Comprehensive factorisation & non-commutative Stone duality

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\textsuperscript{1} joint with Mai Gehrke and Ralph Kaufmann
1 Introduction

2 Consistent comprehension schemes

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4 Distributive bands and distributive skew-lattices

5 Non-commutative Stone duality
Examples (notions of covering)

- topological covering $\X \rightsquigarrow \Pi_1(X)$-set
- discrete fibration $\C \rightsquigarrow$ set-valued presheaf on $\C$

Purpose of the talk

- general notion of covering & associated factorisation system using Lawvere's comprehension schemes '70.
- apply to idempotent semigroups to get non-commutative versions of Stone duality '37.
Examples (notions of covering)

- topological covering/$X \leftrightarrow \Pi_1(X)$-set
- discrete fibration/$\mathcal{C} \leftrightarrow$ set-valued presheaf on $\mathcal{C}$

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Introduction

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- discrete fibration/$\mathcal{C}$-set-valued presheaf on $\mathcal{C}$

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- topological covering/\(X\) \(\leftrightarrow\) \(\Pi_1(X)\)-set
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- general notion of covering & associated factorisation system using Lawvere’s comprehension schemes ’70.
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**Definition (category of adjunctions)**

Objects of $\text{Adj}_*$ are categories with a distinguished terminal object. Morphisms of $\text{Adj}_*$ are adjunctions $(f_!, f^*)$.

**Definition (comprehension scheme)**

A *comprehension scheme* on $\mathcal{E}$ is a pseudo-functor $P : \mathcal{E} \to \text{Adj}_*$ such that for each object $B$ of $\mathcal{E}$ the functor

$$
\begin{array}{ccc}
\mathcal{E}/B & \longrightarrow & PB \\
(f : A \to B) & \longmapsto & f_!(\star_{PA})
\end{array}
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has a *fully faithful* right adjoint $e_{\downarrow B} : PB \to \mathcal{E}/B$. 
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has a *fully faithful* right adjoint \( e_B : PB \to \mathcal{E}/B \).
**Definition**

- A morphism $f : A \to B$ is a $P$-covering if it belongs to the essential image of $e|_B$.
- A comprehension scheme is consistent if $P$-coverings compose and are left cancellable: $gf, g \in \text{Cov}_B \implies f \in \text{Cov}_B$.
- A morphism $f : A \to B$ is $P$-connected if $f_!(\star_{PA}) \cong \star_{PB}$.

**Theorem (B-Kaufmann ’17)**

There is a 1-1 correspondence between consistent comprehension schemes and complete orthogonal factorisation systems.

**Proof.**

- $ccs$ induces $(P$-connected, $P$-covering)-factorisation.
- $(\mathcal{L}, \mathcal{R})$-factorisation induces $ccs$ with $e|_B = \mathcal{R}/B$. 
Definition

- A morphism \( f : A \to B \) is a \( P \)-covering if it belongs to the essential image of \( e|_B \).
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- A morphism \( f : A \to B \) is a \( P \)-covering if it belongs to the essential image of \( \text{el}_B \).
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**Definition**

- A morphism $f : A \to B$ is a **$P$-covering** if it belongs to the essential image of $\text{el}_B$.
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- A morphism \( f : A \to B \) is a \( P \)-covering if it belongs to the essential image of \( e_1 B \).
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- A morphism $f : A \rightarrow B$ is a $P$-covering if it belongs to the essential image of $\text{el}_B$.
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Remark (Frobenius)

A ccs satisfies Frobenius reciprocity (Lawvere ’70) if and only if $P$-connected maps are stable under pullback along $P$-coverings.

Examples (comprehensive factorisation systems)

- $\text{Sets} \to \text{Adj} : X \mapsto (P_X, \subset)$ induces epi/mono-factorisation.
- $\text{Cat} \to \text{Adj} : C \mapsto PC = [C^{\text{op}}, \text{Sets}]$ induces the comprehensive factorisation of a functor (Street-Walters ’73).
- $PC$ restricts to $\text{Posets} \subset \text{Cat}$ and $\text{Gpd} \subset \text{Cat}$ (Bourn ’87).
- $\exists$ ccs $\text{Multicat} \to \text{Adj}$ and $\text{Feyn} \to \text{Adj}$ (B-Kaufmann ’17).
- $\text{Top}_{slsc} \to \text{Adj} : X \mapsto \text{Sh}_{loc}(X)$ yields a comprehensive factorisation of a continuous map of slsc spaces.
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Examples (comprehensive factorisation systems)

- $\text{Sets} \to \text{Adj}^\ast : X \mapsto (\mathcal{P}X, \subset)$ induces epi/mono-factorisation.
- $\text{Cat} \to \text{Adj}^\ast : \mathcal{C} \mapsto \mathcal{P}\mathcal{C} = [\mathcal{C}^{\text{op}}, \text{Sets}]$ induces the comprehensive factorisation of a functor (Street-Walters ’73).
- $\mathcal{P}\mathcal{C}$ restricts to $\text{Posets} \subset \text{Cat}$ and $\text{Gpd} \subset \text{Cat}$ (Bourn ’87).
- $\exists$ ccs Multicat $\to \text{Adj}^\ast$ and Feyn $\to \text{Adj}^\ast$ (B-Kaufmann ’17).
- $\text{Top}_{\text{slsc}} \to \text{Adj}^\ast : X \mapsto \text{Sh}_{\text{loc}}(X)$ yields a comprehensive factorisation of a continuous map of slsc spaces.
### Remarks (Frobenius)

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### Examples (comprehensive factorisation systems)

- **Sets** → $\text{Adj} \_*$ : $X \mapsto (\mathcal{P}X, \subset)$ induces epi/mono-factorisation.
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- $\mathcal{P}C$ restricts to $\text{Posets} \subset \text{Cat}$ and $\text{Gpd} \subset \text{Cat}$ (Bourn ’87).
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- ∃ccs Multicat → **Adj**$_*$ and Feyn → **Adj**$_*$ (B-Kaufmann ’17).
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- \( \exists \text{ccs Multicat} \rightarrow \text{Adj}^* \) and \( \text{Feyn} \rightarrow \text{Adj}^* \) (B-Kaufmann ’17).
- \( \text{Top}_{slsc} \rightarrow \text{Adj}^* : X \mapsto \text{Sh}_{loc}(X) \) yields a comprehensive factorisation of a continuous map of slsc spaces.
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### Remark (Frobenius)

A *ccs* satisfies *Frobenius reciprocity* (Lawvere ’70) if and only if *P*-connected maps are stable under pullback along *P*-coverings.

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- $PC$ restricts to Posets $\subset$ Cat and Gpd $\subset$ Cat (Bourn ’87).
- $\exists ccs$ Multicat → Adj* and Feyn → Adj* (B-Kaufmann ’17).
- **Top* *slsc** → Adj*: $X \mapsto \text{Sh}_{loc}(X)$ yields a comprehensive factorisation of a continuous map of slsc spaces.
Remark (espace étalé)

The equivalence $\mathsf{Sh}(X) \simeq \{\text{local homeomorphisms}/X\}$ restricts to an equivalence $\mathsf{Sh}_{loc}(X) \simeq \{\text{topological coverings}/X\}$.

Lawvere '70: ... we remark that although our discussion below of comprehension hinges on the operation $\Sigma$, there is one structure in which all features of hyperdoctrines except $\Sigma$ exist ..., but in which there is clearly a kind of “extension”, namely the espace étalé.

Proposition ($f!$ for locally constant sheaves on slsc spaces)

For any slsc space, monodromy induces an equivalence of categories $\mathsf{Sh}_{loc}(X) \simeq \Pi_1(X)$-sets. In particular for $f : X \to Y$,

\[
\begin{array}{c}
\mathsf{Sh}_{loc}(X) \xrightarrow{f!} \mathsf{Sh}_{loc}(Y) \\
\simeq \\
\downarrow \\
\Pi_1(X)\text{-sets} \xrightarrow{\Pi_1(f)_!} \Pi_1(Y)\text{-sets}
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\simeq \quad \simeq
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**Proposition (homotopical characterisation of connected maps)**

A map of slsc spaces \( f : X \to Y \) is connected iff \( \pi_0(f) \) is bijective and \( \pi_1(f, x) : \pi_1(X, x) \to \pi_1(Y, f(x)) \) is surjective \( \forall x \in X \).

**Corollary (existence of universal coverings)**

For any based slsc space \((X, x)\) the comprehensive factorisation

\[
\begin{array}{ccccccc}
\star & \xrightarrow{\text{connected}} & U(X, x) & \xrightarrow{\text{covering}} & X \\
\downarrow & & \downarrow & & \\
 x & \xrightarrow{f} & X
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\]

produces the *universal covering* of \( X \) at \( x \).
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Definition

A *band* (=idempotent semigroup) is a set $(X, \cdot)$ with an associative multiplication such that $x^2 = x$ for all $x \in X$.

Lemma (meet-semilattices)

*Commutative* bands are the same as posets with binary meets.

Lemma (Green’s $D$-relation)

Each band is partially ordered by $x \leq y \iff x = yxy$. The commutative bands form a reflective subcategory. The reflection is given by $X \to X/D$ where $xDy \iff x = xyx$ and $y = yxy$. 
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Definition (Schützenberger ’47)

A band is left (resp. right) regular if \( xy = xyx \) (resp. \( yx = xyx \)).

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The category of right regular bands admits a comprehensive factorisation system lifted along the functor \( (X, \cdot) \mapsto (X, \leq) \).

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Definition

A band $X$ is called right distributive if

(i) $X$ is right normal;
(ii) $X/\mathcal{D}$ is a (bounded) distributive lattice;
(iii) for any finite subset $S$ of $X$ consisting of pairwise commuting elements the join $\bigvee S$ in $(X, \leq)$ exists.

Example (the local sections of a sheaf form a distributive band)

We define $(U, \sigma)(V, \tau) = (U \cap V, \tau|_{U \cap V})$. Local sections commute iff they glue. $(U, \sigma) \leq (V, \tau)$ iff $U \subset V$ and $\sigma = \tau|_U$. (iii) expresses sheaf condition w/to finite open covers.
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Definition (skew-lattice, Leech '89)

A skew lattice \((S, \land, \lor)\) consists of two bands \((S, \land)\) and \((S, \lor)\) such that the following four absorption laws hold:

(i) \((y \land x) \lor x = x = x \land (x \lor y)\);
(ii) \(x \lor (x \land y) = x = (y \lor x) \land x\).

Remark (lattice reflection)

The order relation of \((S, \land)\) is dual to the order relation of \((S, \lor)\). Green's \(D\)-relation yields a lattice \(S/D\), the lattice reflection of \(S\). \((S, \land)\) is right regular iff \((S, \lor)\) is left regular.

Definition (variety of distributive skew-lattices)

A skew-lattice is symmetric if \(x \land y = y \land x \iff x \lor y = y \lor x\).
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Definition (skew-lattice, Leech ’89)

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The order relation of $(S, \preceq)$ is *dual* to the order relation of $(S, \succ)$. Green’s $D$-relation yields a lattice $S/D$, the *lattice reflection* of $S$. $(S, \preceq)$ is right regular iff $(S, \succ)$ is left regular.

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### Definition (skew-lattice, Leech '89)

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There is a duality between the category of distributive lattices and the category of spectral spaces.

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