\Omega\cap B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}|dg|^2}{\int_{\Omega\cap B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}g^2},\frac{\int_{\Omega\setminus B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}|dh|^2}{\int_{\Omega\setminus B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}h^2}\Big)\\
&\qquad\le \frac{\int_{\Omega\cap B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}|dg|^2+\int_{\Omega\setminus B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}|dh|^2}{\int_{\Omega\cap B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}g^2+\int_{\Omega\setminus B_{\frac{r_{k_0}+r_{k_0+1}}{2}}}h^2}\\
&\qquad\le \frac{1}{(1-R^{-\alpha})^2}\Big[\frac{\int_\Omega|du|^2}{\int_\Omega u^2}+\frac{8}{(1-\gamma)^2R^{2(1-\alpha)}}\Big]
\end{aligned}$$

This lemma implies the following

Lemma 5.2. *For any $V_0>0$ there exist $C(V_0,n)>\lambda_1^*(V_0)$ and $R(V_0,n)>0$ such that for any bounded convex set Ω which satisfies $vol(\Omega)\le V_0$ and $\lambda_1(\Omega)\in[\lambda_1^*(V_0),C(V_0,n)]$, we have*

$$\lambda_1(\Omega)\le\lambda_1\big(\Omega\cap B(x_0,R)\big)\le\frac{1}{(1-R^{-\alpha})^2}\Big[\lambda_1(\Omega)+\frac{32}{R^{2(1-\alpha)}}\Big]$$

for any $\alpha\in]0,1[$, $R\ge R(V_0,n,\alpha)$ and x_0 such that $B\big(x_0,\mathrm{Inrad}(\Omega)\big)\subset\overline{\Omega}$.

Proof. We set $r(V_0,n)=\frac{\pi}{\sqrt{2\lambda_1^*(V_0)+n-1}}$ and

$$C(V_0,n)=\min\big[2\lambda_1^*(V_0),\frac{\lambda_1^*(V_0)+\lambda_1^*\big(V_0-\mathrm{Vol}\,B(x_0,r(V_0,n)/2)\big)}{2}\big]$$

then by Proposition 5.1 we have $B(x_0,r(V_0,n)/2)\subset\Omega$ and so $\mathrm{Vol}\,\big(\Omega\setminus B(x_0,R/2)\big)\le V_0-\mathrm{Vol}\,B(x_0,r(V_0,n)/2)$ for any $R\ge r(V_0,n)$. By the Faber-Krahn inequality, this implies that $\lambda_1\big(\Omega\setminus B(x_0,R/2)\big)$ is larger than $\lambda_1^*\big(V_0-\mathrm{Vol}\,B(x_0,r(V_0,n)/2)\big)$. Now we can chose $R(V_0,n)$ large enough to have $\frac{1}{(1-R^{-\alpha})^2}\Big[C(V_0,n)+\frac{32}{R^{2(1-\alpha)}}\Big]\le\lambda_1\big(\Omega\setminus B(x_0,R/2)\big)$ for any $R\ge R(V_0,n)$. The Lemma 5.1 then applies.

Now, we prove (simultaneously) the coercivity of λ_1 and (7). By definition of the bottom of the spectrum, it is sufficient to prove that every sequence of bounded convex domains (Ω_i) such that $vol(\Omega_i)\le V_0$ and $\lim_i\lambda_1(\Omega_i)=\lambda_1^*(V_0)$, converges, up to isometries and extraction, to B_{V_0}.

Let (Ω_i) be such a sequence. Up to isometries, we can suppose that the fixed point $x_0\in\mathbb{H}^n$ satisfies the condition $B\big(x_0,\mathrm{Inrad}(\Omega_i)\big)\subset\Omega_i$

for every i. By the lemma above and Corollary 5.1, the sequence $\big(\Omega_i \cap B_R\big)_{i\in\mathbb{N}}$ is precompact in $\mathcal{C}$ for all $R \geq R(V_0, n)$. Up to a diagonal extraction, we can now suppose that for any $n \in \mathbb{N}$ the sequence $\big(\Omega_i \cap B_n\big)_{i\in\mathbb{N}}$ converges to an element U_n of $\mathcal{C}$. Using the continuity of λ_1 and of the volume on $\mathcal{C}$, we have

$$\lambda_1^*(V_0) \leq \lambda_1(U_n) \leq \frac{1}{(1-n^{-1/2})^2}\Big[\lambda_1^*(V_0) + \frac{32}{n}\Big]$$
$$\text{and} \quad \operatorname{Vol}(U_n) \leq V_0$$

So $\lambda_1(U_n)$ tends to $\lambda_1^*(V_0)$ and by the Faber-Krahn inequality, we must have $\operatorname{Vol} U_n \to V_0$. Moreover, $(U_n)_{n\in\mathbb{N}}$ is a nondecreasing sequence of convex sets for the inclusion. As a consequence, $\Omega = \bigcup_n U_n$ is a convex domain of volume V_0 and first eigenvalue $\lambda_1(\Omega) = \lambda_1^*(V_0)$. By Proposition 4.2, $\Omega = B(x_0, R_{V_0})$ and we infer that the sequence Ω_i converges to $B(x_0, R_{V_0})$ in $\mathcal{C}$.

6. Coercivity of the λ_1/λ_2 functional

6.1. case $\delta = 0$

We show that, on $\mathcal{C}_{V_0}$, λ_1/λ_2 tends to 1 when λ_1 tends to ∞. By section 5.2.2, by $\inf_{\mathcal{C}_{V_0}} \lambda_1/\lambda_2 < 1$ and by the fact that λ_1/λ_2 is invariant under homothetie on the domains, this implies Theorem 1.4 in $\mathbb{R}^n$.

By a classical result due to Jones, for any $\Omega \in \mathcal{C}_{V_0}$ there exists an ellipsoid $\mathcal{E}$ such that $\mathcal{E} \subset \Omega \subset \sqrt{n}\mathcal{E}$. We easily infer that there is a n-rectangle R with edges of lengths $L_1 \leq \cdots \leq L_n$, such that $R \subset \Omega \subset nR$. This gives

$$\lambda_1(\Omega) \leq \lambda_1(R) \leq \frac{n\pi^2}{L_1^2} \quad \text{and} \quad V_0 \leq L_n^{n-1} n^n L_1, \tag{8}$$

and so $L_n \geq \Big(\frac{V_0\sqrt{\lambda_1}}{\pi n^{n+\frac{1}{2}}}\Big)^{\frac{1}{n-1}}$. Following [11] we can translate and rotate Ω so that R be centred in $(0,\ldots,0)$ and the edge of R of length L_n be parallel to the last coordinate axis. We denote $\Omega(y) = \{x \in \mathbb{R}^{n-1} / \, (x,y) \in \Omega\}$ and $\lambda(y) = \lambda_1\big(\Omega(y)\big)$. Then, if f is an eigenfunction of Ω associated to the first eigenvalue, we have

$$\int_\Omega f^2 = \int_\Omega \frac{|\nabla f|^2}{\lambda_1(\Omega)} \geq \int_{\mathbb{R}} \int_{\Omega(y)} \frac{|\nabla_x f(x,y)|^2}{\lambda_1(\Omega)} dxdy$$
$$\geq \int_{\mathbb{R}} \frac{\lambda_1\big(\Omega(y)\big)}{\lambda_1(\Omega)} \int_{\Omega(y)} |f(x,y)|^2 dxdy$$

Thus there is y such that $\lambda_1(\Omega) \geq \lambda_1\big(\Omega(y)\big)$. By convexity of Ω we deduce that $\big(1-(\frac{L_n}{2})^{-\frac{2}{3}}\big)\Omega(y)\times[y-(\frac{L_n}{2})^{\frac{1}{3}}, y+(\frac{L_n}{2})^{\frac{1}{3}}]$ is contained in Ω and so that

$$\lambda_1(\Omega)\leq\lambda_2(\Omega)\leq\lambda_2\Big(\big(1-(\frac{L_n}{2})^{-\frac{2}{3}}\big)\Omega(y)\times[y-(\frac{L_n}{2})^{\frac{1}{3}}, y+(\frac{L_n}{2})^{\frac{1}{3}}]\Big)$$

$$\leq\frac{\lambda_1\big(\Omega(y)\big)}{\big(1-(\frac{L_n}{2})^{-\frac{2}{3}}\big)^2}+\frac{2\pi^2}{(\frac{L_n}{2})^{\frac{2}{3}}}\leq\frac{\lambda_1(\Omega)}{\big(1-(\frac{L_n}{2})^{-\frac{2}{3}}\big)^2}+\frac{2\pi^2}{(\frac{L_n}{2})^{\frac{2}{3}}} \tag{9}$$

Since we have shown above that $L_n\to\infty$ when $\lambda_1\to\infty$ we obtain that λ_1/λ_2 tend to 1 when λ_1 tends to ∞.

Remark 6.1. The same method could be used to show that for any integers $p\leq q$, λ_p/λ_q tends to 1 when λ_1 tends to ∞ on $\mathcal{C}_{V_0}$. We conlude that for any $p\leq q$ there exists a convex domain (to determine) which minimizes the quotient λ_p/λ_q.

Remark 6.2. The inequations (9) implies that for any convex domain Ω of $\mathbb{R}^n$, any $x_0\in\Omega$ and any $R>0$ such that $B(x_0,R)$ does not contain Ω, we have

$$\lambda_1(\Omega\cap B(x_0,R))\leq\lambda_2(\Omega\cap B(x_0,R))\leq\frac{\lambda_1(\Omega\cap B(x_0,R))}{\big(1-C(n)R^{-\frac{2}{3}}\big)^2}+\frac{C(n)}{R^{\frac{2}{3}}}$$

and so $\lambda_2(\Omega\cap B(x_0,R))$ tends to $\lambda_1(\Omega)$ when R tends to ∞. We conclude that a convex Euclidean domain with spectral gap is bounded (hence has a discrete spectrum) and that its diameter is bounded from above by $C(n)\big(\frac{1+\lambda_1}{\lambda_2-\lambda_1}\big)^{3/2}$. This implies readily the coercivity of λ_1/λ_2 on the set of convex Euclidean domain of fixed λ_1, from which we infer the Theorem 1.5 in $\mathbb{R}^n$.

6.2. Case $\delta=1$

The coercivity λ_1/λ_2 on the set of convex domains with $\lambda_1=\lambda$ follows from Lemma 5.1. On $\mathcal{C}_{V_0}$ it follows from the compactness of $\mathcal{C}_{V_0}$ which, by Corollary 5.1 is a consequence of the inequality $\mathrm{Inrad}\,\Omega\geq C(n)\mathrm{Vol}\,\Omega$, valid for any convex domain of $\mathbb{S}^n$:

First remark that from the inequality (8), based on the Jones ellipsoid, we get easily that for any convex domain contained in a geodesic ball of radius R in $\mathbb{R}^n$ we have $\mathrm{Vol}\,\Omega\leq n^nR^{n-1}\mathrm{Inrad}\,\Omega$. Now, since $\mathbb{S}^n$ can be covered by $2(n+1)$ balls of radius $R_n=\arccos(\frac{1}{\sqrt{n+1}})$ we infer that there is a point x_0 in $\mathbb{S}^n$ such that $\mathrm{Vol}\big(\Omega\cap B(x_0,R_n)\big)\geq\frac{1}{2(n+1)}\mathrm{Vol}\,\Omega$. Using the canonical embedding of $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$, we can project $B(x_0,R_n)$ onto the tangent space $T_{x_0}\mathbb{S}^n$ (using the origin of the Euclidean space). This map P_0 is a quasi-isometry from the ball $B(x_0, R_n)$ in $\mathbb{S}^n$ to the geodesic ball $B(x_0, \sqrt{n})$ in the Euclidean space, which preserves the convexity. Then, we have

$$\begin{aligned}\operatorname{Inrad}_{\mathbb{S}^n} \Omega &\geq \operatorname{Inrad}_{\mathbb{S}^n} \big(\Omega \cap B(x_0, R_n)\big) \\ &\geq C_1(n) \operatorname{Inrad}_{T_{x_0}\mathbb{S}^n} P_0\big(\Omega \cap B(x_0, R_n)\big) \\ &\geq C_2(n) \operatorname{Vol}_{T_{x_0}\mathbb{S}^n} P_0\big(\Omega \cap B(x_0, R_n)\big) \geq C(n) \operatorname{Vol}_{\mathbb{S}^n} \Omega.\end{aligned}$$

6.3. Case $\delta = -1$.

In this section, we prove simultaneously the Theorem 1.5, the coercivity of the functional λ_1/λ_2 on bounded convex domains whose the first eigenvalue is fixed and the property

$$\sup_{\mathcal{C}'} \lambda_2(\Omega) < \lambda_2^*(\lambda), \tag{10}$$

where $\mathcal{C}' = \{\Omega \text{ unbounded convex sets; } \lambda_1(\Omega) = \lambda\}$.

We will use the

Lemma 6.1. *Let Ω be a convex domain in $\mathbb{H}^n$ such that the bottom of the spectrum is an eigenvalue. Then for any fixed point $x_0 \in \mathbb{H}^n$, we have*

$$\lim_{R\to\infty} \lambda_i(\Omega \cap B(x_0, R)) = \lambda_i(\Omega), \quad \text{for } i = 1, 2.$$

Thanks to this lemma (which follows easily from the min-max principle, see Theorem XIII.1 of [16]) these two properties reduce to the following fact: Every sequence $(\Omega_i) \in \mathcal{C}$ such that $\lim_i \lambda_1(\Omega_i) = \lambda$ and $\lim_i \lambda_2(\Omega_i) = \lambda_2^*(\lambda)$, converges (up to an extraction) to a ball such that $\lambda_1(B) = \lambda$.

First, we show that a lower bound on the spectral gap implies some estimates on the first eigenfunction.

Lemma 6.2. *Let Ω be a bounded domain of $\mathbb{H}^n$. If $u \in H_0^1(\Omega)$ is such that $\triangle u = \lambda_1(\Omega) u$ then there is a point $x_m \in \mathbb{H}^n$ such that*

$$\Big(\lambda_2(\Omega) - \lambda_1(\Omega) - \frac{n-1}{\sinh^2(R)}\Big) \int_{\Omega \setminus B(x_m,R)} u^2 \leq \frac{n}{R^2} \int_{\Omega \cap B(x_m,R)} u^2$$

for any $R > 0$. This implies that for any $R \geq 2\sqrt{\frac{n-1}{\lambda_2(\Omega)-\lambda_1(\Omega)}}$ we have

$$\begin{aligned}\lambda_1(\Omega) &\leq \lambda_1\big(\Omega \cap B(x_m, R)\big) \\ &\leq \frac{(1+\frac{1}{R^2})}{1 - \frac{4n}{(\lambda_2(\Omega)-\lambda_1(\Omega))R^2+4}} \Big(\lambda_1(\Omega) + \frac{4n}{(\lambda_2(\Omega) - \lambda_1(\Omega))R^2 + 4}\Big).\end{aligned}$$

Proof. Proposition 4.3 applied to $g(s) = s/R$ on $[0, R]$ and $g(s) = 1$ on $[R, \infty[$ gives a point $x_m \in \mathbb{H}^n$ such that

$$\lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\frac{n}{R^2}\int_{\Omega\cap B(x_m,R)} u^2(x)dx + \frac{n-1}{\sinh^2 R}\int_{\Omega\setminus B(x_m,R)} u^2(x)dx}{\int_{\Omega\setminus B(x_m,R)} u^2(x)dx}$$

which gives the first estimate.

For the second estimate we set ψ the non-increasing Lipschitzian function defined on $\mathbb{R}^+$ by $\psi = 1$ on $[0, R/2]$, $\psi = 0$ on $[R, \infty[$ and $\|\nabla\psi\|\infty = \frac{2}{R}$. Then we have

$$\begin{aligned}|d(\psi u)|^2 &= \psi^2|du|^2 + 2u\psi(d\psi, du) + u^2|d\psi|^2\\ &\leq (1 + \frac{1}{R^2})\psi^2|du|^2 + (1 + R^2)|d\psi|^2u^2\end{aligned}$$

So we infer

$$\begin{aligned}\lambda_1\big(\Omega \cap B(x_m, R)\big) &\leq \frac{\int_{\Omega\cap B(x_m,R)} |d(\psi u)|^2}{\int_{\Omega\cap B(x_m,R)}(\psi u)^2}\\ &\leq (1 + \frac{1}{R^2})\frac{\int_\Omega \psi^2|du|^2}{\int_{\Omega\cap B(x_m,R/2)} u^2} + (1 + R^2)\frac{\int_\Omega |d\psi|^2u^2}{\int_{\Omega\cap B(x_m,R/2)} u^2}\end{aligned}$$

$$\leq (1+\frac{1}{R^2})\frac{\int_\Omega |du|^2}{\int_\Omega u^2}\Big(1+\frac{\int_{\Omega\setminus B(x_m,R/2)} u^2}{\int_{\Omega\cap B(x_m,R/2)} u^2}\Big)+4(1+\frac{1}{R^2})\frac{\int_{\Omega\setminus B(x_m,R/2)} u^2}{\int_{\Omega\cap B(x_m,R/2)} u^2}.$$

By the first estimate we have

$$\frac{\int_{\Omega\setminus B(x_m,R/2)} u^2}{\int_{\Omega\cap B(x_m,R/2)} u^2} \leq \frac{4n}{(\lambda_2(\Omega) - \lambda_1(\Omega))R^2 - 4(n-1)}$$

from which we infer

$$\begin{aligned}&\lambda_1\big(\Omega \cap B(x_m, R)\big)\\ &\leq \frac{(1 + \frac{1}{R^2})\big((\lambda_2(\Omega) - \lambda_1(\Omega))R^2 + 4\big)}{(\lambda_2(\Omega) - \lambda_1(\Omega))R^2 - 4(n-1)}\Big(\lambda_1(\Omega)+\frac{4n}{(\lambda_2(\Omega) - \lambda_1(\Omega))R^2 + 4}\Big)\end{aligned}$$

Let $\Omega_i \in \mathcal{C}$ such that $\lim_i \lambda_1(\Omega_i) = \lambda$ and $\lim_i \lambda_2(\Omega_i) = \lambda_2^*(\lambda)$. We can assume that $\lambda_2(\Omega_i) - \lambda_1(\Omega_i) > \frac{\lambda_2^*(\lambda)-\lambda}{2} > 0$. Note that by the preceeding lemma we infer that for any $R \geq 4\sqrt{\frac{n-1}{\lambda_2^*(\lambda)-\lambda}}$ we have $\lambda_1\big(\Omega_i \cap B(x_m^i, R)\big) \leq C(\lambda, n)$ (where $C(\lambda, n)$ is a universal funtion and x_m^i is the center of mass of Ω_i). This implies, by Proposition 5.1, that we can suppose (up to isometry) $x_m^i \in B\big(x_0, 4\sqrt{\frac{n-1}{\lambda_2^*(\lambda)-\lambda}}\big)$ and

$B\big(x_0, r(\lambda, n)\big) \subset \Omega_i$ for all i. Then the sequence $\big(\Omega_i \cap B(x_0, R)\big)$ has its values in a compact set of $\mathcal{C}$ (see Corollary 5.1). By diagonal extraction we can suppose that for any $k \in \mathbb{N}$ the sequence $\big(\Omega_i \cap B(x_0, k)\big)$ converge to an element U_k of $\mathcal{C}$. By continuity of λ_1 on $\mathcal{C}$ we have

$$\lambda \leq \lambda_1(U_k) = \lim_i \lambda_1\big(\Omega_i \cap B(x_0, k)\big)$$
$$\leq \lim_i \lambda_1\Big(\Omega_i \cap B(x_m^i, k - 4\sqrt{\frac{n-1}{\lambda_2^*(\lambda) - \lambda}})\Big) \leq f(k, \lambda, n)$$

where $f(k, \lambda, n)$ is a universal function given by the preceding lemma and that converge to λ when k tends to ∞. So $\lambda_1(U_k)$ tends to λ. As in the subsection 5.2.3 we set $\Omega = \cup_k U_k$. Then Ω is a convex domain with $\lambda_1(\Omega) = \lim_k \lambda_1(U_k) = \lambda$ ($U_k = \Omega \cap B(x_0, k)$) and

$$\lambda_2(\Omega) = \lim_k \lambda_2(U_k) = \lim_k \lim_i \lambda_2(\Omega_i \cap B(x_0, k)) \geq \lim_i \lambda_2(\Omega_i) = \lambda_2^*(\lambda).$$

We then conclude by Theorem 4.2.

A. A trigonometric calculus

In this appendix we perform the calculus of $l'(0)$ and $L'(0)$ used in the proof of lemma 2.1. We denote by β the half angle at y_0 of the geodesic double cone tangent to the ball $B(x_0, r)$. By the law of sinus we have $\sin\beta = \frac{s_\delta(r)}{s_\delta\left(d(x_0,y_0)\right)}$ and $\frac{s_\delta\left(l(t)\right)}{\sin\beta} = \frac{s_\delta\left(l_1(t)\right)}{\sin t}$, where we have set $l_1\big(d(u_0, v)\big) = d(x_0, Z(v))$. By making t tend to 0 we get $l_1'(0) = \frac{s_\delta^2\left(d(x_0,y_0)\right)}{s_\delta(r)}$. On the other hand, the cosinus laws gives us the equation $c_\delta(l) = c_\delta(l_1)c_\delta\big(d(x_0, y_0)\big) + \delta s_\delta(l_1)s_\delta\big(d(x_0, y_0)\big)\cos\beta$ (resp. $l^2 = l_1^2 + \big(d(x_0, y_0)\big)^2 - 2l_1 d(x_0, y_0)\cos\beta$ if $\delta = 0$), whose the derivative at $t = 0$ gives the relation $l'(0) = -l_1'(0)\cos\beta$. We easily deduce the relation $l'(0) = -s_\delta\big(d(x_0, y_0)\big)\sqrt{\left(\frac{s_\delta\left(d(x_0,y_0)\right)}{s_\delta(r)}\right)^2 - 1}$. Note that for $L'(0)$ we just have to replace β by $\pi - \beta$ in what preceed.

B. A Moser iteration

In this section we prove the inequality (6) used in the proof of lemma 4.2.

Set $0 \leq \eta \leq 1$ a $\mathcal{C}^1$ function such that $\eta \equiv 1$ on $B(x_0, \alpha r)$ (for $\alpha \in]0, 1[$ and $1 \geq r > 0$), $\eta \equiv 0$ on $X^\delta \setminus B(x_0, r)$ and $|d\eta| \leq 2/(1-\alpha)r$.

We fix $m > 0$ and $\beta \geq 0$ and set $h = \inf(m, f_1)$, $u = f_1 h^{\frac{\beta}{2}}$ and $\phi = \eta^2 h^\beta f_1 \in H_0^1(\Omega)$. Then we have

$$\lambda_1 \int_\Omega \eta^2 u^2 = \lambda_1 \int_\Omega \phi f_1 \geq \int_\Omega (df_1, d\phi)$$
$$\geq \beta \int_\Omega \eta^2 h^\beta |dh|^2 + \frac{1}{2} \int_\Omega \eta^2 h^\beta |df_1|^2 - 2 \int_\Omega |d\eta|^2 h^\beta f_1^2,$$

where we used $2\eta f_1 (df_1, d\eta) \geq -\frac{1}{2}\eta^2 |df_1|^2 - 2f_1^2 |dh|^2$. This inequality, combined with the inequalities $|d(\eta u)|^2 \leq 2u^2|d\eta|^2 + 2\eta^2|du|^2$ and $|du|^2 \leq (1+\beta)h^\beta(2\beta|dh|^2 + |df_1|^2)$, gives

$$\int_\Omega |d(u\eta)|^2 \leq (10 + 4\lambda_1)(1+\beta) \int_\Omega u^2(\eta^2 + |d\eta|^2).$$

Hence the Sobolev inequality $(*)$ applied to $u\eta$ implies

$$\Big(\int_{B(x_0,\alpha r)} h^{\frac{(2+\beta)n}{n-2}}\Big)^{\frac{n-2}{n}} \leq \frac{5C(n)(10+4\lambda_1)(1+\beta)}{(1-\alpha)^2 r^2} \int_{B(x_0,r)} f_1^{(2+\beta)}$$

Then we let m tends to ∞ and set $r_k = \frac{1}{2\sqrt{k}}$, $\alpha_k = 2^{\sqrt{k}-\sqrt{k+1}}$ and $\beta_k = 2\big(\frac{n}{n-2}\big)^k - 2$. By multiplying the $(2+\beta_k)$-th square root of the inequalities obtained for $1 \leq k \leq K-1$ we infer

$$\Big(\int_{B(x_0,r_K)} h^{2(\frac{n}{n-2})^K}\Big)^{\frac{1}{2(n/n-2)^K}} \leq A(n,K)(1+\lambda_1)^{\gamma(K)} \int_{B(x_0,r)} f_1$$

With our choice of r_K we have $\Big[\int_{B(x_0,r_K)} f^{2(\frac{n}{n-2})^K}\Big]^{\frac{1}{2(n/n-2)^K}}$ tends to $f(x_0)$ when K tends to $+\infty$, meanwhile $A(n,K)$ and $\gamma(K)$ converge, which gives (6).

C. Proof of lemma 4.3

This lemma is essentially proven in [1,2,4] for u with compact support (which include the case $\delta = 1$) but we will apply it for u an eigenfunction of a convex (unbounded) domain Ω and so we have to extend it in the case $\delta = 0, -1$.

In the remaining of the proof X denotes $\mathbb{R}^n$ or $\mathbb{H}^n$. We fix $x_0 \in X$ and define

$$F \;:\; T_{x_0}X \to T_{x_0}X$$
$$v \mapsto d(\exp_{x_0}^{-1})\Big(\int_X g\big(d(\bar{v},y)\big)\frac{\exp_{\bar{v}}^{-1}(y)}{d(\bar{v},y)} u^2(y)\,dy\Big)$$

where we have set $\bar{v}=\exp_{x_0}(u)$. We set $m=\liminf_{+\infty} g$. Let $R_1>0$ such that $\int_{X\setminus B(x_0,R_1)} u^2 \leq \min\big(\frac{m}{32\|g\|_\infty},\frac{1}{2}\big)$. Then for any $v\in T_{x_0}X$ with $|v|\geq R_1$ we easily have

$$\Big|F(v)-d(\exp_{x_0}^{-1})\Big(\int_{B(x_0,R_1)} g\big(d(\bar{v},y)\big)\frac{\exp_{\bar{v}}^{-1}(y)}{d(\bar{v},y)}u^2(y)\,dy\Big)\Big|\leq\frac{m}{32}.$$

Note that $d(\exp_{x_0}^{-1})\circ\exp_{\bar{v}}^{-1}(x_0)=-v$ and so we infer that for any $v\in T_{x_0}X$ with $|v|\geq R_1$ we have

$$\Big|F(v)+\lambda(v)\frac{v}{|v|}\Big|\leq\|g\|_\infty\int_{B(x_0,R_1)}\Big|\frac{\exp_{\bar{v}}^{-1}(x_0)}{d(\bar{v},x_0)}-\frac{\exp_{\bar{v}}^{-1}(y)}{d(\bar{v},y)}\Big|\,dy+\frac{m}{32}$$

where we have set $\lambda(v)=\int_{B(x_0,R_1)} g\big(d(\bar{v},y)\big)u^2(y)\,dy$ and used the fact that $d(\exp_{x_0}^{-1})$ is a contraction. Then we have $\lambda(v)\geq\frac{m}{4}>0$ for any v with $|v|\geq R_2\geq R_1$. Remark also that the quantity under the integral sign above mesure the difference between the unit tangent vectors at $\bar{v}$ to the minimizing geodesic from $\bar{v}$ to x_0 and $y\in B(x_0,R_1)$. By the law of cosinus we can easily show that this quantity uniformely tends to zero on $B(x_0,R_1)$ when $|v|$ tends to $+\infty$. Hence there exists $R_3>0$ such that for any $v\in T_{x_0}X$ such that $|v|\geq R_3$, we have

$$\Big|F(v)+\lambda(v)\frac{v}{|v|}\Big|\leq\frac{m}{16}\quad\text{and}\quad\lambda(v)\geq\frac{m}{4}$$

We have to show that F is zero somewhere. If it where not the case then the following application (with $R>R_3$)

$$\begin{aligned} G\ :\ B(0,1)\subset T_{x_0}X &\to S(0,1)\subset T_{x_0}X\\ v&\mapsto\frac{F(-Rv)}{|F(-Rv)|}\end{aligned}$$

would be continuous and satisfy $|G(v)-v|\leq\frac{2\left|F(-Rv)+\lambda(-Rv)\right|}{|F(-Rv)|}\leq 4/3$ for any $v\in S(0,1)$. We could then easily construct a retraction from $B(0,2)$ to $S(0,2)$.

D. A result of Li and Yau

Lemma D.1 (Li-Yau). *Let Ω a bounded and smooth domain with positive mean curvature (for the exterior normal). If f an eigenfunction associated to the first eigenvalue of the Dirichlet problem on Ω, then we have:*

$$|\nabla f|^2\leq\lambda_1(\|f\|_\infty^2-f^2)$$

(resp.

$$|\nabla f|^2 \leq (\lambda_1 + n - 1)(\|f\|_\infty^2 - f^2)$$

if $\delta = -1$*).*

Proof. Let $F = \frac{|\nabla f|^2}{\beta - f^2}$ where $\beta = (1+\epsilon)\|f\|_\infty^2$. Then we have

$$dF(v) = \frac{2|\nabla f|^2}{\beta - f^2}\Big(\frac{\operatorname{Hess} f(\frac{\nabla f}{|\nabla f|}, v)}{|\nabla f|} + \frac{f df(v)}{\beta - f^2}\Big)$$

If x_0 is a point of $\partial\Omega$ then $\nu = \nabla f(x_0)/|\nabla f(x_0)|$ is well defined (by the strong maximum principle applied to f) and is the interior normal to Ω at x_0. We then have

$$dF(\nu) = \frac{2|\nabla f|^2}{\beta - f^2}\Big(\frac{\operatorname{Hess} f(\nu, \nu)}{|\nabla f|} + \frac{f|\nabla f|}{\beta - f^2}\Big) \geq 0,$$

since $\frac{\operatorname{Hess} f(\nu,\nu)}{|\nabla f|} = -\frac{\triangle f}{|\nabla f|} + \mu(x_0)$, where $\mu(x_0)$ is the mean curvature of $\partial\Omega$ at x_0. We infer by the strong maximum principle that at a point x_0 where F reaches its maximum on $\bar{\Omega}$ we must have

$$dF(x_0) = 0 \qquad \text{and} \qquad \triangle F(x_0) \geq 0$$

The first equation and our computation of dF implies that $\nabla f/|\nabla f|$ is an eigenvector of $\operatorname{Hess} f(x_0)$ with respect to $g(x_0)$ associated to the eigenvalue $-\frac{f|\nabla f|^2}{\beta - f^2}$ (we have $\nabla f(x_0) \neq 0$ since $F \neq 0$). So we have $|\operatorname{Hess} f(x_0)|^2 \geq f^2 F^2$.

From the Bochner formula $\frac{1}{2}\triangle|\nabla f|^2 = \lambda_1|\nabla f|^2 - |\operatorname{Hess} f|^2 - \operatorname{Ric}(\nabla f, \nabla f)$ (where Ric denote the Ricci curvature tensor of X^δ) we infer that, at x_0, we have

$$\begin{aligned}|\nabla f|^2 F - f\triangle f F + \frac{(\beta - f^2)}{2}\triangle F &= \frac{1}{2}\triangle\big((\beta - f^2)F\big)\\ &\leq \lambda_1|\nabla f|^2 - f^2F^2 - \delta(n-1)|\nabla f|^2.\end{aligned}$$

Since $\triangle F(x_0) \geq 0$ and $|\nabla f|^2 = F(\beta - f^2)$ we readily obtain the estimate $F(x_0) \leq \lambda_1$ (resp. $F(x_0) \leq \lambda_1 + n - 1$ if $\delta = -1$). We then just have to make ϵ tend to 0.

To get Proposition 5.1 in case of a smooth convex domain, let f denote a positive eigenfunction associated to λ_1 and $z_0 \in \Omega$ a point where $f(z_0) = \|f\|_\infty$. Set γ a normal geodesic from z_0 to a point $y \in \partial\Omega$. By the lemma D we have

$$\big(\arcsin(f\circ\gamma/\|f\|_\infty)\big)' \geq -\sqrt{\lambda_1 + \delta(n-1)}$$

and so that $f\circ\gamma(t) \geq \|f\|_\infty \cos\big(\sqrt{\lambda_1 + \delta(n-1)}\,t\big)$. Since $f(y) = 0$ we have infer that the geodesic ball $B(z_0, \frac{\pi}{2\sqrt{\lambda_1 + \delta(n-1)}})$ is included in Ω.

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Erwann AUBRY
Université de Nice Sophia-Antipolis
Laboratoire J.-A. Dieudonné
UMR6621 (UNSA-CNRS)
Parc Valrose
F-06108 Nice Cedex (France)
eaubry@math.unice.fr

Jérôme BERTRAND
Scuola Normale Superiore
Piazza dei Cavalieri, 7
I-56100 Italia
j.bertrand@sns.it

Bruno COLBOIS
Institut de mathématiques
Université de Neuchâtel
Rue Émile Argand, 11
Case postale 158
CH-2009 Neuchâtel
bruno.colbois@unine.ch