AN INVITATION TO QUASIHOMOGENEOUS RIGID GEOMETRIC STRUCTURES

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Dedicated to Alex Dimca and Stefan Papadima

ABSTRACT. This is a survey article dealing with quasihomogeneous geometric structures, in the sense that they are locally homogeneous on a nontrivial open set, but not on all of the manifold. Our motivation comes from Gromov's open-dense orbit theorem which asserts that, if the pseudogroup of local automorphisms of a *rigid geometric structure* acts with a dense orbit, then this orbit is open. Fisher conjectured that the maximal open set of local homogeneity is all of the manifold as soon as the following three conditions are fulfilled: the automorphism group of the manifold acts with a dense orbit, the geometric structure is a G-structure (meaning that it is locally homogeneous at the first order) and the manifold is compact. In a recent joint work, with Adolfo Guillot, we succeeded to prove Fisher's conjecture for real analytic torsion free affine connections on surfaces: we construct and classify those connections which are quasihomogenous; their automorphism group never acts with a dense orbit.

1. INTRODUCTION

Following the work of Lie [Lie80] and then Ehresmann [Ehr36] , the study of *locally homogeneous rigid* geometric structures (think of pseudo-riemannian metrics or connections) became a classical subject in differential geometry. Recall that these locally homogeneous geometric structures are closely related to manifolds *locally modelled* on homogeneous spaces in the following way.

Let G be a connected Lie group and I a closed subgroup of G. A manifold M is said to be *locally modelled* on the homogeneous space G/I if it admits an atlas with open sets diffeomorphic to open sets in G/I, such that the transition maps are given by restrictions of elements in G. Any G-invariant geometric structure $\tilde{\phi}$ on G/I uniquely defines a locally homogeneous geometric structure ϕ on M which is locally isomorphic to $\tilde{\phi}$.

In [Mos50], Mostow gave modern proofs for Lie's classification of surfaces locally modelled on homogeneous spaces (see also the more recent [Olv95]). It is interesting to notice that the dimension of the Lie group G is unbounded. Locally homogeneous connections on surfaces were studied in [Opo04, AMK08, KOV04]. In [BMM08], answering a question of Lie, the authors classified Riemannian metrics on surfaces whose underlying Levi-Civita connections are projectively locally homogeneous.

In dimension three, the locally homogeneous Riemannian metrics formed the context on Thurston's geometrization program [Thu97].

The celebrated open-dense orbit theorem of M. Gromov [DG91, Gro88] (see also [Ben97, CQB03, Fer02]) asserts that *a rigid geometric structure admitting a pseudogroup of local automorphisms which acts with a dense orbit is locally homogeneous on an open dense set.*

This maximal locally homogeneous open-dense set appears to be mysterious and it might very well happen that it coincides with all of the (connected) manifold in many interesting geometric backgrounds. This was proved, for instance, for Anosov flows with differentiable stable and unstable foliations and transverse contact

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structure [BFL92] and for three dimensional compact Lorentz manifolds admitting a nonproper one parameter group acting by automorphisms [Zeg96]. In these papers, the authors make use of the nontrivial dynamics of the automorphism group of the geometric structure, to prove that the geometric structure needs to be locally homogeneous on all of the manifold.

Surprisingly, the extension of a locally homogeneous open dense subset to all of the (connected) manifold might stand *even without assuming the existence of a big automorphism group*. This is known to be true in the Riemannian setting [PTV96], as a consequence of the fact that all scalar invariants are constant. This was also proved in the frame of three dimensional real analytic Lorentz metrics [Dum08], for real analytic unimodular affine connections on surfaces [Dum12] and for complete real analytic pseudo-Riemannian metrics admitting a semi-simple Killing Lie algebra in [Mel09].

In [BF05], the authors deal with this question and their results indicate ways in which some rigid geometric structures cannot degenerate off the open dense set. In [Fis11], Fisher conjectured that the maximal open set of local homogeneity is all of the manifold as soon as the following three conditions are fulfilled: the automorphism group of the manifold acts with a dense orbit, the geometric structure is a G-structure (meaning that it is locally homogeneous at the first order) and the manifold is compact.

In a recent joint work, with Adolfo Guillot [DG13], we succeeded to prove Fisher's conjecture for torsion free affine connections on surfaces. We classify torsion free real-analytic affine connections on compact oriented real-analytic surfaces which are *quasihomogeneous*, in the sense that they are locally homogeneous on a nontrivial open set, without being locally homogeneous on all of the surface. In particular, we prove that such connections exist, but their automorphism group never acts with a dense orbit, which gives a positive answer to Fisher's conjecture for analytic connections on surfaces.

This classification relies in a local result that gives normal forms for germs of torsion free real-analytic affine connections on a neighborhood of the origin in the plane which are *quasihomogeneous*, in the sense that they are locally homogeneous on an open set containing the origin in its closure, but not locally homogeneous in the neighborhood of the origin.

Another motivation to study quasihomogeneous rigid geometric structures, is the result proved in [Dum01] (see also [Bog79]) which asserts, in particular, that *holomorphic rigid geometric structures on compact complex manifolds of algebraic dimension zero are locally homogeneous on an open dense set.* As we will see in the last section of this survey, the holomorpic rigidity implies that many rigid holomorphic geometric structures on compact complex manifolds are locally homogeneous, as it was shown in [Dum01, Dum07, Dum10].

2. RIGID GEOMETRIC STRUCTURES AND KILLING FIELDS

We give here the defintions in the framework of smooth real manifolds and smooth geometric structures. The definitions are similar in the holomorphic category.

Consider an n-manifold M and, for all integers $r \geq 1$, consider the associated bundle $R^r(M)$ of r-frames, which is a $D^r(\mathbb{R}^n)$-principal bundle over M, with $D^r(\mathbb{R}^n)$ the real algebraic group of r-jets at the origin of local diffeomorphisms of $\mathbb{R}^n$ fixing 0.

Let us consider, as in [DG91, Gro88], the following

Definition 1. A *geometric structure* (of order r and of algebraic type) ϕ on a manifold M is a $D^r(\mathbb{R}^n)$-equivariant smooth map from $R^r(M)$ to a real algebraic variety Z endowed with an algebraic action of $D^r(\mathbb{R}^n)$.

Riemannian and pseudo-Riemannian metrics, affine and projective connections and the most encountered geometric objects in differential geometry are known to verify the previous definition [DG91, Gro88, Ben97, CQB03, Fer02]. For instance, if the image of ϕ in Z is exactly one orbit, this orbit identifies with a homogeneous space $D^r(\mathbb{R}^n)/G$, where G is the stabilizer of a chosen point in the image of ϕ. We get then a reduction of the structure group of $R^r(M)$ to the subgroup G. This is exactly the classical definition of a G-structure (of order r): the case $r=1$ and $G=O(n,\mathbb{R})$ corresponds to a Riemannian metric and that of $r=2$ and $G=GL(n,\mathbb{R})$ gives a torsion free affine connection.

Definition 2. A (local) Killing field of ϕ is a (local) vector field on M whose canonical lift to $R^r(M)$ preserves ϕ.

For all $k \in \mathbb{N}$, the k-th prolongation of ϕ is a geometric structure of order $r+k$ (and of algebraic type), given by an equivariant smooth map $\phi^{(k)} : R^{(r+k)}(M) \to J^{n,k}(Z)$, where $J^{n,k}(Z)$ is the bundle of k-jets at 0 of smooth maps from $\mathbb{R}^n$ to Z. The real algebraic variety $J^{n,k}(Z)$ admits an algebraic action of $D^{(r+k)}(\mathbb{R}^n)$ which is obtained in a canonical way from the $D^r(\mathbb{R}^n)$-action on Z (see, for example, [Fer02]).

For all $m \in M$, we denote by $Is^{(r+k)}(m)$ the group of $(r+k)$-isometric jets. They are $(r+k)$-jets at m of local diffeomorphisms f of M fixing m such that, for any $\xi \in R^{(r+k)}(M)$ above m, we have $\phi^{(r+k)}(f\cdot\xi) = \phi^{(r+k)}(\xi)$. Remark that the previous relation only depends on the $(r+k)$-jet of f at m.

For every point m in M, there exists a natural homomorphism

$$Is^{(r+k+1)}(m) \to Is^{(r+k)}(m).$$

Following Gromov [Gro88, DG91] we define rigidity as:

Definition 3. The geometric structure ϕ (of order r) is rigid at order $(r+k)$, if for every point $m \in M$, the homomorphism $Is^{(r+k+1)}(m) \to Is^{(r+k)}(m)$ is injective.

A consequence of the previous definition is that the Lie algebra of Killing fields of a rigid geometric structure is finite dimensional in the neighborhood of every point in M [Fer02].

Recall that (pseudo)-Riemannian metrics, as well as affine and projective connections, or conformal structures in dimension ≥ 3 are known to be rigid [DG91, Gro88, Ben97, CQB03, Fer02].

Definition 4. The geometric structure ϕ is said to be locally homogeneous on the open subset $U \subset M$ if for every $u \in U$ and every tangent vector $V \in T_uU$ there exists a local Killing field X of ϕ such that $X(u) = V$.

The Lie algebra of Killing fields is the same at the neighborhood of any point of a locally homogeneous geometric structure ϕ. This still holds for any real analytic rigid geometric structure (not necessarly locally homogeneous). In this case, one need to make use of an extendibility result for local Killing fields proved first for Nomizu in the Riemannian setting [Nom60] and generalized then for rigid geometric structures by Amores et Gromov [Amo79, Gro88] (see also [CQB03, DG91, Fer02]). This phenomenon states roughly that a local Killing field of a *rigid analytic* geometric structure can be extended along any curve in M. We then get a multivalued Killing field defined on all of M or, equivalently, a global Killing field defined on the universal cover. In particular, the Killing algebra in the neighborhood of any point is the same (as long as M is connected). It will be simply called *the Killing algebra of* ϕ.

Notice also that Nomizu's extension phenomenon doesn't imply that the extension of a family of linearly independent Killing fields, stays linearly independent. In general, the extension of a localy transitive Killing algebra, fails to be transitive on a nowhere dense analytic subset S in M. This is exactly what happens for quasihomogeneous geometric structures (see later our examples of quasihomogenous connections on surfaces). Moreover, this explains also that *a real analytic rigid geometric structure which is locally homogeneous on some nontrivial open set, is also locally homogeneous away from an analytic subset S of positive codimension* (S might be empty).

We recall that there exists locally homogeneous Riemannian metrics on 5-dimensional manifolds which are not locally isometric to a invariant Riemannian metric on a homogeneous space. However this phenomenon cannot happen in lower dimension:

Theorem 5. *Let M be a manifold of dimension ≤ 4 bearing a locally homogeneous rigid geometric structure ϕ with Killing algebra $\mathfrak{g}$. Then M is locally modelled on a homogeneous space G/I, where G is a connected Lie group with Lie algebra $\mathfrak{g}$ and I is a closed subgroup of G. Moreover, (M,ϕ) is locally isomorphic to a G-invariant geometric structure on G/I.*

Proof. Let $\mathfrak{g}$ be the Killing algebra of ϕ. Denote by $\mathfrak{I}$ the (isotropy) subalgebra of $\mathfrak{g}$ composed by Killing fields vanishing at a given point in M. Let G be the unique connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Since $\mathfrak{I}$ is of codimension ≤ 4 in $\mathfrak{g}$, a result of Mostow [Mos50] (chapter 5, page 614) shows that the Lie subgroup I in G associated to $\mathfrak{I}$ is *closed*. Then ϕ induces a G-invariant geometric structure $\tilde{\phi}$ on G/I locally isomorphic to it. Moreover, M is locally modelled on G/I. □

Remark 6. By the previous construction, G is simply connected and I is connected (and closed), which implies that G/I is simply connected (see [Mos50], page 617, Corollary 1). In general, the G-action on G/I admits a nontrivial discrete kernel. We can assume that this action is effective considering the quotient of G and I by the maximal normal subgroup of G contained in I (see proposition 3.1 in [Sha97]).

We give now a last definition:

Definition 7. A geometric structure ϕ on M is said to be of *Riemannian type* if there exists a Riemannian metric on M preserved by all Killing fields of ϕ.

Roughly speaking a locally homogeneous geometric structure is of Riemannian type if it is constructed by putting together a Riemannian metric and any other geometric structure (e.g. a vector field). Since Riemannian metrics are rigid, a geometric structure of Riemannian type it is automaticaly rigid.

With this terminology we have the following corollary of theorem 5.

Theorem 8. *Let M be a compact manifold of dimension ≤ 4 equipped with a locally homogeneous geometric structure ϕ of Riemannian type. Then M is isomorphic to a quotient of a homogeneous space G/I, endowed with a G-invariant geometric structure, by a lattice in G.*

Proof. By theorem 5, M is locally modelled on a homogeneous space G/I. Since ϕ is of Riemannian type, G/I admits a G-invariant Riemannian metric. This implies that the isotropy I is compact.

On the other hand, compact manifolds locally modelled on homogeneous space G/I with compact isotropy group I are classically known to be complete (meaning exactly that M is isomorphic to a quotient of G/I by a lattice in G) : this is a consequence of the Hopf-Rinow's geodesical completeness [Sha97]. □

Remark 9. A G-invariant geometric structure on G/I is of Riemannian type if and only if I is compact.

Recall that a homogeneous space G/I is said to be *imprimitive* if the canonical G-action preserves a non trivial foliation.

Proposition 10. *If M is a compact surface locally modelled on an imprimitive homogeneous space, then M is a torus.*

Proof. The G-invariant one dimensional foliation on G/H descends on M to a non singular foliation. Hopf-Poincaré's theorem implies then that the genus of M equals one: M is a torus. □

Note that the results of [KOV04, AMK08] imply in particular:

Theorem 11. *A locally homogeneous affine connection on a surface which is neither torsion free and flat, nor of Riemannian type, is locally modelled either on an imprimitive homogeneous space or on $SL(2,\mathbb{R})/I$, where I the diagonal one parameter subgroup.*

Indeed, T. Arias-Marco and O. Kowalski study in [AMK08] all possible local normal forms for locally homogeneous affine connections on surfaces with the corresponding Killing algebra. Their results are summarized in a nice table (see [AMK08], pages 3-5). In all cases, except for the Killing algebra of the (standard) torsion free affine connection and for Levi-Civita connections of Riemannian metrics of constant sectional curvature, either there exists at least one Killing field non contained in the isotropy algebra which is normalized by the Killing algebra, or the Lie algebra is the standard $sl(2,\mathbb{R})$. In the first case, the normalized Killing field direction defines a G-invariant line field on G/I; in the second case the corresponding homogeneous space is $SL(2,\mathbb{R})/I$, where I is the diagonal one parameter subgroup.

The previous result combined with proposition 10 imply the main result in [Opo04]:

Theorem 12. *A compact surface M bearing a locally homogeneous affine connection of non Riemannian type is a torus.*

Recall first that a well known result of J. Milnor shows that a compact surface bearing a flat torsion free affine connection is a torus.

Proof. In the case of a non flat connection, theorem 11 shows that M is locally modelled on an imprimitive homogeneous space or on $SL(2,\mathbb{R})/I$. Proposition 10 finishes the proof in the first case. In the second case, M inherits of a flat torsion free affine connection and we conclude using Milnor's theorem. □

As an application of the Nomizu's phenomenon we give (compare with theorem 5):

Theorem 13. *Let M be a compact simply connected real analytic manifold admitting a real analytic locally homogeneous rigid geometric structure. Then M is isomorphic to a homogeneous space G/I endowed with a G-invariant geometric structure.*

Proof. Since ϕ is locally homogeneous and M is simply connected and compact, the local transitive action of the Killing algebra extends to a global action of the associated simply connected Lie group G (we need compactness to ensure that vector fields on M are complete). All orbits have to be open, so there is only one orbit: the action is transitive and M is a homogeneous space. □

3. Quasihomogeneous real-analytic connections

We present now the recent results obtained in collaboration with Adolfo Guillot [DG13].

The main results in [DG13] are stated in Theorem 16 (the local classification) and in Theorem 19 (the global classification). In particular, we show that such (strictly) quasihomogeneous connections exist.

Motivated by these results, Theorem 19 constructs and characterizes torsion free real-analytic affine connections on compact surfaces which are quasihomogeneous (but not homogeneous). Even if we cannot say that a real analytic connection on a compact surface which is locally homogeneous somewhere is locally homogeneous everywhere, we fully understand the connections that do not satisfy this property.

Lets first give the following example of a quasihomogeneous (unimodular) real analytic affine connection.

Proposition 14. *Let ∇ be a connection on $\mathbb{R}^2$ such that $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \mu y^3 \frac{\partial}{\partial x}$, with μ a nonzero real constant, and all other Christoffel symbols vanish. Then the (torsion free) unimodular affine connection determined by ∇ and the volume form $vol = dx \wedge dy$ is locally homogeneous on $y > 0$ and on $y < 0$, but not on all of $\mathbb{R}^2$.*

Proof. We check easily that ∇ and the volume form are invariant by the flows of $\frac{\partial}{\partial x}$ and of $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$.

Remark that the invariant vector fields of the previous action are $A = y\frac{\partial}{\partial y}$ and $B = \frac{1}{y}\frac{\partial}{\partial x}$. They are of constant volume and ∇ has constant Christoffel symbols with respect to A and B:

$$\nabla_A A = A, \nabla_B B = \mu A, \nabla_B A = 0, \nabla_A B = [A,B] = -B.$$

Thus the unimodular affine connection is locally homogeneous on the open sets $y > 0$ and $y < 0$, where A and B are pointwise linearly independent.

The only nonzero component of the curvature tensor is

$$R(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\frac{\partial}{\partial x} = \nabla_{\frac{\partial}{\partial x}} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x} - \nabla_{\frac{\partial}{\partial y}} \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -3\mu y^2 \frac{\partial}{\partial x}.$$

The curvature tensor vanishes exactly on $y = 0$. Thus ∇ is not locally homogeneous on all of $\mathbb{R}^2$. □

The main theorem in [Dum12] shows that the previous germ of connection cannot be extended on a compact surface:

Theorem 15. *Let ∇ be a real analytic unimodular affine connection on a compact connected real analytic surface M. If ∇ is locally homogeneous on a nontrivial open set in M, then ∇ is locally homogeneous on all of M.*

The main local ingredient of Theorem 19 is the following classification of (all) germs of torsion free real-analytic affine connections which are quasihomogeneous. It is, in some sense, the quasihomogeneous analogue to the local results in [KOV04].

Theorem 16. *Let ∇ be a torsion free real-analytic affine connection in a neighborhood of the origin in $\mathbf{R}^2$. Suppose that the maximal open set where $\mathfrak{K}(\nabla)$, the Lie algebra of Killing vector fields of ∇, is transitive, contains the origin in its closure, but does not contain the origin. Then, up to an analytic change of coordinates, the germ of ∇ at the origin is one of the following:*

Type $\mathrm{I}(n)$**,** $n \in \frac{1}{2}\mathbf{Z}$**,** $n \geq \frac{1}{2}$**:** *The germ at $(0,0)$ of*

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = 0,\ \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = -\gamma x^n \frac{\partial}{\partial x},\ \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\tfrac{1}{n}\varepsilon x^{2n+1} \frac{\partial}{\partial x} - \phi x^n \frac{\partial}{\partial y},$$

with $\phi = 0$ and $\gamma = 0$, if $n \notin \mathbf{Z}$ and $(n,\phi,\varepsilon) \neq (1,-\gamma,-\gamma^2)$. For these, $\mathfrak{K}(\nabla) = \langle x\partial/\partial x - ny\partial/\partial y, \partial/\partial y\rangle$.

Type $\mathrm{II}(n)$**,** $n \in \frac{1}{2}\mathbf{Z}$**,** $n \geq \frac{5}{2}$**:** *The germ at $(0,0)$—Type $\mathrm{II}^0(n)$—or the germ at $(0,1)$—Type $\mathrm{II}^1(n)$—of*

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \left(-\tfrac{1}{n}\varepsilon x^{2n-3}y^2 + 2\gamma x^{n-2}y\right)\frac{\partial}{\partial x} + \left(-\tfrac{1}{n}\varepsilon x^{2n-4}y^3 + [2\gamma - \phi]x^{n-3}y^2\right)\frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \left(\tfrac{1}{n}\varepsilon x^{2n-2}y - \gamma x^{n-1}\right)\frac{\partial}{\partial x} + \left(\tfrac{1}{n}\varepsilon x^{2n-3}y^2 + [\phi - \gamma]x^{n-2}y\right)\frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\tfrac{1}{n}\varepsilon x^{2n-1}\frac{\partial}{\partial x} - \left(\tfrac{1}{n}\varepsilon x^{2n-2}y + \phi x^{n-1}\right)\frac{\partial}{\partial y},$$

with $\phi = 0$ and $\gamma = 0$, if $n \notin \mathbf{Z}$. For these, $\mathfrak{K}(\nabla) = \langle x\partial/\partial x + (1-n)y\partial/\partial y, x\partial/\partial y\rangle$.

Type III**:** *The germ at* $(0,0)$ *of*

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \left(-\tfrac{1}{2}\varepsilon xy^2 + 2\gamma y\right) \frac{\partial}{\partial x} - \tfrac{1}{2}\varepsilon y^3 \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = \left(\tfrac{1}{2}\varepsilon x^2 y - \gamma x\right) \frac{\partial}{\partial x} + \left(\tfrac{1}{2}\varepsilon xy^2 + \gamma y\right) \frac{\partial}{\partial y},$$

$$\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\tfrac{1}{2}\varepsilon x^3 \frac{\partial}{\partial x} - \left(\tfrac{1}{2}\varepsilon x^2 y + 2\gamma x\right) \frac{\partial}{\partial y},$$

for which $\mathfrak{K}(\nabla)$ *is the Lie algebra of divergence-free linear vector fields in* $\mathbf{R}^2$.

In Types I *and* II*,* $(\gamma,\phi,\varepsilon) \in \mathbf{R}^3 \setminus \{(0,0,0)\}$ *and the connection with parameters* $(\gamma,\phi,\varepsilon)$ *is equivalent to the one with parameters* $(\mu\gamma,\mu\phi,\mu^2\varepsilon)$*, for* $\mu > 0$ *in the case of Type* II^1 *and, for* $\mu \in \mathbf{R}^*$*, in the other cases. In Type* III*,* $(\gamma,\varepsilon) \in \mathbf{R}^2 \setminus \{(0,0)\}$ *and the connection with parameters* (γ,ε) *is equivalent to the one with parameters* $(\mu\gamma,\mu^2\varepsilon)$ *for* $\mu \in \mathbf{R}^*$*. Apart from this, all the above connections are inequivalent.*

Remark 17. For the connections of Type III, $\mathfrak{K}(\nabla) \approx \mathfrak{sl}(2,\mathbf{R})$ and there is one two-dimensional orbit of the Killing algebra (the complement of the origin). In the other cases, $\mathfrak{K}(\nabla) \approx \mathfrak{aff}(\mathbf{R})$, the Lie algebra of the affine group of the real line, and the components of the complement of the geodesic $\{x = 0\}$ in $\mathbf{R}^2$ are the two-dimensional orbits of the Killing algebra. In particular, all these germs admit nontrivial open sets on which ∇ is locally isomorphic to a translation invariant connexion on the connected component of the affine group of the real line. *The closed set where* ∇ *is not locally homogeneous is either a geodesic, or a point.* Moreover, in all cases, every vector field in the Killing algebra $\mathfrak{K}(\nabla)$ is an affine one and thus these Killing Lie algebras also preserve a flat torsion free affine connection.

The previous theorem admits the following corollary:

Corollary 18. *If a surface S admits a quasihomogeneous real analytic torsion free affine connection* ∇*, then S also admits a flat torsion free affine connection preserved by the Killing algebra of* ∇*. In particular, if S is compact, then S is diffeomorphic to a torus.*

The idea for the proof of Theorem 16 is the following. We prove that for a germ of quasihomogeneous real-analytic affine connection, we can always find a two-dimensional subalgebra of the Killing algebra which is transitive on a nontrivial open set, but not at the origin. Consequently, there exists a nontrivial open set where the connection is locally isomorphic either to a translation-invariant connection on $\mathbf{R}^2$, or to a left-invariant connection on the affine group. Then we show that a quasihomogeneous connection cannot be locally isomorphic (on a nontrivial open set) to a translation invariant connection on $\mathbf{R}^2$ (without being locally homogeneous everywhere). In order to deal with the affine case, we will have to study the invariant affine connections and their Killing algebras in the affine group. The method consists in considering normal forms at the origin of left invariant vector fields X,Y on the affine group, with respect to which we compute Christoffel coefficients (in general X and Y are meromorphic at the origin).

Our global classification is the following one:

Theorem 19. (1) *For integers* n_1, n_2*, with* $n_2 \geq n_1 \geq 2$*, there exists a unique (up to automorphism) real-analytic torsion free affine connection* ∇_{n_1,n_2} *on* $\mathbf{R}^2$ *such that*

(a) ∇_{n_1,n_2} *is locally homogeneous on a nontrivial open set, but not on all of* $\mathbf{R}^2$*. For* $i = 1,2$*, there exists a point* $p_i \in \mathbf{R}^2$ *such that* ∇_{n_1,n_2} *is, in a neighborhood of* p_i*, given by a normal form of type* $\mathrm{II}^1(n_i)$*, if* n_i *is odd and by a normal form of type* $\mathrm{I}(n_i)$*, if* n_i *is even (in particular, the Killing Lie algebra of* ∇_{n_1,n_2} *is isomorphic to that of the affine group of the real line).*

(b) *There exist groups of automorphisms of* ∇_{n_1,n_2} *acting freely, properly discontinuously and cocompactly on* $\mathbf{R}^2$.

(2) *Let S be a compact orientable analytic surface endowed with a real-analytic torsion free affine connection that is locally homogeneous on some nontrivial open set, but not on all of S. Then* (S,∇) *is isomorphic to a quotient of* $(\mathbf{R}^2,\nabla_{n_1,n_2})$.

(3) *The moduli space of compact quotients of* $(\mathbf{R}^2,\nabla_{n_1,n_2})$ *is* $\Xi=\mathbf{N}\times\mathbf{R}\times\mathbf{R}/\mathbf{Z}$. *Every compact quotient of* $(\mathbf{R}^2,\nabla_{n_1,n_2})$ *is a torus. For the torus T corresponding to* $(k,\tau,\theta)\in\Xi$ *we have:*

(a) *The open set of local homogeneity is dense and is a union of* $2k$ *(if* $n_1\neq n_2$*) or* k *(if* $n_1=n_2$*) cylinders bounded by simple closed geodesics.*

(b) *There exists a globally defined Killing field A on T, unique up to multiplication by a constant. When normalized such that there exists a Killing vector field B defined in some open subset such that* $[A,B]=B$, *A is periodic with period* τ.

(c) *If* γ_1 *and* γ_2 *are generators of the fundamental group of T and* γ_1 *is homotopic to an orbit of A, then the analytic continuations of B along* γ_1 *and* γ_2 *are, respectively,* $e^{\tau}B$ *and* $e^{(\theta+k)\tau}B$.

Example 20. Let us describe explicitly the quotients of $\nabla_{2,2}$ with parameters $(1,\tau,[\theta])\in\Xi$. This will also give a self-contained proof of the fact that tori admit quasihomogeneous connections. Consider, in $\mathbf{R}^2$, the torsion free affine connection ∇ such that

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}=0,\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}=\frac{3}{4}x^2\frac{\partial}{\partial x},\ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y}=\frac{3}{8}x^5\frac{\partial}{\partial x}+\frac{3}{4}x^2\frac{\partial}{\partial y}$$

(it is a connection of type I for $n=2$ and $\varepsilon=\phi=\gamma=-\frac{3}{4}$). Remark that $B=\partial/\partial y$ is a Killing vector field for ∇. Consider the commuting meromorphic vector fields

$$A=\frac{1}{2}x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y},\ Z=\frac{1}{2}x\frac{\partial}{\partial x}+x^{-2}\frac{\partial}{\partial y}.$$

If we let $h(x,y)=x^2y$, we have

$$\nabla_A Z=\frac{3}{4}A-Z,\ \nabla_Z Z=-\frac{1}{4}Z,\ \nabla_A A=\frac{3}{4}(h^2+2h)A+\left(\frac{1}{4}-\frac{3}{8}[h^2+2h]\right)Z. \tag{1}$$

Notice that, since $[A,Z]=0$ and h is a first integral of A, the vector field A is a Killing field of ∇. The Lie algebra of Killing vector fields of ∇ contains the subalgebra generated by A and by B. The rank of this Lie algebra of vector fields is one in $\{x=0\}$ and two in its complement. A direct computation shows that the curvature tensor of ∇ vanishes exactly on $\{x=0\}$. The connection is thus locally homogeneous in the half planes $\{x<0\}$ and $\{x>0\}$, but not on all of $\mathbf{R}^2$.

Consider the orientation-preserving birational involution $\sigma(x,y)=(-x,-y-2x^{-2})$. It preserves the vector fields A and Z. Moreover, $h\circ\sigma=-2-h$ and hence $(h^2+2h)\circ\sigma=h^2+2h$. This implies, by (1), that ∇ is preserved by σ. Let $\Omega=\{(x,y);y<0,h>-2\}$, $U^+=\Omega\cap\{x>0\}$, $U^-=\Omega\cap\{x<0\}$. Notice that $\sigma|_{U^+}:U^+\to U^-$ is an analytic diffeomorphism. Let $\phi:\Omega\to\Omega$ be the diffeomorphism generated by the flow of A in time τ. The diffeomorphism ϕ preserves ∇ and commutes with σ. The quotient of Ω under the group generated by ϕ is a cylinder containing a simple closed curve coming from $\{x=0\}$, whose complement is the union of the two cylinders $U^+/\langle\phi\rangle$ and $U^-/\langle\phi\rangle$. Let $K:U^+/\langle\phi\rangle\to U^-/\langle\phi\rangle$ be given by, first, the restriction to $U^+/\langle\phi\rangle$ of the flow of A in time $\theta\tau$ and then composing with σ (notice that adding an integer to θ yields the same result). By identifying $U^+/\langle\phi\rangle$ and $U^-/\langle\phi\rangle$ (as open subsets of $\Omega/\langle\phi\rangle$), via K, we obtain a torus S, naturally endowed with a connection ∇_s, coming from ∇, a globally defined Killing vector field for ∇_s, induced by A, and a multivalued one, induced by B. There is one simple closed curve in S coming from $\{x=0\}/\langle\phi\rangle$. The rank of the Killing algebra of vector fields of ∇_s is one along this curve and two in the complement: the connection is not locally homogeneous everywhere (since the curvature tensor vanishes exactly on $\{x=0\}$), but is locally homogeneous in a dense open subset.

Note that in the previous example the connected component of the automorphism group of ∇_s is the flow generated by A, all of whose orbits are closed. The proof of Theorem 19 shows that, in general, the automorphism group of a quasihomogeneous connection is, up to a finite group, the flow of a Killing field, all of whose orbits are simple closed curves. Hence, the automorphism group doesn't admit dense orbits (our quasihomogeneous connections are not counter-examples to Fisher's conjecture, but they give a positive answer).

Another framework where Fisher's conjecture hold is real analytic Lorentz geometry in dimension three. In this context we proved in [Dum08]:

Theorem 21. *Let g be a real analytic Lorentz metric on a compact connected real analytic threefold M. If g is locally homogeneous on a nontrivial open set in M, then g is locally homogeneous on all of M.*

Later, in a common work with Zeghib [DZ10], we completely classified those locally homogeneous compact Lorentz threefolds. All of them have to be quotients of Lorentz homogeneous spaces (completeness). We also proved that there are exactly 4 essential and maximal Lorentz homogeneous spaces which admit compact quotients: the flat Lorentz space, the anti-de Sitter space (constant negative curvature), a left invariant Lorentz metric on the Heisenberg group and a left invariant Lorentz metric on the Sol group. This is a Lorentz version of Thurston's classification of 8 maximal Riemannian geometries in dimension three and leads to the following uniformization result:

Corollary 22. *If a compact manifold of dimension three is locally modeled on a Lorentz homogeneous space (of non Riemannian kind), then it admits, on a finite cover, a Lorentz metric of constant (non positive) sectional curvature.*

4. HOLOMORPHIC GEOMETRIC STRUCTURES

Recall that a complex manifold M is of algebraic dimension zero if there are no nonconstant meromorphic functions on M. In this context the following theorem was proved in [Dum01].

Theorem 23. *Let M be a connected complex manifold of algebraic dimension zero, endowed with a meromorphic rigid geometric structure ϕ. Then ϕ is locally homogeneous on an open dense subset in M.*

Moreover, if ϕ is unimodular and holomorphic and its Killing Lie algebra $\mathcal{G}$ is unimodular and simply transitive, then ϕ is locally homogeneous on all of M.

Proof. Let us denote by n the complex dimension of M and by $Is^{loc}(\phi)$ the pseudogroup of local isometries of ϕ.

Let ϕ be of order r, given by a map $\phi : R^r(M) \to Z$. For each positive integer s we consider the s-jet $\phi^{(s)}$ of ϕ. This is a $D^{(r+s)}(\mathbb{C}^n)$-equivariant meromorphic map $R^{(r+s)}(M) \to Z^{(s)}$, where $Z^{(s)}$ is the algebraic variety of the s-jets at the origin of holomorphic maps from $\mathbb{C}^n$ to Z. One can find the expression of the (algebraic) $D^{(r+s)}(\mathbb{C}^n)$-action on $Z^{(s)}$ in [CQB03, Fer02].

Since ϕ is rigid, there exists a nowhere dense analytic subset S' in M, containing the poles of ϕ, and a positive integer s such that two points m, m' in $M \setminus S'$ are related by local automorphisms if and only if $\phi^{(s)}$ sends the fibers of $R^{(r+s)}(M)$ above m and m' on the same $D^{(r+s)}(\mathbb{C}^n)$-orbit in $Z^{(s)}$ [Gro88, DG91].

Rosenlicht's theorem shows that there exists a $D^{(r+s)}(\mathbb{C}^n)$-invariant stratification

$$Z^{(s)} = Z_0 \supset \ldots \supset Z_l,$$

such that Z_{i+1} is Zariski closed in Z_i, the quotient of $Z_i \setminus Z_{i+1}$ by $D^{(r+s)}(\mathbb{C}^n)$ is a complex manifold and rational $D^{(r+s)}(\mathbb{C}^n)$-invariant functions on Z_i separate orbits in $Z_i \setminus Z_{i+1}$.

Consider the open dense $Is^{loc}(\phi)$-invariant subset U in $M \setminus S'$, such that $\phi^{(s)}$ is of constant rank above U and the image of $R^{(r+s)}(M)|_U$ through $\phi^{(s)}$ lies in $Z_i \setminus Z_{i+1}$, but not in Z_{i+1}. Then the orbits of $Is^{loc}(\phi)$ in U are the fibers of a fibration of constant rank (on the quotient of $Z_i \setminus Z_{i+1}$ by $D^{(r+s)}(\mathbb{C}^n)$). Obviously, $U = M \setminus S$, where S is a nowhere dense analytic subset in M.

Assume now, by contradiction, that m and m' are two points in U which are not in the same $Is^{loc}(\phi)$-orbit, then the corresponding fibers of $R^{(r+s)}(M)|_U$ are sent by $\phi^{(s)}$ on two distinct $D^{(r+s)}(\mathbb{C}^n)$-orbits in $Z_i \setminus Z_{i+1}$. By Rosenlicht's theorem there exists a $D^{(r+s)}(\mathbb{C}^n)$-invariant rational function $F : Z_i \setminus Z_{i+1} \to \mathbb{C}$, which takes distincts values at these two orbits.

The meromorphic function $F \circ \phi^{(s)} : R^{(r+s)}(M) \to \mathbb{C}$ is $D^{(r+s)}(\mathbb{C}^n)$-invariant and descends in a $Is^{loc}(\phi)$-invariant meromorphic function on M which takes distincts values at m and at m': a contradiction.

We proved that the Killing Lie algebra $\mathcal{G}$ of ϕ is transitive on a maximal open dense subset U in M. Suppose now that ϕ is unimodular and also that $\mathcal{G}$ is unimodular and simply transtitive.

Pick up a point m in M. We want to show that m is in U. The point m admits an open neighborhood U_m in M such that any local holomorphic Killing field of ϕ defined on a connected open subset in U_m extends on all of U_m [Nom60, Amo79].

Since $\mathcal{G}$ acts transitively on U, choose local linearly independent Killing fields $X_1, \ldots, X_n$ on a connected open set included in $U \cap U_m$. As ϕ is unimodular, it determines a holomorphic volume form vol on U_m (if necessary we restrict to a smaller U_m). But $Is^{loc}(\phi)$ acts transitively on $U \cap U_m$ and $\mathcal{G}$ is supposed to be unimodular. This implies that the function $vol(X_1, \ldots, X_n)$ is $\mathcal{G}$-invariant and, consequently, a (non-zero) constant on $U \cap U_m$.

On the other hand $X_1, \ldots, X_n$ extend in some holomorphic Killing fields $\tilde{X}_1, \ldots, \tilde{X}_n$ defined on all of U_m. The holomorphic function $vol(\tilde{X}_1, \ldots, \tilde{X}_n)$ is a non-zero constant on U_m: in particular, $\tilde{X}_1(m), \ldots, \tilde{X}_n(m)$ are linearly independent. We proved that $\mathcal{G}$ acts transitively in the neighborhood of m and thus $m \in U$. □

As a consequence of theorem 23, we proved in [Dum10] the following result:

Theorem 24. *Any holomorphic geometric structure τ on a Inoue surface is locally homogeneous.*

Indeed, it is known that Inoue surfaces admit holomorphic affine structures (locally modelled on $(GL(2,\mathbb{C}) \ltimes \mathbb{C}^2, \mathbb{C}^2)$). We put together this holomorphic affine structure and any other holomorphic geometric structure τ in some extra *rigid* holomorphic geometric structure ϕ, for which theorem 23 applies (since Inoue surfaces have algebraic dimension zero). It remains to prove that ϕ is locally homogeneous on all of the surface. The next step of the proof is to use the fact that Inoue surfaces don't admit nontrivial complex submanifolds and to show that ϕ is locally homogeneous at least in the complementary of a finite number of points. The last step of the proof uses the fact that nontrivial holomorphic vector fields on Inoue surfaces don't vanish to conclude that ϕ is locally homogeneous. In particular, τ is also locally homogeneous.

Let us give an example where the open set of local homogeneity is not all of M.

Example 25. Consider the Hopf surface M of algebraic dimension zero, which is the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the group $\mathbb{Z}$ generated by $T(z_1, z_2) = (\frac{1}{2}z_1, \frac{1}{3}z_2)$. The only curves on M are the two elliptic curves obtained as projections of the lines $\{z_1 = 0\}$ and $\{z_2 = 0\}$.

The Hopf surface M inherits of the standard affine structure of $\mathbb{C}^2$ and also of the vector fields $X_1 = z_1 \frac{\partial}{\partial z_1}$ et $X_2 = z_2 \frac{\partial}{\partial z_2}$ (they are T-invariant). Let us denote by ϕ the holomorphic rigid geometric structure which is the juxtaposition of the affine structure and of the vector fields X_1 and X_2.

The open dense set U of local homogeneity is the complement of the two elliptic curves in M. This is exactly the open set where the vector fields X_1 and X_2, which are Killing fields of ϕ (since they are linear and commute), are linearly independent. On the other hand, since X_1 and X_2 are part of the geometric structure, ϕ is not locally homogeneous in the neighborhood of points where X_1 and X_2 are collinear (on each elliptic curve, either X_1, or X_2 vanishes).

Let us also remind that Inoue, Kobayashi and Ochiai classified in [IKO80] all compact complex surfaces admitting holomorphic affine connections. In particular, they proved the following:

Theorem 26. *If a complex compact surface M admits a holomorphic affine connection, then it also admits a flat holomorphic affine connection (locally modelled on $(GL(2,\mathbb{C}) \ltimes \mathbb{C}^2, \mathbb{C}^2)$).*

Moreover, if M is Kaehler then M admits a finite cover which is a complex torus.

Following their work, in [Dum10] we determined the local structure of all holomorphic torsion free affine connections on compact complex surfaces. It turns out that the Killing Lie algebra of those connections is either of dimension one, or it is isomorphic to $\mathbb{C}^2$ acting transitively (these connections are locally homogeneous). In particular, we conclude that *there are no quasihomogeneous torsion free holomorphic affine connections on compact complex surfaces*. Precisely, the classsification is given by the following:

Theorem 27. *Let (M,∇) be a compact complex surface endowed with a torsion free holomorphic affine connection.*

(i) If M is not biholomorphic to a principal elliptic bundle over a Riemann surface of genus $g \geq 2$, with odd first Betti number, then ∇ is locally isomorphic to a translation invariant connection on $\mathbb{C}^2$. In particular, ∇ is locally homogeneous.

(ii) In the other case, ∇ is invariant by the fundamental vector field of the elliptic fibration. If ∇ is non flat, then the corresponding Killing algebra is of dimension one.

Corollary 28. *Normal holomorphic projective connections on compact complex surfaces are flat (locally modeled on $(PGL(3,\mathbb{C}), P^2(\mathbb{C}))$.*

Let us now define a new holomorphic rigid geometric structure which is the complex analogous of a pseudo-Riemannian metric.

Definition 29. A holomorphic Riemannian metric on M is a holomorphic section q of the bundle $S^2(T^*M)$ such that at any point m in M, the complex quadratic form $q(m)$ is non degenerated.

For this geometric structure, we proved in [Dum07] the following:

Theorem 30. *A holomorphic Riemannian metric on a compact complex connected threefold is locally homogeneous.*

Later, in a joint work with Zeghib [DZ09], we proved a holomorphic analogous of our Lorentz classification in dimension three:

Theorem 31. *Let M be a compact complex connected threefold endowed with a holomorphic Riemannian metric.*

(i) If the Killing Lie algebra of g admits a nontrivial semi-simple part , then it preserves a holomorphic Riemannian metric of constant sectional curvature on M.

(ii) If the Killing Lie algebra is solvable, then M admits a finite cover which is a quotient of the complex Heisenberg group or of the complex SOL group by a lattice.

Corollary 32. *M possesses a finite cover which admits a holomorphic Riemannian metric of constant sectional curvature.*

Contrary to the situation in Lorentz geometry, the classification of the compact complex 3-manifolds endowed with a holomorphic Riemannian metric of constant sectional curvature is still an open problem. Two different situations could appear:

Flat case. In this case M is locally modelled on $(O(3,\mathbb{C}) \ltimes \mathbb{C}^3, \mathbb{C}^3)$. The challenge remains:
1) *Markus conjecture:* Is M complete (i.e. is M a quotient of the model) ?
2) *Auslander conjecture:* Assuming M as above, is Γ solvable?
Note that these questions are settled in the setting of (real) flat Lorentz manifolds [Car89, FG83], but unsolved for general (real) pseudo-Riemannian metrics. The real part of the holomorphic Riemannian metric is a (real) pseudo-Riemannian metric of signature $(3,3)$ for which both previous conjectures are still open.

Non flat case. In this case, M is locally modelled on G/I, with $G = SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ and $I = SL(2,\mathbb{C})$ diagonally embedded in G. The completeness of this geometry on compact complex manifolds is still an open problem (i.e. whether our manifold M is isomorphic to a quotient of the model G/I, or not), despite a positive local result of Ghys [Ghy95].

Nevertheless recent results of Tholozan [Tho13] together with a theorem of Kassel [Kas08] show that the space of complete structures is a union of connected components (in the space of group homomorphisms from the fundamental group of M into $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$). The question now is to find out if some connected component formed by (exotic) non complete structures does exist.

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