The lattice path operad*

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Part 1. Iterated loop spaces and $E_n$-operads.

Let $(X, \ast)$ be a based topological space and $(S^n, \ast)$ be the $n$-sphere. Then

$$\Omega^n X = \text{Top}_* (S^n, X)$$

is an algebra over the coendomorphism operad

$$\text{Coend}(S^n)(k) = \text{Top}_* (S^n, \underbrace{S^n \vee \cdots \vee S^n}_k)$$

where the action is given by composition:

$$\text{Coend}(S^n)(k) \times (\Omega^n X)^k \longrightarrow \Omega^n X$$

$$\text{Top}_* (S^n, (S^n)^\vee k) \times \text{Top}_* ((S^n)^\vee k, X) \longrightarrow \text{Top}_* (S^n, X)$$
The operad \((C_n(k))_{k \geq 0}\) of little \(n\)-cubes is a suboperad of \((\text{Coend}(S^n)(k))_{k \geq 0}\). Therefore any \(n\)-fold loop space is a \(C_n\)-algebra.

**Theorem 1.** (Boardman-Vogt, May, Segal) Any \(C_n\)-algebra is up to group completion an \(n\)-fold loop space. In particular, \(\Omega^n S^m X\) is the group completion of the free \(C_n\)-algebra on \(X\).

**Theorem 2.** (F. Cohen) \(H_\ast(\Omega^n S^m X, \mathbb{Z}/p\mathbb{Z})\) is a \(H_\ast(C_n, \mathbb{Z}/p\mathbb{Z})\)-algebra on \(H_\ast(X, \mathbb{Z}/p\mathbb{Z})\) equipped with certain Dyer-Lashof operations.

For any field \(k\), a \(H_\ast(C_2, k)\)-algebra is called a Gerstenhaber \(k\)-algebra. The Hochschild cohomology \(HH^\ast(A; A)\) of an associative \(k\)-algebra \(A\) is a Gerstenhaber algebra, whence Deligne’s conjecture: Is this structure induced by a dg-\(E_2\)-operad action on \(CC^\ast(A; A)\) ?

Proofs of the Deligne conjecture have been given by Tamarkin, McClure-Smith, Kontsevich-Soibelman and Berger-Fresse.
Part 2. Enriched categories.

Let $\mathcal{E} = (\mathcal{E}, \otimes_{\mathcal{E}}, I_{\mathcal{E}}, \mathcal{E}(-,-))$ be a closed symmetric monoidal category, for instance $(\text{Top}, \times, *, \text{Top}(-,-))$ or $(\text{Ch}(k), \otimes_k, k, \text{Hom}_k(-,-))$.

**Def. 1.** An $\mathcal{E}$-category $\mathcal{A}$ consists of objects $A, A', \cdots \in \mathcal{A}_0$ and (for each pair of objects) hom-objects $\mathcal{A}(A, A') \in \mathcal{E}_0$, together with

- **units** $u_A : I_{\mathcal{E}} \to \mathcal{A}(A, A)$, $A \in \mathcal{A}_0$,

- **compositions** $\mathcal{A}(A', A'') \otimes_{\mathcal{E}} \mathcal{A}(A, A') \to \mathcal{A}(A, A'')$

fulfilling unit and associativity axioms.

Any closed symmetric monoidal category $\mathcal{E}$ is an $\mathcal{E}$-category. There is a 2-category of $\mathcal{E}$-categories, $\mathcal{E}$-functors and $\mathcal{E}$-natural transformations.
Lemma 1. The iterated tensor-product $\bigotimes^k E : E \times \cdots \times E \to E$ is an $E$-functor, i.e. there are canonical maps $\mathcal{E}(X, Y)^{\otimes_k} \to \mathcal{E}(X^{\otimes k}, Y^{\otimes k})$.

The coendomorphism operad of an object $X$ of $\mathcal{E}$ is given by

$$\text{Coend}(X)(k) = \mathcal{E}(X, X^{\otimes k}), \quad k \geq 0,$$

with the obvious structural maps.

Proposition 1. Let $X, Y$ be two objects of $\mathcal{E}$. Assume that $Y$ is a commutative monoid in $\mathcal{E}$. Then $\mathcal{E}(X, Y)$ is a Coend($X$)-algebra.

The Coend($X$)-algebra structure is given by

$$\text{Coend}(X)(k) \otimes \mathcal{E}(X, Y)^{\otimes k} \to \mathcal{E}(X, Y)$$

$\downarrow$ enrichment $\quad \downarrow$ multiplication

$$\mathcal{E}(X, X^{\otimes k}) \otimes \mathcal{E}(X^{\otimes k}, Y^{\otimes k}) \xrightarrow{\text{composition}} \mathcal{E}(X, Y^{\otimes k}).$$
Part 3. Condensation of coloured operads.

An $N$-coloured operad in $\mathcal{E}$ is given by a collection of objects $O(n_1, \ldots, n_k; n)$ of $\mathcal{E}$, where $(n_1, \ldots, n_k, n) \in N^{k+1}$, together with units, $\Sigma_k$-actions and composition maps

$$O(n_1, \ldots, n_k; n) \otimes_{\mathcal{E}} O(m_1, \ldots, m_l; n_i) \xrightarrow{o_i} O(n_1, \ldots, n_{i-1}, m_1, \ldots, m_l, n_{i+1}, \ldots, n_k; n),$$

which are unital, associative and equivariant.

If $N = \{\ast\}$ then $O(k) = O(\ast, \ldots, \ast; \ast)$ is a symmetric operad in $\mathcal{E}$.

Each $N$-coloured operad $O$ defines a category $O_u$ of unary operations with object-set $N$:

$$O_u(n, n') = O(n; n').$$
A coloured operad $\mathcal{O}$ in $\mathcal{E}$ can also be presented as a \textit{multitensor} on $\mathcal{O}_u$ with values in $\mathcal{E}$:

$$\mathcal{O}^{\text{op}}_u \times \cdots \times \mathcal{O}^{\text{op}}_u \times \mathcal{O}_u \mathcal{O}(-, \ldots, -; -) \mathcal{E}$$

This defines a \textit{lax symmetric monoidal structure} on $\mathcal{E}^{\mathcal{O}_u}$ by the coend formula:

$$(X_1 \otimes \mathcal{O} \cdots \otimes \mathcal{O} X_k)(n) = \int^{n_1, \ldots, n_k} \mathcal{O}(-, \ldots, -; n) \otimes \mathcal{E} X_1(-) \otimes \mathcal{E} \cdots \otimes \mathcal{E} X_k(-).$$

In particular, for each object $\delta \in \mathcal{E}^{\mathcal{O}_u}$, we get a \textit{coendomorphism operad}

$$\text{Coend}_{\mathcal{O}}(\delta)(k) = \text{Hom}_{\mathcal{O}_u}(\delta, \delta \otimes \mathcal{O} \cdots \otimes \mathcal{O} \delta).$$

**Proposition 2.** Let $X$ be an algebra over the coloured operad $\mathcal{O}$ in $\mathcal{E}$. Let $\delta \in \mathcal{E}^{\mathcal{O}_u}$. Then $\text{Hom}_{\mathcal{O}_u}(\delta, X)$ is a $\text{Coend}_{\mathcal{O}}(\delta)$-algebra.
\( \mathcal{E} = \textbf{Top} \) or \( \mathcal{E} = \text{Ch}(\mathbb{Z}) \) contains \( \text{Sets} \) as the subcategory of \textit{discrete objects} via the \textit{strong monoidal} functor \( S \mapsto \bigcup S I_{\mathcal{E}} \) (\( I_{\mathcal{E}} \equiv \text{unit of } \mathcal{E} \)).

We shall construct a \textit{coloured operad} \( \mathcal{L} \) in \( \text{Sets} \), parametrizing the combinatorial structure of \textit{iterated loop spaces} in the following sense:

1. \( \mathcal{L} = \bigcup_{m \geq 0} \mathcal{L}_m \) and \( \mathcal{L}_u = \Delta = (\mathcal{L}_m)_u \);

2. For the standard object \( \delta : \Delta \to \mathcal{E} \), \( \text{Coend}_{\mathcal{L}_m}(\delta) \) is an \( E_m \)-\textit{operad} in \( \mathcal{E} \).

In particular, any \( \mathcal{L}_m \)-algebra \( X \) in \( \mathcal{E} \) gives rise to an \( E_m \)-algebra \( \text{Hom}_\Delta(\delta, X) \). Being an \( \mathcal{L}_m \)-algebra in \( \mathcal{E} \) is a combinatorial property!!

The funny tensor product of categories $\mathcal{A} \otimes \mathcal{B}$ has $(A, B) \in \mathcal{A}_0 \times \mathcal{B}_0$ as objects, and “free” compositions of $(f, 1_B) : (A, B) \to (A', B)$ and $(1_A, g) : (A, B) \to (A, B')$ as morphisms.

**Def. 2.** The lattice path operad is the $\mathbb{N}$-coloured operad in sets defined by

$$\mathcal{L}(n_1, \ldots, n_k; n) = \text{Cat}_*, *[n+1], [n_1+1] \otimes \cdots \otimes [n_k+1]).$$

**Example.** Let $x \in \mathcal{L}(2, 1; 3)$ be the lattice path:

![Diagram of the lattice path](image)

The path is determined by the sequence of “directions” and “stops”: $x = 1|21|1|2$. 

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\( \mathcal{L}(n_1, \ldots, n_k; n) \) may be identified with the set of finite sequences containing \( n_1 + 1 \) times 1, \( n_2 + 1 \) times 2, \ldots, \( n_k + 1 \) times \( k \), and \( n \) (possibly multiple) stop’s. Under this identification, the operad composition map is given by renumbering and substitution:

\[
1\|12\|3\|2 \circ_2 1\|32 = 1\|12\|5\|43.
\]

**Lemma 2.** \( \mathcal{L}_u = \Delta. \) (Joyal-duality)

\[
\mathcal{L}_u(n', n) = \text{Cat}_{\ast, \ast}([n+1], [n'+1]) = \Delta([n'], [n]).
\]

Let \( \Delta \Sigma \) be the category of finite sets and finite set mappings equipped with total orderings of the fibers, cf. Feigin-Tsygan, Krasauskas and Fiedorowicz-Loday. (Crossed simplicial group).

**Proposition 3.** (Extended Joyal-duality)

\[
\mathcal{L}(n_1, \ldots, n_k; n) = \{ x \in \Delta \Sigma([n_1] \ast \cdots \ast [n_k], [n]) \text{ sth. } \forall i : x|_{[n_i]} \in \Delta([n_i], [n]) \},
\]

where the operad composition is given by join and composition in \( \Delta \Sigma \).
**Def. 3. (Filtration by complexity)**
For $1 \leq i < j \leq k$, let $p_{ij}$ be the projection

$$[n_1 + 1] \otimes \cdots \otimes [n_k + 1] \rightarrow [n_i + 1] \otimes [n_j + 1].$$

Let $a_{ij}(x)$ be the number of angles in the lattice path $p_{ij} \circ x$, and $c(x) = \max_{i<j} a_{ij}(x)$. Then,

$$\mathcal{L}_m(n_1, \ldots, n_k; n) = \{ x \in \mathcal{L}(n_1, \ldots, n_k; n) \mid c(x) \leq m \}$$

defines a suboperad $\mathcal{L}_m$ of $\mathcal{L}$ with $(\mathcal{L}_m)_u = \Delta$.

**Proposition 4. (Batanin)** The category of $\mathcal{L}_1$-algebras is isomorphic to the category of cosimplicial $\Box$-monoids ($\Box$ is induced by ordinal sum).

**Proposition 5. (Tamarkin)** The category of $\mathcal{L}_2$-algebras in $\mathcal{E}$ is isomorphic to the category of multiplicative non-symmetric operads in $\mathcal{E}$.

**Example.** The Hochschild cochain complex of an associative algebra is an $\mathcal{L}_2$-algebra.
**Proposition 6.** For each simplicial set $X$, the norm. cochain complex $N^*(X)$ is an $L$-algebra.

The dual coaction is given by

$$\mathcal{L}(n_1, \cdots, n_k; n) \otimes N_n(X) \to N_{n_1}(X) \otimes \cdots \otimes N_{n_k}(X)$$

$$x \otimes [\alpha] \mapsto [x_1^*(\alpha)] \otimes \cdots \otimes [x_k^*(\alpha)]$$

where $(x_1, \ldots, x_k)$ are the components of $x : [n_1] \ast \cdots \ast [n_k] \to [n]$.

**Proposition 7.** Let $S^m$ be $\Delta[m]/\partial \Delta[m]$ and $X$ be a pointed object of $\mathcal{E}$. Then, the cosimplicial $\mathcal{E}$-object $(X, \ast)(S^m, \ast)$ is an $L_m$-algebra.

There is an $L$-coaction on $S^m$:

$$\mathcal{L}(n_1, \cdots, n_k; n) \times (S^m)_n \to (S^m)_{n_1} \times \cdots \times (S^m)_{n_k}$$

$$x \times \alpha \mapsto (x_1^*(\alpha), \ldots, x_k^*(\alpha)).$$

If $c(x) \leq m$, the image is in $(S^m)_{n_1} \vee \cdots \vee (S^m)_{n_k}$. 
We now consider the case $\mathcal{E} = \text{Top}$. Let $\delta : \Delta \to \text{Top}$ be the standard cosimplicial object. 

$$\text{Hom}_\Delta(\delta, (X, *)^{(S^m, *)}) \cong \text{Top}_*(|S^m|, X) = \Omega^m X.$$ 

Thus, any $m$-fold loop space is an algebra over the coendomorphism-operad 

$$\mathcal{D}_m(k) = \text{Hom}_\Delta(\delta, \delta \otimes \mathcal{L}_m \cdots \otimes \mathcal{L}_m \delta)$$ 

$$= \text{Tot}_\delta(Y_{m,k}), \ k \geq 0.$$ 

**Theorem 3.** *(McClure-Smith)* For $1 \leq m \leq \infty$, $\mathcal{D}_m$ is a topological $E_m$-operad. 

$$\text{Tot}_\delta(Y_{m,k}) \cong Y_{m,k}(0) \times \text{Tot}_\delta(\delta) \cong Y_{m,k}(0)$$ and $Y_{m,k}(0)$ is the realization of the $k$-simplicial set $\mathcal{L}_m(-, \ldots, -; 0)$ of surjections with codomain $\{1, \ldots, k\}$ and complexity $\leq m$. 

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We now turn to the case $\mathcal{E} = \text{Ch}(\mathbb{Z})$ with $\delta : \Delta \to \text{Ch}(\mathbb{Z}) : [n] \mapsto N_*(\Delta[n])$.

**Totalization** $\text{Hom}_{\Delta}(\delta, -)$ takes a cosimplicial module to the $dg$-module with differential $d = \sum(-1)^i \partial_i$. Thus the cochain complex $N^*(X)$ is a $\bar{X}_\infty$-algebra, and the Hochschild cochain complex $CC^*(A; A)$ is a $\bar{X}_2$-algebra, where $\bar{X}_m$ is the coendomorphism operad

$$\bar{X}_m(k) = \text{Hom}_{\Delta}(\delta, \delta \otimes L_m \cdots \otimes L_m \delta), \quad k \geq 0.$$

“Summing up the elements of the fibers” of

$$L_m(-, \ldots, -; n) \to L_m(-, \ldots, -; 0)$$

defines a cosimplicial dg-submodule of

$$|L_m(-, \cdots, -; n)|_{\delta \otimes \cdots \otimes \delta},$$

and by totalization a dg-suboperad $\mathcal{X}_m$ of $\bar{X}_m$:

$$\mathcal{X}_m(k) = |L_m(-, \cdots, -; 0)|_{\delta \otimes \cdots \otimes \delta}, \quad k \geq 0.$$

This suboperad is the $m$-th filtration stage of the so-called *surjection operad* $\mathcal{X}$. 

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Theorem 4. (McClure-Smith, Berger-Fresse) For $1 \leq m \leq \infty$, $\mathcal{X}_m$ is a dg-$E_m$-operad.

This yields an $E_\infty$-structure on $N^*(X)$ as well as an $E_2$-structure on $CC^*(A; A)$, solving the Deligne conjecture.

We finally consider the case $\mathcal{E} = \text{Sets}^{\Delta^\text{op}}$ with $\delta : \Delta \to \text{Sets}^{\Delta^\text{op}}$ the Yoneda-embedding.

Theorem 5. (Berger-Fresse) The diagonal of the $k$-simplicial set $\mathcal{L}(-, \cdots, -; 0)$ is the universal $\Sigma(k)$-bundle $E\Sigma(k)$. There is a weak equivalence of filtered dg-operads $N_*(E_m\Sigma) \to \mathcal{X}_m$, where $E_m\Sigma$, $m \geq 1$, denotes the Smith filtration of Barratt-Eccles’ $E_\infty$-operad $E\Sigma$.

Theorem 6. (Kashiwabara, Berger) For $1 \leq m \leq \infty$, $E_m\Sigma$ is a simplicial $E_m$-operad.
The simplicial isomorphism
\[ \alpha : E\Sigma(k)_d \cong \mathcal{L}(d, \ldots, d; 0) \]
is given by a “shuffle” which increases the filtration degree in a minimal way. For instance,
\[ \alpha((123, 213, 231, 321)) = 122213333121. \]
For \( k = 2 \), this \( \alpha \) is a filtration-preserving equivariant simplicial isomorphism.

The map of filtered dg-operads \( N_*(E\Sigma) \to \mathcal{X} \) is induced by Alexander-Whitney maps
\[ N_*(\Delta[n_1] \times \cdots \times \Delta[n_k]) \to N_*(\Delta[n_1]) \otimes \cdots \otimes N_*(\Delta[n_k]) \]
via the identifications
\[ N_*(E\Sigma(k)) = |\mathcal{L}(-, \cdots, -; 0)|_{N_*(\delta \times \cdots \times \delta)} \]
\[ \mathcal{X}(k) = |\mathcal{L}(-, \cdots, -; 0)|_{N_*(\delta) \otimes \cdots \otimes N_*(\delta)} \]

The compatibility with the operad structures and filtrations follows from a cellular decomposition of \( E\Sigma(k) \) compatible with these data, which is induced by the complete graph operad \( \mathcal{K}(k), k \geq 0 \).
Tamarkin’s 2-operad action on $\mathcal{E}$–Cat.

Given two small $\mathcal{E}$-categories $A$, $B$ and two $\mathcal{E}$-functors $F, G : A \to B$, one defines a cosimplicial object of natural transformations $R^\bullet(F, G)$ where $R^n(F, G)$ is given by

$$\prod \mathcal{E}(A(x_0, x_1) \otimes \cdots \otimes A(x_{n-1}, x_n), B(F(x_0), G(x_n)))$$

where the product is over $(x_0, \ldots, x_n) \in A_0^{n+1}$.

The derived object is then by definition

$$R(F, G) = \text{Tot}_\delta(R^\bullet(F, G)).$$

If $A$ is a one-object dg-category with $A(\ast, \ast) = A$, then $R(Id_A) = CC^*(A; A)$.

Tamarkin constructs an $\mathbb{N}$-coloured 2-operad $T_2$ whose symmetrization is $\mathcal{L}_2$ and whose totalization is a contractible 2-operad in dgMod. He shows that $T_2$ acts on dgCat. This yields (by a theorem of Batanin) a “global” proof of the Deligne conjecture.