SINGULARITIES AND THEIR DEFORMATIONS: HOW THEY CHANGE THE SHAPE AND VIEW OF OBJECTS

ALEXANDRU DIMCA

Abstract. We show how the presence of singularities affect the geometry of complex projective hypersurfaces and of their complements. We illustrate the general principles and the main results by a lot of explicit examples involving curves and surfaces.

1. The setting and the problem

Let $\mathbb{P}^{n+1}$ be the complex projective $(n+1)$-dimensional projective space. It can be regarded as the set of complex lines passing through the origin of $\mathbb{C}^{n+2}$ or, alternatively, as the simplest compactification of the affine space $\mathbb{C}^{n+1}$. The homogeneous coordinates of a point $x \in \mathbb{P}^{n+1}$ are denoted by

$$x = (x_0 : x_1 : ... : x_{n+1}).$$

Let $\mathbb{C}[X_0, X_1, ..., X_{n+1}]$ be the corresponding ring of polynomials in $X_0, X_1, ..., X_{n+1}$ with complex coefficients. For a homogeneous polynomial $f \in \mathbb{C}[X_0, X_1, ..., X_{n+1}]$ we define the corresponding projective hypersurface by

$$V(f) = \{x \in \mathbb{P}^{n+1}; f(x) = 0\}$$

i.e. $V(f)$ is the zero set of the polynomial f in the complex projective $(n+1)$-dimensional projective space. We consider $\mathbb{P}^{n+1}$ endowed with the strong complex topology (coming from the metric topology on $\mathbb{C}^{n+1}$) and all subsets in $\mathbb{P}^{n+1}$ are topological spaces with the induced topology. Note that this topology is quite different from the Zariski topology used in Algebraic Geometry over an arbitrary algebraically closed field.

A point $x \in V(f)$ is a singular point if the tangent space of $V(f)$ at x is not defined. Formally the set of such singular points of $V(f)$ is called the *singular locus* of $V(f)$ and is given by

$$Sing(V(f)) = \{x \in \mathbb{P}^{n+1}; f_0(x) = ... = f_{n+1}(x) = 0\}$$

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where f_j denotes the partial derivative of f with respect to X_j. We assume in the sequel that the hypersurface $V(f)$ is reduced (i.e. we have chosen a simple equation for $V(f)$, without multiple factors) and then $dimSing(V(f)) < dimV(f) = n$.

In this survey we will investigate an algebraic view of the *shape* of the hypersurface $V(f)$, expressed by various invariants from Algebraic Topology such as the homology groups, cohomology groups, fundamental groups. For the definition of these invariants we refer to [13], [28]. This will give a precise idea about the intrinsic geometry of the hypersurface and helps a lot in understanding the possible deformation of that object.

To understand the topology of a space A it is usual to give its homology group with integer coefficients $H_j(A, \mathbb{Z})$ or at least the corresponding Betti numbers

$$b_j(A) = rankH_j(A, \mathbb{Z})$$

defined when the rank of this $\mathbb{Z}$-module is finite. To give the Betti numbers of a space A is the same as giving its rational homology groups $H_j(A, \mathbb{Q})$. Indeed, one has

$$b_j(A) = dim_{\mathbb{Q}} H_j(A, \mathbb{Q}).$$

A weaker invariant is the Euler characteristic of the space A given by

$$\chi(A) = \sum_j (-1)^j b_j(A)$$

when these Betti numbers exist and are all trivial except finitely many. For algebraic varieties these numerical invariants are always defined since a quasi-projective n-dimensional complex algebraic variety has the homotopy type of a finite CW-complex of (real) dimension $2n$.

In order to understand the position of $V(f)$ inside the complex $(n+1)$-dimensional projective space, in other words *its view from outside*, we have to study the topology of the complement

$$M(f) = \mathbb{P}^{n+1} \setminus V(f).$$

This will tell us how much freedom we have to move around the hypersurface $V(f)$. This idea was very fruitful in Knot Theory. Here one studies various embeddings of the circle S^1 into the sphere S^3. The image of such an embedding is a knot K and the fundamental group of the complement $\pi_1(S^3 \setminus K)$ is called *the group of the knot* K.

For any knot K one has

$$H_1(S^3 \setminus K, \mathbb{Z}) = \mathbb{Z}$$

and $S^3 \setminus K$ is a $K(\pi, 1)$-space, i. e. all the topological information about it is contained in its fundamental group. Refer to [28] for a formal definition.

Note that the homology says nothing about the view of our knot K. A key result due to Papakyriakopoulos says that

$$\pi_1(S^3 \setminus K, \mathbb{Z}) = \mathbb{Z}$$

if and only if the knot K is trivial, i.e. isotopic to a linear embedding of the circle. For all these results concerning Knot Theory we refer to [25].

This trip into the realm of knot theory is related to the above discussion through the following construction. Let $n = 1$ and O be any point on the curve $V(f)$ such that $V(f)$ has just one branch at O. A small closed ball B in $\mathbb{P}^2$ centered at O has a boundary ∂B homeomorphic to the sphere S^3. Moreover we have that the intersection $V(f) \cap \partial B$ is homeomorphic to the circle S^1. The corresponding knot is trivial if and only if O is not a singular point on the curve $V(f)$.

The main message of our paper is that the larger the dimension of the singular locus of $V(f)$, the more difficult it is to give accurate answers to the above problems concerning the shape and the view of the hypersurface $V(f)$.

We warn the reader that the setting discussed here is the simplest possible one. We will show by examples that the answers to the above questions become much more complicated in either of the following three apparently simpler settings.

(RS) The *real setting* consists of replacing all the objects above by the corresponding real objects. This study is clearly more interesting for applications than the complex setting (CS) considered above. However, usually, a real problem is first solved in the complex setting and then we try to get as much real information out of the complex solution. For more on this see [2], [4], [20] [24], [27].

(AS) The *affine setting* consists of working in an affine (or numerical) space $\mathbb{C}^{n+1}$. The objects are easier to define but the behavior at infinity causes many technical problems. For more on this see [7], Chapter 6, section 3, [12], [11], [8].

(RB) The *real bounded setting* consists of studying bounded pieces of real algebraic varieties, e.g. the intersections of real affine algebraic varieties with balls or cubes.

2. The smooth case

In this section we consider only smooth hypersurfaces $V(f)$, i.e. hypersurfaces with an empty singular locus

$$Sing(V(f)) = \emptyset.$$

The first result says that in this case the coefficients of the polynomial f play no role in determining the shape and the view of the smooth hypersurface $V(f)$, see [6], p.15. In terms of deformations, we can say that a small deformation of a smooth hypersurface is smooth and its shape and view are unchanged.

Theorem 2.1. *Let f and g be two homogeneous polynomials in $\mathbb{C}[X_0, X_1, ..., X_{n+1}]$ of the same degree d such that the corresponding hypersurfaces $V(f)$ and $V(g)$ are smooth. Then the following hold.*
(i) The hypersurfaces $V(f)$ and $V(g)$ are diffeomorphic. In particular they have exactly the same invariants coming from Algebraic Topology.
(ii) The complements $M(f)$ and $M(g)$ are diffeomorphic.

Example 2.2. (i) Consider first the case of complex projective plane curves, i.e. $n = 1$. Such a curve C is the same as an oriented Riemann surface, so topologically it is obtained from the 2-dimensional sphere by adding a number of handles. This number is called the genus $g(C)$ of the curve C. In the case of a plane curve $C = V(f)$ one can easily show using the above theorem and taking $g = X_0^d + X_1^d + X_2^d$ (a Fermat type equation) that there is the following celebrated *genus-degree formula*

$$g(V(f)) = \frac{(d-1)(d-2)}{2}.$$

Hence for $d = 1$ and $d = 2$ we get the sphere $S^2 = \mathbb{P}^1$, for $d = 3$ we get an elliptic curve which is diffeomorphic to a torus $S^1 \times S^1$. One can also show that

$$H_0(V(f)) = H_2(V(f)) = \mathbb{Z} \text{ and } H_1(V(f)) = \mathbb{Z}^{2g}.$$

(ii) Consider now the case of real projective plane curves. The example of $f = X_0^2 + X_1^2 + X_2^2$ and $g = X_0^2 - X_1^2 + X_2^2$ shows that the above theorem is false in the real setting. A smooth real curve $V(f)$ is a collection of circles, but their exact number and relative position depends heavily on the coefficients of f and this is an area of active research, see [4].
(iii) Consider now the affine setting, i.e. complex curves in $\mathbb{C}^2$. The example of $f = X^3 + Y^3 - 1$ and $g = X + X^2Y - 1$ shows that the above theorem is false in this setting. Indeed, topologically $V(f)$ is a torus with 3 deleted points, while $V(g)$ is a punctured plane. Hence

$$b_1(V(f)) = 4 \neq 1 = b_1(V(g)).$$

There is a similar description of the homology of a smooth hypersurface $V(f)$ in general, see for instance [6], p.152.

Proposition 2.3. *Let V be an n-dimensional smooth hypersurface of degree n. Then the integral homology of V is torsion free and the corresponding Betti numbers are as follows.*
(i) $b_j(V) = 0$ for $j \neq n$ odd or $j \notin [0, 2n]$;
(ii) $b_j(V) = 1$ for $j \neq n$ even and $j \in [0, 2n]$;
(ii) $\chi(V) = \frac{(1-d)^{n+2}-1}{d} + n + 2$.

Example 2.4. Let S_3 be a smooth cubic surface in $\mathbb{P}^3$. Then the corresponding sequence of integral homology groups $H_j(S_3, \mathbb{Z})$ for $0 \leq j \leq 4$ is the following: $\mathbb{Z}, 0, \mathbb{Z}^7, 0, \mathbb{Z}$.

Now we turn to the study of the complement $M(f)$ in this case. One way to study it is to consider the *Milnor fiber* $F(f)$ associated to the polynomial f. This is the following *affine* hypersurface

$$F(f) = \{x \in \mathbb{C}^{n+2}; f(x) = 1\}.$$

If d is the degree of f as above, then there is a monodromy automorphism of $F(f)$ given by

$$h : F(f) \to F(f), x = (x_0, ..., x_{n+1}) \mapsto (\lambda x_0, ..., \lambda x_{n+1})$$

with $\lambda = exp(2\pi i/d)$. Let G be the cyclic group of order d spanned by h. Then the quotient $F(f)/G$ can be identified to the complement $M(f)$. This gives the second part of the following.

Theorem 2.5. *Assume that $n > 0$. Then the following hold.*
(i) The Milnor fiber $F(f)$ is homotopy equivalent to a bouquet of $(n+1)$-dimensional spheres. In particular, $F(f)$ is simply-connected and the reduced integral homology groups of $F(f)$ vanish in degrees up-to n.
(ii) The complement $M(f)$ has $\pi_1(M(f)) = \mathbb{Z}/d\mathbb{Z}$ and the reduced rational homology groups of $F(f)$ vanish in degrees up-to n.

Using the homotopy exact sequence of the fibration $G \to F(f) \to M(f)$, refer to [28] for a definition, one gets information on the higher homotopy groups, namely $\pi_j(M(f)) = \pi_j(F(f)) = 0$ for $1 < j < n+1$ and $\pi_j(M(f)) = \pi_j(F(f)) = \mathbb{Z}^\mu$ for $j = n+1$ where

$$\mu = (d-1)^{n+2}$$

is the *Milnor number* of f. In particular, even in the simplest case, the complement $M(f)$ is not a $K(\pi, 1)$-space.

In conclusion, in the case of smooth hypersurfaces, the spaces $V(f)$ and $M(f)$ are not very complicated. They depend only on the dimension n and the degree d, and their topological invariants can be computed to a large extent.

3. The isolated singularities case

In this section we consider hypersurfaces $V(f)$ having at most isolated singularities, i.e. hypersurfaces with a finite singular locus

$$Sing(V(f)) = \{a_1, ..., a_m\}.$$

In order to study the topology of such an object we have to study first the local situation, i.e. the topology of an isolated hypersurface singularity $(V, 0)$ defined at the origin of $\mathbb{C}^{n+1}$ by a reduced analytic function germ

$$g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0).$$

The topological study was essentially done by Milnor in [21] where the following facts are obtained.

Theorem 3.1. *Let B_ϵ be a closed ball of radius $\epsilon > 0$, centered at the origin of $\mathbb{C}^{n+1}$ with boundary the sphere S_ϵ.*
(i) For all $\epsilon > 0$ small enough, the intersection $V \cap B_\epsilon$ is the cone over the link $K = V \cap S_\epsilon$ of the singularity $(V, 0)$. This link is an $(n-2)$-connected submanifold of the sphere S_ϵ and $dim K = 2n - 1$.
(ii) For all $\epsilon >> \delta > 0$ small enough, the Milnor fiber of the singularity $(V, 0)$, defined as $F = B_\epsilon \cap g^{-1}(\delta)$ is a smooth manifold, homotopy equivalent to a bouquet of n-spheres. The number of spheres in this bouquet is the Milnor number $\mu(V, 0)$ of the singularity $(V, 0)$ and is given by

$$\mu(V, 0) = dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J(g)}$$

where $\mathcal{O}_{n+1}$ is the ring of germs of analytic function germs at the origin of $\mathbb{C}^{n+1}$ and $J(g)$ is the Jacobian ideal of g, i.e. the ideal spanned by all the partial derivatives of g. Alternatively, the Milnor number $\mu(V, 0)$ is given by the degree of the gradient mapping germ

$$grad(g) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0).$$

The Milnor fiber should be regarded as a smooth deformation of the singular fiber $g^{-1}(0)$ of g over the origin.

By the above theorem, it follows that the only interesting homology group of the Milnor fiber F is the group

$$L(V, 0) = H_n(F, \mathbb{Z}) = \mathbb{Z}^{\mu(V,0)}.$$

This group is endowed with a $(-1)^n$-symmetric bilinear form $<,>$ coming from the intersection of cycles. Regarded with this additional structure, the free abelian group $L(V, 0)$ is called the *Milnor lattice* of the singularity $(V, 0)$. It is known that this intersection form $<,>$ is non-degenerate exactly when the link K is a $\mathbb{Q}$-homology sphere, i.e. $H_*(K, \mathbb{Q}) = H_*(S^{2n-1}, \mathbb{Q})$, see for details [6], p. 93.

Similarly, the Milnor lattice is unimodular (i.e. the corresponding bilinear form has as determinant $+1$ or -1) if and only if the link has the same integral homology as the sphere S^{2n-1}.

The Milnor lattice tells a lot about the possible deformations of an isolated hypersurface singularity. Indeed, the singularity $(V, 0)$ can be deformed into the singularity $(W, 0)$ only if there is an embedding of lattices

$$L(W, 0) \to L(V, 0).$$

The Milnor number $\mu(V, 0)$ is also called *the number of vanishing cycles* at the singularity $(V, 0)$. Ample justification for this name is given below.

One may ask which singularities among the isolated ones are the simplest. The answer depends on our interests, but in a lot of questions the class of *simple singularities* introduced by Arnold, see for details [1], are very useful. These singularities are by definition the singularities which can be deformed only into finitely many isomorphism classes of singularities. Their classification, up to isomorphism, is given in dimension $n = 2$ by the following list of local equation at the origin, see [5] where the possible deformations are discussed in detail.

$$A_k : x^{k+1} + y^2 + z^2,\ for\ k > 0;$$

$$D_k : x^2y + y^{k-1} + z^2,\ for\ k > 3;$$

$$E_6 : x^3 + y^4 + z^2,\ E_7 : x^3 + xy^3 + z^2\ and\ E_8 : x^3 + y^5 + z^2.$$

To get the corresponding equation for the curve singularities, i.e. $n = 1$, we have just to discard the last term z^2 from the above equations. Note that for curves A_1 is just a *node*, while A_2 is just a *cusp*. These names are used for higher dimensional singularities A_k as well.

In the above list of simple surface singularities, all the associated Milnor lattices are non-degenerate and only the lattice $L(E_8)$ is unimodular.

Remark 3.2. Using the real parts of some of the above expressions defining the simple singularities, one can obtain *simple real equations for hypersurfaces in the real projective space* $\mathbb{R}P^3$ which represent up to diffeomorphism all the surfaces, i.e. all the compact, connected 2-manifolds. For example, any compact, connected 2-manifold M which is orientable can be constructed up to diffeomorphism from the sphere S^2 by attaching g handles, where $g \geq 0$ is the genus of M exactly as in Example 2.2. This integer g is completely determined by the equality

$$\chi(M) = 2 - 2g.$$

Let X, Y, Z and W be the homogeneous coordinates on the real projective space $\mathbb{R}P^3$. Then the equation

$$Re(X + iY)^{2g} + (X^2 + Y^2 + Z^2 + W^2)^{g-1} Re(Z + iW)^2 = 0,$$

which is essentially the real part of the simple singularity A_{2g-1}, defines a compact, connected 2-manifold of genus g. For more details and a similar formula for non-orientable surfaces based on the real part of the simple singularity D_k, see [10].

It is quite natural to look for equations of the non-orientable surfaces in the real projective space $\mathbb{R}P^3$ since they are not embeddable in the usual affine space $\mathbb{R}^3$. See [14], p. 181 for a topological immersion of the projective plane $\mathbb{R}P^2$ in $\mathbb{R}^3$ having as singularity a cross cap (also called a Whitney umbrella) described as the image of the mapping

$$\phi : \mathbb{R}^2 \to \mathbb{R}^3, (x, y) \mapsto (x^2, y, xy).$$

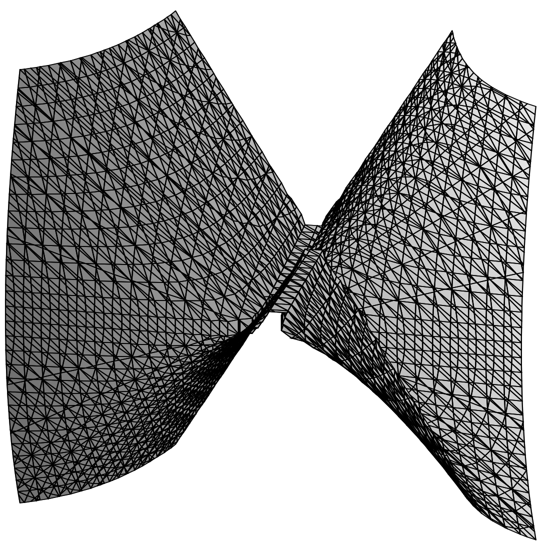

FIGURE 1. A cross cap

The next result compares the topology of the hypersurface $V(f)$ having at most isolated singularities to the known topology of a smooth hypersurface $V(f)_{smooth}$ having the same dimension n and degree d as $V(f)$. For a proof we refer to [6], p. 162.

Theorem 3.3. *(i)* $H_j(V(f), \mathbb{Z}) = H_j(V(f)_{smooth}, \mathbb{Z})$ *for all* $j \notin \{n, n+1\}$. *In addition,* $H_{n+1}(V(f), \mathbb{Z})$ *is torsion free.*
(ii) $\chi(V(f)) = \chi(V(f)_{smooth}) + (-1)^{n-1} \sum_{k=1,m} \mu(V, a_k)$.

Example 3.4. For a plane curve C, the above result coupled with the following easy facts gives a complete description of the integral homology.
(a) $b_2(C)$ is equal to the number of irreducible components of C;
(b) the first homology group is torsion free.

As an explicit example, consider a 3 cuspidal quartic curve C_4.

Any such curve is *projectively equivalent* to the curve defined by the equation

$$X^2Y^2 + Y^2Z^2 + Z^2X^2 - 2XYZ(X + Y + Z) = 0.$$

The corresponding smooth curve has genus $g = 3$ and hence $b_1 = 6$. The singular curve C_4 is irreducible (since the only singularities are cusps, hence locally irreducible!) and has 3 cusps located at the points $(1:0:0), (0:1:0), (0:0:1)$. Hence one would expect a loss of $6 = 3 \times \mu(A_2)$ cycles due to the presence of singularities. Using Theorem 3.3 and the above remarks, it follows that indeed $b_1(C_4) = 0$. This result is confirmed by the known fact that the normalization of C_4 is the projective line $\mathbb{P}^1$ and the normalization morphism is a homeomorphism in this situation.

For any curve C, its homology is determined by its degree, the list of singularities on C and the number of irreducible components of C.

Beyond the curve case, new phenoma may occur. First of all torsion can appear in the homology, see for details [6], p. 161.

Theorem 3.5. *(i) If all the Milnor lattices $L(V, a_k)$ for $k = 1, ..., m$ are unimodular, then $H_j(V(f), \mathbb{Z}) = H_j(V(f)_{smooth}, \mathbb{Z})$ for all $j \neq n$ and $H_n(V(f), \mathbb{Z})$ is torsion free of rank $b_n(V(f)_{smooth}) - \sum_{k=1,m} \mu(V, a_k)$. In this situation, $V(f)$ is an integral homology manifold and in particular the Poincaré Duality holds over $\mathbb{Z}$.*
(ii) If all the Milnor lattices $L(V, a_k)$ for $k = 1, ..., m$ are nondegenerated, then $H_j(V(f), \mathbb{Z}) = H_j(V(f)_{smooth}, \mathbb{Z})$ for all $j \neq n$, $b_n(V(f)) = b_n(V(f)_{smooth}) - \sum_{k=1,m} \mu(V, a_k)$ and the torsion part of $H_n(V(f), \mathbb{Z})$ is determined by a lattice morphism defined on the direct sum of the lattices $L(V, a_k)$.

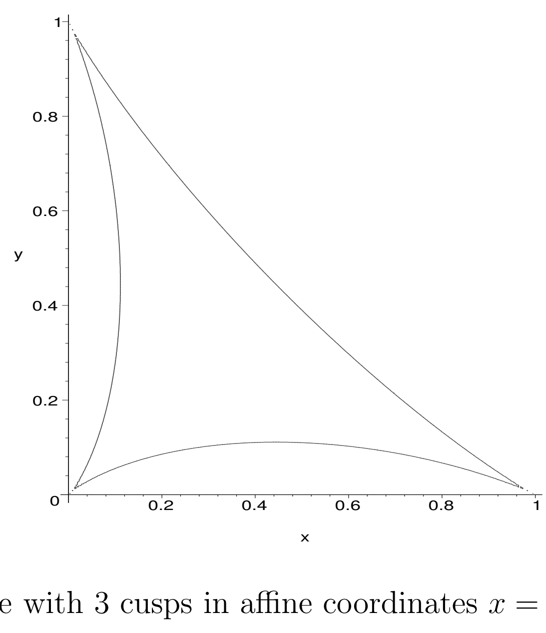

FIGURE 2. A curve with 3 cusps in affine coordinates $x = \frac{X}{X+Y+Z}, y = \frac{Y}{X+Y+Z}$

In the second case, the hypersurface $V(f)$ is a rational homology manifold and in particular the Poincaré Duality holds over $\mathbb{Q}$. For precise information on the determinant of the cup-product in this case see [6], p. 171. For general facts on Poincar Duality and cup-product, see [13].

Example 3.6. The list of cubic surfaces with isolated singularities can be found in [3]. We list some of the cases below.

(a) A cubic surface S can have s nodes A_1, for $s = 1, 2, 3, 4$. The only case which produces torsion is $s = 4$ and then the torsion part of the second homology group of S is given by $TorsH_2(S) = \mathbb{Z}/2\mathbb{Z}$.

(b) A cubic surface S can have s cusps A_2, for $s = 1, 2, 3$. The only case which produces torsion is $s = 3$ and then $TorsH_2(S) = \mathbb{Z}/3\mathbb{Z}$.

For a complete discussion and proofs we refer to [6], p. 165.

Note also that the determinant of the cup-product can be used to distinguish hypersurfaces having the same integral homology. For instance, the three cubic surfaces with singularity type $3A_1$, A_1A_2 and A_3 have all the same integral homology, but they are not homotopy equivalent since the cup-products are different, see [6], p. 171.

A second major phenomenon is the dependence of the Betti numbers of the hypersurface $V(f)$ on the position of singularities.

Example 3.7. The classical example here, going back to Zariski in the early '30's, is that of sextic surfaces

$$S_6 : f(X, Y, Z) + T^6 = 0$$

where $f(X, Y, Z) = 0$ is a plane sextic curve C_6 having 6 cusps. Two situations are possible here.

(a) The six cusps of the sextic curve C_6 are all situated on a conic. This is the case for instance for

$$f(X, Y, Z) = (X^2 + Y^2)^3 + (Y^3 + Z^3)^2.$$

Then it can be shown that $b_2(S_6) = 2$, see for instance [6], p.210.

(b) The six cusps of the sextic curve C_6 are not situated on a conic. Then it can be shown that $b_2(S_6) = 0$, see loc.cit.

The explanation of this difference is that the two types of sextic curves cannot be deformed one into the other even though they are homeomorphic.

A good way to understand this strange behaviour of the Betti numbers is to use *algebraic differential forms defined on $M(f)$ and with poles along $V(f)$* to describe the topology of $V(f)$ and of the complement $M(f)$. See [6], Chapter 6, for details on this approach and a lot more examples.

This remark has brought into discussion the complement $M(f)$. For $n = 1$, the main topological invariant is the fundamental group $\pi_1(M(f))$. Usually this group is highly non-commutative.

Example 3.8. (a) For the 3 cuspidal quartic C_4 considered in Example 3.4, the fundamental group $\pi_1(M(f))$ is the metacyclic group of order 12 which can be described by generators and relations as the group

$$G = \{u, v; u^2 = v^3 = (uv)^2\}.$$

(b) For the two types of 6 cuspidal sextic curves discussed in Example 3.7, one has

$$\pi_1(M(f)) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z},$$

a free product, for C_6 of the first type, and

$$\pi_1(M(f)) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}/6\mathbb{Z},$$

a direct product, for C_6 of the second type, see [6], p.134.

This example shows that the fundamental group $\pi_1(M(f))$ depends on the position of singularities even for plane curves.

In higher dimension, i.e. for $n > 1$, the complement $M(f)$ has a commutative fundamental group, which is cyclic of order d, and the object of study is the homology $H_*(M(f)^c, \mathbb{Q})$ of the infinite cyclic covering $M(f)^c$ of the space $M(f) \setminus H$, where H is a generic hyperplane. Then the groups $H_*(M(f)^c, \mathbb{Q})$ can be regarded in a natural way as Λ-modules of finite type, where $\Lambda = \mathbb{Q}[T, T^{-1}]$ and, as such, they are called the Alexander invariants of the hypersurface $V(f)$. See [12], [15], [16], [19] for more on this beautiful subject.

In conclusion, when the number or the type of singularities on the hypersurface $V(f)$ is small compared to the degree d, then the list of singularities is enough to determine the topology of $V(f)$, even the embedded topology, see [6], pp.17-19. In such a case there is usually no torsion in homology.

On the other hand, when the number or the type of singularities on the hypersurface $V(f)$ is large compared to the degree d, then torsion is likely to occur in homology and the position of singularities may influence the Betti numbers of the hypersurface $V(f)$.

4. The general case

In this section we consider hypersurfaces $V(f)$ having an s-dimensional singular locus, for $0 \leq s \leq n-1$. Note that $s < n-1$ implies that the hypersurfaces $V(f)$ is irreducible. Little is known in general about the homology of such a hypersurface; for the following result see [6], p. 144.

Theorem 4.1. *With the above notation, there are isomorphisms*

$$H_j(V(f)) = H_j(\mathbb{P}^n)$$

for all $j < n$ and $j > n + s + 1$. The complement $M(f)$ has a commutative fundamental group, which is cyclic of order d, if $s < n - 1$.

Remark 4.2. For an arbitrary hypersurface $V(f)$, Parusiński has defined a global Milor number $\mu(V(f))$ such that one has the following generalization of Theorem 3.3, (ii).

$$\chi(V(f)) = \chi(V(f)_{smooth}) - \mu(V(f)).$$

For more details and application see [23]. An alternative approach via vanishing cycles is described in [7], pp.179-183.

For special classes of hypersurfaces the information we have is complete. This is the case for instance when $V(f)$ is a *hyperplane arrangement*, i.e. $V(f)$ is a finite union of hyperplanes in $\mathbb{P}^{n+1}$. Then not only the homology is known, but also the cohomology algebra, see [22].

In general, one can adopt various approaches which we briefly describe below.

4.3. **Using algebraic differential forms.** There is a spectral sequence whose E_2-term consists of various homogeneous components of the homology of the Koszul complex of the partial derivatives of f and converging to the cohomology of $M(f)$, see [9]. Recall that the Koszul complex describes the linear relations involving the partial derivatives of f, then the relations among the relations, and so on. Hence it can be successfully handled by the computer algebra packages.

As an example, using this approach one can determine the Betti numbers of the cubic surface

$$S_3' : X^2Z + Y^3 + XYT = 0.$$

The singular locus here is 1-dimensional (i.e. the line $X = Y = 0$) and it turns out that the surface S_3' has the same rational cohomology as the projective plane $\mathbb{P}^2$, see [9] for very explicit computations.

4.4. **Building the hypersurface inductively out of successive hyperplane sections.** We have seen in Theorem 3.1 that a key role is played by the fact that the Milnor fiber F in that case has a very simple topology, i.e. it is homotopically equivalent to a bouquet of spheres. In the case of projective hypersurfaces we have a similar result, see [11].

Theorem 4.5. *For any hypersurface $V(f)$ and any transversal hyperplane H, the complement $V_a(f) = V(f) \setminus H$, which can be regarded as the affine part of $V(f)$ with respect to the hyperplane at infinity H, is homotopically equivalent to a bouquet of n-spheres. The number of spheres in this bouquet is given by the global Milnor number of f, defined as the degree of the gradient mapping*

$$grad(f) : M(f) \to \mathbb{P}^{n+1}.$$

The remaining difficult problem is to glue the information we have on $V_a(f)$ and on the hyperplane section $V(f) \cap H$ in order to get information on the hypersurface $V(f)$.

Another possibility is to look for the Alexander invariants in this setting, and this was recently done by Maxim, see [19].

4.6. **Cyclic coverings of a projective space.** Let $p : X \to \mathbb{P}^{n+1}$ be a cyclic covering ramified along the hypersurface $V(f)$.

For simplicity, we assume below that the degree of p coincides with the degree d of the hypersurface $V(f)$. The general case can be treated similarly, using weighted projective spaces instead of the usual projective spaces, see [6], Appendix B.

Under our assumption, it follows that X can be identified to the hypersurface $V(\widetilde{f})$ given in the projective space $\mathbb{P}^{n+2}$ by the equation

$$\widetilde{f}(X_0, ..., X_{n+1}, T) = f(X_0, ..., X_{n+1}) + T^d = 0.$$

We have already seen this construction in Example 3.7.

If $M(\widetilde{f})$ denotes the corresponding complement, then we have the following isomorphism for the cohomology with rational coefficients coming from Alexander Duality

$$H_0^{n+1+j}(V(\widetilde{f})) = H^{n+2-j}(M(\widetilde{f}))$$

where H_0^* denotes the primitive cohomology as defined in [6], p. 146. Let now F (resp. $\widetilde{F}$) be the Milnor fibers associated to the homogeneous polynomials f (resp. $\widetilde{f}$) as in Theorem 2.5. It follows from the Thom-Sebastiani Theorem, see [6], p. 196, that one has the following isomorphisms.

$$H^{n+2-j}(M(\widetilde{f})) = H^{n+2-j}(\widetilde{F})_1 = H^{n+1-j}(F)_{\neq 1}.$$

Here $H^{n+2-j}(\widetilde{F})_1$ is the eigenspace of the monodromy corresponding to the eigenvalue 1, and $H^{n+1-j}(F)_{\neq 1}$ has a similar meaning.

Now several results in [7], Section 6.4 obtained via the theory of *perverse sheaves* give sufficient conditions for the vanishing of the groups $H^{n+1-j}(F)_{\neq 1}$ for all $j > 0$, see Example 6.4.14, Corollary 6.4.15, Theorem 6.4.18. As a sample result, we give the following.

Proposition 4.7. *Assume that the hypersurface $V(f)$ is a normal crossing divisor along one of its irreducible components. Then the associated cyclic covering $X = V(\widetilde{f})$ satisfies $H_0^{n+1+j}(V(\widetilde{f})) = 0$ for all $j > 0$. In other words, the Betti numbers of the associated cyclic covering $X = V(\widetilde{f})$ are known once the Euler characteristic $\chi(X)$ is known.*

In conclusion, in the general case we encounter two difficult problems, whose solution is far from being complete even from the theoretical view-point. The first one is to classify the simplest non-isolated singularities and to understand their local topology. For recent progress in this area we refer to [26]. The second one is to glue the local information in order to obtain information on the global topology. Here the most powerful approach is to use the theory of constructible sheaves, [7].

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Laboratoire J.A. Dieudonné, UMR du CNRS 6621, Université de Nice-Sophia-Antipolis, Parc Valrose, 06108 Nice Cedex 02, FRANCE.

E-mail address: `dimca@math.unice.fr`