THE COMPARISON THEOREM

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1. Introduction

Remark 1.1. This note follows Milne's book and notes on étale cohomology and Artin's Exposé XI in SGA 4-III.

The aim is to prove the following

Theorem 1.2 (Artin)**.** *Let X be a smooth $\mathbb{C}$-scheme (think nonsingular variety if you wish) and Λ a finite group, then there is a natural isomorphism*

$$H^i(X_{\acute{e}t}, \Lambda) \longrightarrow H^i(X(\mathbb{C}), \Lambda).$$

We will now interpret the various symbols appearing the statement of Theorem 1.2.

2. Some remarks on analytification

Note that if $U \subset \mathbb{C}^n$ is a domain, then we can form the locally ringed space $(U, \mathcal{H}_U)$, where $\mathcal{H}_U$ denotes the sheaf of analytic functions on U.

A **basic analytic variety** is a subset of a domain $U \subset \mathbb{C}^n$ defined by the zero locus of a family of analytic functions defined on U. An analytic variety can be given the structure of a locally ringed space by restricting analytic functions from U.

An **analytic space** $(X, \mathcal{H}_X)$ is a locally ringed topological space satisying the following condition: X is covered by a family of open sets $\{U_i\}$ such that $(U_i, \mathcal{H}_X|_{U_i})$ is isomorphic (as a locally ringed space) to a basic analytic variety.

Let X be a $\mathbb{C}$-scheme which is locally of finite type, then we can view it as an analytic space. Indeed, X may be covered by affine $\mathbb{C}$-schemes of the form $\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]/(g_1, \dots, g_r)$ and so we can form a basic analytic space from the zeroes of the g_i. Let this analytic space be X_{an}. There is now an obvious morphism of locally ringed spaces $\phi_X : (X_{\mathrm{an}}, \mathcal{H}_{X_{\mathrm{an}}}) \to (X, \mathcal{O}_X)$.

Note that X_{an} satisfies the following universal property: if Y is an analytic space, then for any morphism of locally ringed spaces $\sigma : Y \to X$, σ factors uniquely through $Y \to X_{\mathrm{an}} \to X$. To do this, it suffices to assume that X is an affine $\mathbb{C}$-variety and $Y = \sigma^{-1}X$. In this case, the result is obvious. Hence, in the category of analytic spaces, we have shown that X_{an} is unique up to unique isomorphism. Note that it is immediate that we have a bijection between $|X_{\mathrm{an}}|$ and the $\mathbb{C}$-points of X, $X(\mathbb{C})$ and so there is a well-defined topology on $X(\mathbb{C})$.

3. Why is Theorem 1.2 amazing?

It contains some well-known facts.

- *Cool*: $\Gamma(X, \Lambda) = \mathrm{H}^0(X(\mathbb{C}), \Lambda)$ tells us that the Zariski connected components of X are the same as it's $\mathbb{C}$-connected components. This is implied by the Theorem.
- *Even cooler*: $\mathrm{H}^1(X, \mathbb{Z}/n) = \mathrm{H}^1(X(\mathbb{C}), \mathbb{Z}/n)$ tells us that the finite étale covers of X of degree n are the same as the covering spaces of degree n in the $\mathbb{C}$-topology! This is the Grauert-Remmert Theorem. This doesn't rely on the resolution of singularities, but the shortish proof (given in SGA 1) does. It is tough anyways. We will take this as fact, since it used in the proof of Theorem 1.2.
- *Coolest*: Consider the smooth, projective, elliptic curve $E : y^2 = x^3 + 5x + 1$ which we think of as being defined over $\overline{\mathbb{Z}_7}$. The generic fiber of this curve consists of the $\overline{\mathbb{Q}}_7$ points. The closed fiber of this curve consists of the $\overline{\mathbb{F}}_7$ points of E reduced mod 7. The proper smooth base change theorem (in algebraic geometry) tells us that

  $$\mathrm{H}^i((E_s)_{\text{ét}}, \mathbb{Z}/9) = \mathrm{H}^i((E_g)_{\text{ét}}, \mathbb{Z}/9) = \mathrm{H}^i((E_{\mathbb{C}})_{\text{ét}}, \mathbb{Z}/9) = \mathrm{H}^i(E(\mathbb{C}), \mathbb{Z}/9)!$$

  This result generalises.

4. A general method to compare topologies

The essence of Theorem 1.2 is a comparison of topologies.

Let X_{cx} be the "étale site on X_{an}". That is, it is the category whose objects are analytic spaces $U \to X_{\text{an}}$ which are local isomorphisms and the morphisms are triangles as before. Note that we have continuous maps of sites $s : X_{\text{cx}} \to X_{\text{an}}$ and $h : X_{\text{cx}} \to X_{\text{ét}}$.

Now let us look what happens in the category of sheaves. This gives rise to $s_* : \mathbf{Sh}(X_{\text{cx}}) \to \mathbf{Sh}(X_{\text{an}})$ and $h_* : \mathbf{Sh}(X_{\text{cx}}) \to \mathbf{Sh}\,(X_{\text{ét}})$.

Lemma 4.1. *s_* is an exact functor.*

Proof. It suffices to show that $(s_*\mathscr{F})_x = \mathscr{F}_x$ for all $x \in X(\mathbb{C})$. Now,

$$(s_*\mathscr{F})_x = \varinjlim_{X(\mathbb{C}) \supset U \ni x} (s_*\mathscr{F})(U) = \varinjlim_{X(\mathbb{C}) \supset U \ni x} \mathscr{F}(U \subset X(\mathbb{C})).$$

Also,

$$\mathscr{F}_x = \varinjlim_{p:V \to X \,:\, x \in p(V)} \mathscr{F}(V \to X).$$

Hence, it remains to show that $X(\mathbb{C}) \supset U \ni x$ is cofinal for the above family. The inverse function theorem says that for $p : V \to X$ a local isomorphism at x, then by shrinking V to W, $p\,|_W : W \to p(W)$ is an isomorphism and so $\mathscr{F}(W \to p(W)) = \mathscr{F}(p(W) \subset X(\mathbb{C}))$ and we win. □

In particular, we now observe that there is an isomorphism

$$\mathrm{H}^i(X_{\text{cx}}, \mathscr{F}) = \mathrm{H}^i(X_{\text{an}}, \mathscr{F}).$$

To study h_*, we require a fact, which is not too difficult to prove.

Lemma 4.2. *h_* takes flabby sheaves to flabby/flasque sheaves and is left-exact. (We can use flabby sheaves to compute cohomology, since theirs is trivial).*

The purpose of the Lemma was to enable us to apply a Grothendieck spectral sequence:

$$\mathrm{H}^p(X_{\text{ét}}, \mathrm{R}^q h_* \mathscr{F}) \Longrightarrow \mathrm{H}^{p+q}(X_{\text{cx}}, \mathscr{F}).$$

Hence, Theorem 1.2 will follow from

Proposition 4.3. *Let* Λ *be a finite group and* X *a smooth* $\mathbb{C}$*-scheme, then* $\mathrm{R}^q h_* \Lambda = 0$ *for all* $q > 0$*.*

5. Higher pushforwards and the $q = 1$ case of Proposition 4.3

In general, it is not too difficult to show for a continuous morphism of sites $u : Y \to X$, with the sheaves on Y having enough injectives, sheafification existing on X and that sheaves are determined by points, $\mathrm{R}^q u_* \mathscr{F}$ is the sheafification of the presheaf $U \mapsto \mathrm{H}^q(u^{-1}U, \mathscr{F}\,|_{u^{-1}U})$.

Proof. Let $\pi_u : \mathbf{PreSh}(Y) \to \mathbf{PreSh}(X)$ denote the natural transformation which takes a presheaf $\mathscr{F} \in \mathbf{PreSh}(Y)$ to the presheaf $U \mapsto \mathscr{F}(u^{-1}U)$ on X. Let $\mathscr{I}_\bullet(\mathscr{F})$ be an injective resolution of the sheaf $\mathscr{F} \in \mathbf{Sh}(Y)$. Observe that if $\mathscr{H}$ is a sheaf on Y, then we have a commutative diagram:

$$\begin{array}{ccc} \mathscr{H} & \xrightarrow{u_*} & u_*\mathscr{H} \\ {\scriptstyle\gamma}\downarrow & & \uparrow{\scriptstyle a} \\ \mathscr{H} & \xrightarrow[\pi_u]{} & \pi_u \mathscr{H} \end{array}$$

where a is sheafification and γ is the forgetful functor. Hence, we have

$$\mathrm{R}^p u_* \mathscr{F} = \mathrm{H}^p(u_* \mathscr{I}_\bullet(\mathscr{F})) = \mathrm{H}^p(a \pi_u \gamma \mathscr{I}_\bullet(\mathscr{F})).$$

But since a is exact, we conclude that it commutes with taking cohomology and so

$$\mathrm{R}^p u_* \mathscr{F} = a\mathrm{H}^p(\pi_u \gamma \mathscr{I}_\bullet(\mathscr{F})).$$

We deduce the result. □

In fact, with the Grauert-Remmert Theorem, we can prove the $q = 1$ case of Proposition 4.3. The exact sequence of terms of low degree gives us (since we know that $\mathrm{R}^2 h_* \Lambda = 0$):

$$0 \longrightarrow \mathrm{H}^0(X_{\text{ét}}, \mathrm{R}^1 h_* \Lambda) \longrightarrow \mathrm{H}^1(X_{\text{cx}}, \Lambda) \longrightarrow \mathrm{H}^1(X_{\text{ét}}, \Lambda) \longrightarrow 0\,.$$

By the Grauert-Remmert Theorem, we deduce that $\mathrm{H}^0(X_{\text{ét}}, \mathrm{R}^1 h_* \Lambda) = 0$. But this holds for *any* smooth $X_{\text{ét}}$. Now notice that if $\pi : U \to X$ is étale, then $\mathrm{H}^0(U_{\text{ét}}, \mathrm{R}^1 h_* \Lambda) = \mathrm{H}^0(U_{\text{ét}}, \pi^* \mathrm{R}^1 h_* \Lambda) = \mathrm{H}^0(U_{\text{ét}}, \mathrm{R}^1 h'_* \Lambda) = 0$ where $h' : U_{\text{cx}} \to U_{\text{ét}}$.

6. A remark about sheafification

Another general fact (which follows immediately that sheafification preserves stalks) is that a sheaf on a site is determined by its stalks (when this statement makes sense!). In other words, we have a diagram:

$$\begin{array}{ccc} \mathscr{F}(U) & \longrightarrow & \prod_{u \in U} \mathscr{F}_u \\ & \searrow \quad \nearrow & \\ & \mathscr{F}_{\text{sh}}(U) & \end{array}$$

So, a section $f \in \mathscr{F}(U)$ is zero in its sheafification if and only if there is a cover $(U_i \to U)_{i\in I}$ such that $f\,|_{U_i}= 0$ for all i.

7. Proof of Proposition 4.3 for $q > 1$

By the remarks in the previous two sections, we have reduced Proposition 4.3 to the following

Lemma 7.1. *Let $q > 1$ and $\gamma \in H^q(X_{\mathrm{cx}}, \mathscr{F})$, where $\mathscr{F}$ is a sheaf which is finite locally constant, then for any $x \in X(\mathbb{C})$, there is an étale morphism $U \to X$ whose image contains x and is such that $\gamma\,|_{U_{\mathrm{cx}}}= 0$.* [1]

To see that Lemma 7.1 implies Proposition 4.3, we need to show that $(\mathrm{R}^q h_*\mathscr{F})_{\overline{x}} = 0$ for every geometric point $\overline{x} \to X$. First, we show that it suffices to prove the above for closed points.

Indeed, take $\mathscr{H}$ to be a sheaf on $X_{\text{ét}}$, then suppose that $\mathscr{H}_{\overline{x}}$ is non-zero for some geometric point $\overline{x} \to X$. We claim that this implies that $\mathscr{H}_{\overline{z}} \neq 0$ for some closed point $z \in X$. Let $Z = \overline{\{x\}}$ then for any $z \in Z$ we note that since x is generic point for Z *any* étale neighbourhood of z has image containing x. From this, we may conclude immediately that $\mathscr{H}_{\overline{z}} \neq 0$.

Now, since $\mathrm{R}^q h_*\mathscr{F}$ is the sheafification of $U \mapsto \mathrm{H}^q(U_{\mathrm{cx}}, \mathscr{F}\,|_{U_{\mathrm{cx}}})$, then to show that $\mathrm{R}^q h_*\mathscr{F}$ is the zero sheaf for $q > 0$ it suffices to prove that for any $\gamma \in \mathrm{H}^q(U_{\mathrm{cx}}, \mathscr{F}\,|_{U_{\mathrm{cx}}})$ every $x \in X(\mathbb{C})$ has an étale neighbourhood $U' \to X$ such that $\gamma\,|_{U'_{\mathrm{cx}}}= 0$.

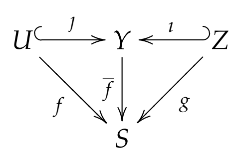
To prove the Lemma, we do induction on $\dim X$. The case for $\dim X = 0$ follows trivially from dimensional vanishing. Note that also, since the statement is local for the étale topology, we may assume that $\mathscr{F}$ is constant. That is, $\mathscr{F}(U) = \mathrm{Hom}_{\mathrm{cts}}(U, \Lambda)$.

Let U be a "small" Zariski open neighbourhood of $x \in X$, then let us embed it as a dense open in some projective scheme Y. Let $Z = Y - U$ be the complement of U in Y, which (for the moment!) we assume to be smooth.

We have a Gysin sequence:

$$\cdots \longrightarrow \mathrm{H}^{q-2}(Z_{\mathrm{an}}, \Lambda) \longrightarrow \mathrm{H}^q(Y_{\mathrm{an}}, \Lambda) \longrightarrow \mathrm{H}^q(U_{\mathrm{an}}, \Lambda) \longrightarrow \cdots.$$

Now suppose we can fiber Y over something smooth of dimension 1 less. In other words, assume we had a diagram at our disposal:



where all maps are over $\mathbb{C}$ and

(1) $Z = Y - U$;
(2) $\jmath$ is an open immersion;
(3) $\imath$ is a closed immersion;

[1] A finite locally constant sheaf is one which is finite constant on a covering.

(4) f is smooth of dimension n;
(5) S is smooth of dimension $n-1$;
(6) g is finite étale with non-empty fibers (this assures that Z is smooth!) (this implies it is proper too!);
(7) $\overline{f}$ is smooth projective with fibers smooth connected curves;
(8) U is dense in every fiber of $\overline{f}$.

If $S' \subset S_{\mathrm{an}}$ is any complex open set, then we will similarly obtain (from the Gysin sequence for these guys) an exact sequence:

$$\cdots \longrightarrow \mathrm{H}^{q-2}(g^{-1}(S'),\Lambda) \longrightarrow \mathrm{H}^{q}(\overline{f}^{-1}(S'),\Lambda) \longrightarrow \mathrm{H}^{q}(f^{-1}(S'),\Lambda) \longrightarrow \cdots .$$

Hence, from our remarks about higher pushforwards, we obtain an exact sequence of sheaves:

$$\cdots \longrightarrow \mathrm{R}^{q-2}g_*\Lambda \longrightarrow \mathrm{R}^{q}\overline{f}_*\Lambda \longrightarrow \mathrm{R}^{q}f_*\Lambda \longrightarrow \cdots$$

Let $s \in S(\mathbb{C})$, then we have a diagram with exact rows and natural maps:

$$\begin{array}{ccccccccc} (\mathrm{R}^{q-2}g_*\Lambda)_s & \longrightarrow & (\mathrm{R}^{q}\overline{f}_*\Lambda)_s & \longrightarrow & (\mathrm{R}^{q}f_*\Lambda)_s & \longrightarrow & (\mathrm{R}^{q-1}g_*\Lambda)_s & \longrightarrow & (\mathrm{R}^{q+1}\overline{f}_*\Lambda)_s \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^{q-2}((Z_{\mathrm{an}})_s,\Lambda) & \longrightarrow & \mathrm{H}^{q}((Y_{\mathrm{an}})_s,\Lambda) & \longrightarrow & \mathrm{H}^{q}((U_{\mathrm{an}})_s,\Lambda) & \longrightarrow & \mathrm{H}^{q-1}((Z_{\mathrm{an}})_s,\Lambda) & \longrightarrow & \mathrm{H}^{q}((Y_{\mathrm{an}})_s,\Lambda) \end{array}.$$

The bottom sequence comes from the Gysin sequence applied to $Z_s \subset Y_s$. Now from the topological proper base change, we know that every vertical map *except* the middle one is an isomorphism. By the 5-Lemma, we obtain that the middle one is an isomorphism too.

Now we can apply dimensional vanishing ($\dim_{\mathbb{C}} U_s = 1$) to conclude that

- $\mathrm{R}^q f_*\Lambda = 0$ for all $q > 1$ (this uses the fact that the fibers non-compact and so the top cohomology group vanishes on all the fibers).

Also, we note that $\mathrm{R}^q f_*\Lambda$ is finite locally constant. This follows from the general

Lemma 7.2. *Let $\phi : X \to S$ be a proper, submersive map of manifolds, then if Λ is a constant sheaf on X, $\mathrm{R}^q\phi_*\Lambda$ is finite locally constant on S for all $q \geq 0$.*

Proof. By the topological proper base change theorem, we know that for any $s \in S$, $(\mathrm{R}^q\phi_*\Lambda)_s = \mathrm{H}^q(\phi^{-1}(s),\Lambda)$. Moreover, since ϕ is smooth and submersive, it is a fibration and so we can find an open ball $W \ni s$ such that $\phi^{-1}W \cong W \times \phi^{-1}(s)$. We now observe that this implies that $(\mathrm{R}^q\phi_*\Lambda)\mid_W$ is the sheaf $\mathrm{H}^q(\phi^{-1}(s),\Lambda)$. ☐

Corollary 7.3. $\mathrm{R}^q f_*\Lambda$ *is finite locally constant for all $q \geq 0$.*

Proof. Putting the above assertions together, we obtain an exact sequence:

$$0 \longrightarrow \overline{f}_*\Lambda \longrightarrow f_*\Lambda \longrightarrow 0 .$$

Since $\overline{f}_*\Lambda$ is finite locally constant, $f_*\Lambda$ is finite locally constant.
For the only other non-zero guy, note too that f is locally fibration and so by homotopy equivalence we can produce a canonical isomorphism $\mathrm{H}^p(U_s,\Lambda) \to \mathrm{H}^p(U_u,\Lambda)$ for all u in some small open disk containing s. In particular, our 5-Lemma argument from before showed us that $(\mathrm{R}^p f_*\Lambda)_s = \mathrm{H}^p(U_s,\Lambda)$ and so we have the local constancy. ☐

We now have a Leray spectral sequence for U over S:

$$\mathrm{H}^p(S_{\mathrm{cx}}, \mathrm{R}^q f_* \Lambda) \Longrightarrow \mathrm{H}^{p+q}(U_{\mathrm{cx}}, \Lambda).$$

By our results above, this degenerates to an exact sequence

$$\cdots \longrightarrow \mathrm{H}^p(S_{\mathrm{cx}}, f_*\Lambda) \longrightarrow \mathrm{H}^p(U_{\mathrm{cx}}, \Lambda) \longrightarrow \mathrm{H}^{p-1}(S_{\mathrm{sx}}, \mathrm{R}^1 f_* \Lambda) \longrightarrow \cdots .$$

Let $\gamma \in \mathrm{H}^p(U_{\mathrm{cx}}, \Lambda)$ map to $t \in \mathrm{H}^{p-1}(S_{\mathrm{cx}}, \mathrm{R}^1 f_* \Lambda)$. Since $\mathrm{R}^1 f_* \Lambda$ is finite locally constant and S_{cx} is smooth of dimension $\dim U_{\mathrm{cx}} - 1$, then we may apply the inductive hypothesis and obtain an étale map $W \to S$ whose image contains $f(x)$ and is such that $t\,|_{W_{\mathrm{cx}}} = 0$. Replace U by $f^{-1}(\mathrm{im}(W \to S))$ and by repeating the Leray spectral sequence above, we obtain an exact sequence

$$\cdots \longrightarrow \mathrm{H}^p(W_{\mathrm{cx}}, f'_*\Lambda) \longrightarrow \mathrm{H}^p(U_{\mathrm{cx}} \times_{S_{\mathrm{cx}}} W_{\mathrm{cx}}, \Lambda) \longrightarrow \mathrm{H}^{p-1}(W_{\mathrm{sx}}, \mathrm{R}^1 f'_* \Lambda) \longrightarrow \cdots .$$

Now note that $\gamma\,|_{U_{\mathrm{cx}} \times_{S_{\mathrm{cx}}} W_{\mathrm{cx}}}$ belongs to the kernel of the map from H^p to H^{p-1} and so is in the image of some $\theta \in \mathrm{H}^p(W_{\mathrm{cx}}, f'_*\Lambda)$. Now apply the inductive hypothesis to θ and obtain an étale $V \to W$ with its image containing $f'(x)$ and $\theta\,|_V = 0$. Take $U' = U \times_S V$, then $\gamma\,|_{U'} = 0$.

Hence, we only owe the existence of that complicated curve fibering business.

8. Existence of Elementary Fibrations

To prove the existence of an elementary fibration, we need a Bertini Theorem.

8.1. **A Bertini Theorem.** The following Lemma is Exercise (III,11.3) from Hartshorne.

Lemma 8.1. *Let W be a normal, projective $\Bbbk$-variety and $\mathfrak{d}$ a linear system, then if $f : W \to \mathbb{P}^n$ is the morphism associated to the system and* $\dim f(W) > 1$, *every divisor is connected.*

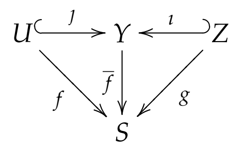
From now on, $\Bbbk$ is an algebraically closed field of characteristic 0. We have the following "Bertini" theorem.

Theorem 8.2 (Bertini)**.** *Let V be a smooth, quasi-projective $\Bbbk$-variety of dimension n with an embedding $V \hookrightarrow \mathbb{P}^r_{\Bbbk}$ such that $\overline{V}$ is normal. If $n > 1$, then there is an open set of hyperplanes $H_1, \ldots, H_s$ for $s < n$ such that if $W = H_1 \cap \cdots \cap H_s$, $W \cap V$ is smooth, connected and $W \cap \overline{V}$ is normal (hence connected) of the appropriate dimensions.*

If $n = 1$, then for any finite subset of points $W \subset \mathbb{P}^r_{\Bbbk}$. the generic hyperplane H_0 avoids W and meets $\overline{V}$ transversely.

8.2. **The fibration.**

Theorem 8.3. *Let X be a smooth scheme of relative dimension n over an algebraically closed field $\Bbbk$, then for every closed point $x \in X$, there is a Zariski open neighbourhood U which fits into a diagram:*



where all maps are over $\operatorname{Spec} \Bbbk$ *and*

(1) $Z = Y - U$;
(2) $\jmath$ *is an open immersion;*
(3) $\imath$ *is a closed immersion;*
(4) f *is smooth of dimension* 1*;*
(5) S *is smooth over* $\operatorname{Spec} \mathbb{k}$ *of relative dimension* $n-1$*;*
(6) g *is finite étale with non-empty fibers;*
(7) $\overline{f}$ *is projective with fibers smooth connected curves;*
(8) U *is dense in every fiber of* $\overline{f}$*.*

Proof. We may assume that X is a quasi-projective variety and by Bertini's Lemma, we can find an embedding $X \hookrightarrow \mathbb{P}^r_{\mathbb{k}}$ together with hyperplanes $H_1, \ldots, H_{n-1}$, containing x, such that if $W = H_1 \cap \cdots \cap H_{n-1}$, then $W \cap X$ is smooth, connected and $W \cap \overline{X}$ is normal of dimension 1. Hence, it is a smooth curve.

Now pick another hyperplane H_0 which meets $W \cap X$ transversely and avoids the finite set of points $(\overline{X} - X) \cap W$. Now consider the projection $\mathbb{P}^r - \rightarrow \mathbb{P}^{n-1}$, which is a rational map, defined away from the finite set $D = H_0 \cap C$, which doesn't contain x. Blowing up D, we get projective morphisms $\overline{f} : B \rightarrow \mathbb{P}^{n-1}$, $\varepsilon : B \rightarrow \mathbb{P}^r$ Let $U_1 = \epsilon^{-1}(X - D)$ (the proper transform of U) and take Y_1 to be its closure in B, $Z_1 = Y_1 - U_1$ and let $\overline{f}$ be the restriction of ϵ to Y_1 and similarly for f and g. Note that the fiber over $f(x)$ is C and in fact, since the morphism is a projection, the fiber over an open neighbourhood S_2 of $f(x)$ is C. Now take $U_2 := f^{-1}(S_2)$, $Y_2 := \overline{U_2}$, $Z_2 = Y_2 - U_2$ and everything is satisfied except (6).

Let $P_1, \ldots, P_t$ denote the divisors mapping to the points of D under ε. Note now that $Z_2 = \varepsilon^{-1}(\overline{X} - X) \coprod Q_1 \coprod \cdots \coprod Q_t$ where $Q_i = P_i \cap Z_2$ and since H_0 meets C transversely and misses $(\overline{X} - X) \cap C$, then g is étale at $f(x)$. More over, since being étale is an open condition, we conclude that g is étale on an open neighbourhood S of $f(x)$. Take $U := f^{-1}(S)$, $Y := \overline{U}$, $Z = Y \setminus U$, then since Z is quasi-finite and projective over S, it is finite and so Z is finite étale and all conditions are satisfied.

□