Hyperplane arrangements, graphic monoids and moment categories

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Purpose of the talk

(hyperplane arrangements) $\overset{\textit{algebraisation}}{\rightsquigarrow}$ (graphic monoids)

(graphic monoids) $\overset{\textit{categorification}}{\rightsquigarrow}$ (moment categories)

(unital moment categories) $\overset{\textit{semantics}}{\rightsquigarrow}$ (operads)

Examples (Fox-Neuwirth, Salvetti, McClure-Smith, Berger-Fresse)

(braid arrangements) $\leftrightsquigarrow$ (symmetric groups) $\leftrightsquigarrow$ (E_n-operads)

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Definition (hyperplane arrangements in $\mathbb{R}^n$)

A linear hyperplane arrangement $\mathcal{A} = \{H_\alpha \subset \mathbb{R}^n, \alpha \in |\mathcal{A}|\}$ is

- *essential* iff $\bigcap_{\alpha\in|\mathcal{A}|} H_\alpha = (0)$;
- *Coxeter* iff $\forall \alpha, \beta \in |\mathcal{A}| : s_\alpha(H_\beta) \in \mathcal{A}$ where s_α is the orthogonal reflection with respect to the hyperplane H_α.

Proposition (Coxeter, Tits)

There is a one-to-one correspondence

$$(\text{essential Coxeter arrangements}) \overset{\cong}{\leftrightarrow} (\text{finite Coxeter groups})$$

$$\mathcal{A}_G \overset{\cong}{\leftrightarrow} G$$

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Example (symmetric group $\mathfrak{S}_3$ and its $\mathcal{A}_{\mathfrak{S}_3}$ in $\mathbb{R}^2$)

Definition (face poset $\mathcal{F}_{\mathcal{A}}$)

$$\mathcal{F}_{\mathcal{A}_{\mathfrak{S}_3}} = \{6 \text{ facets of dim } 2, 6 \text{ facets of dim } 1, 1 \text{ facet of dim } 0\}$$

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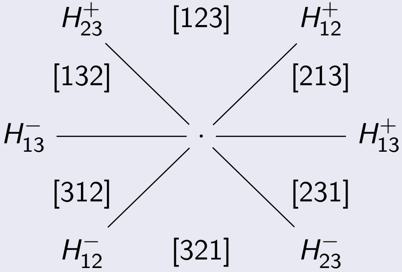

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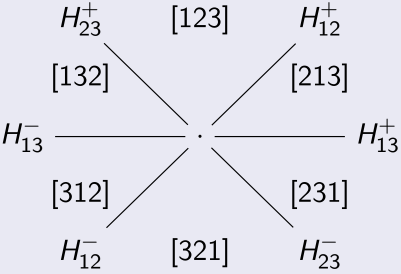

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Lemma (face monoid $\mathcal{F}_{\mathcal{A}}$ with facets x, y, z)

$$xy = z \overset{def}{\iff} \forall s \in x, t \in y : s + \epsilon(t - s) \in z \text{ for } \epsilon > 0 \text{ small}$$

- (0) is neutral element;
- $xyx = xy \quad \forall x, y \in \mathcal{F}_{\mathcal{A}}$;
- $x \subset \bar{y} \iff xy = y$;
- the *univ. comm. quotient* of $\mathcal{F}_{\mathcal{A}}$ is a *geometric lattice* $\mathcal{L}_{\mathcal{A}}$.

Definition (k-th complement of an arrangement)

$$M_k(\mathcal{A}) = \mathbb{R}^n \otimes \mathbb{R}^k - \bigcup_{\alpha \in |\mathcal{A}|} H_\alpha \otimes \mathbb{R}^k$$

Theorem (Orlik-Solomon, Salvetti)

$\mathcal{L}_{\mathcal{A}}$ ($\mathcal{F}_{\mathcal{A}}$) determines cohomology (homotopy type) of $M_2(\mathcal{A})$.

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Definition (skew lattice, left regular band, graphic monoid)

A monoid $(M, \cdot, 1)$ is called *graphic* iff $\forall x, y \in M : xyx = xy$.

Lemma

In any graphic monoid M one has

- $x^2 = x$ (all elements are idempotent);
- $x \preceq y \overset{def}{\iff} yx = x$ is a *partial order* (the right Green order);
- $xy = yx$ if and only if $x \wedge y$ exists in $(M, \preceq)$;
- $x \simeq y \overset{def}{\iff} xy = x$ and $yx = y$ is a *congruence* on $(M, \cdot)$. The quotient $M/\simeq$ is the universal comm. quotient of M (the so-called *support semi-meet lattice* of M).

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Example (graphic line $L = \mathcal{F}_{\mathcal{A}_{\mathfrak{S}_2}}$)

The three-element set $L = \{0, \pm\}$ is a graphic monoid for $++ = +, -- = -, -+ = -, +- = +$ with neutral element 0.

Definition (abstract hyperplanes)

A *hyperplane* of a graphic monoid M is any epimorphism $M \twoheadrightarrow L$. M is said to have *enough hyperplanes* if any two elements $x, y \in M$ can be distinguished by their values on hyperplanes.

Lemma (relationship with oriented matroids)

For each hyperplane arrangement $\mathcal{A}$ the face monoid $\mathcal{F}_{\mathcal{A}}$ is a graphic submonoid of $L^{|\mathcal{A}|}$. More generally, any graphic monoid M with enough hyperplanes embeds into a product of graphic lines.

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Definition (centric elements)

An element $x \in M$ is said to be *centric* if $x \simeq y \implies x = y$.

Lemma

A graphic monoid is commutative iff all its elements are centric.

Remark

There are graphic monoids (e.g. the graphic line) in which the only centric element is the neutral element. Such graphic monoids will be called *primitive* provided they also have non-centric elements.

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Definition (moment structures)

A *moment structure* on a category $\mathcal{M}$ consists of

- a set m_A of special endo's (*moments*) for each object A
- an operation $f_* : m_A \to m_B$ for each $f : A \to B$ such that

1. $1_A \in m_A$
2. $\phi_*(\psi) = \phi\psi \quad (\forall\, \phi, \psi \in m_A)$
3. $(gf)_* = g_* f_* \quad (\forall\, A \xrightarrow{f} B \xrightarrow{g} C)$
4. $f\phi = f_*(\phi)f \quad (\forall\, \phi \in m_A, f : A \to B)$

Axioms 1 and 2 imply: m_A is a *submonoid* of $\mathcal{M}(A, A)$.
Axioms 2 and 4 imply: m_A is *graphic*: $\psi\phi = \psi_*(\phi)\psi = \psi\phi\psi$.
Axioms 2, 3, 4 imply: $f_*(\phi\psi) = f_*(\phi)f_*(\psi)$.
In general: $f_*(1_A) \neq 1_B$.

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- an operation $f_* : m_A \to m_B$ for each $f : A \to B$ such that

1. $1_A \in m_A$
2. $\phi_*(\psi) = \phi\psi \quad (\forall\, \phi, \psi \in m_A)$
3. $(gf)_* = g_* f_* \quad (\forall\, A \xrightarrow{f} B \xrightarrow{g} C)$
4. $f\phi = f_*(\phi) f \quad (\forall\, \phi \in m_A, f : A \to B)$

Axioms 1 and 2 imply: m_A is a *submonoid* of $\mathcal{M}(A, A)$.
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In general: $f_*(1_A) \neq 1_B$.

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Definition (active/inert maps of a moment structure)

A map $f : A \to B$ is called *active* (resp. *inert*) if $f_*(1_A) = 1_B$ (resp. there exists $r : B \to A$ such that $rf = 1_A$ and $fr \in m_B$).

Lemma

- Epimorphisms are active; inert maps have *unique* retractions;
- A map $f : A \to B$ admits a factorization $f = f_{inert} f_{active}$ if and only if the idempotent moment $f_*(1_A)$ splits.

Definition (moment categories)

A *moment category* is a category with an abstract *active/inert factorization system* such that

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Proposition

A category $\mathcal{M}$ is a moment category if and only if $\mathcal{M}$ admits a moment structure in which all moments split.

Proof.

$\Leftarrow$ done

$\Rightarrow$ Define $m_A = \{\phi \in \mathcal{M}(A, A) \mid \phi_{act}\phi_{in} = 1\}$.

For $f : A \to B$ define $f_* : m_A \to m_B$ by

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \phi_{act} \downarrow \uparrow \phi_{in} & & \psi_{in} \uparrow \downarrow \psi_{act} \\ A_\phi & \xrightarrow[\quad f' \quad]{} & B_\psi \end{array} \quad \text{with} \quad f_*(\phi_{in}\phi_{act}) = \psi_{in}\psi_{act}.$$

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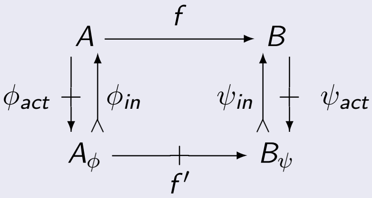

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Example (graphic monoids)

Graphic monoids correspond one-to-one to one-object categories with moment structure such that all morphisms are moments.

Example (corestriction categories – Cockett-Lack)

Corestriction categories correspond one-to-one to categories with *centric* moment structure.

Example (idempotent completion)

Each category with moment structure admits a canonical idempotent completion into a moment category.

Example (simplex category Δ and Segal's category Γ)

- $[m] \xrightarrow{\phi} [n]$ is active/inert iff ϕ endpoint/distance -preserving.
- $\underline{m} \xrightarrow{(\underline{n}_1,\dots,\underline{n}_m)} \underline{n}$ active/inert iff $\underline{n}_1 \cup \cdots \cup \underline{n}_m = \underline{n} / |\underline{n}_i| = 1 \; \forall i$.

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Lemma

For any object A of a moment category, the poset $(m_A, \preceq)$ of moments of A is isomorphic to the poset of *inert* subobjets of A.

Definition (unital moment categories, e.g. Δ and Γ)

A *unit* of a moment category is an object U such that m_U is primitive, and every active map with target U admits exactly one inert section.

A moment is *elementary* if it splits over a unit.

NOTATION FOR ELEMENTARY MOMENTS: $e_\alpha \in el_A \subset m_A$.

A *nilobject* N is an object such that $el_N = \emptyset$.

A moment category is said to be *unital* if it has units and for every active map $f : A \longrightarrow B$: if A is a nilobject then B as well.

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Definition ($\mathcal{M}$-operads for unital moment categories $\mathcal{M}$)

An $\mathcal{M}$-operad $\mathcal{O}$ in a symmetric monoidal category $(\mathcal{E}, \otimes, I)$ assigns to each object A of $\mathcal{M}$ an object $\mathcal{O}(A)$ of $\mathcal{E}$, equipped with

- a unit $I \to \mathcal{O}(U)$ in $\mathcal{E}$ for each unit U in $\mathcal{M}$;
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Definition (wreath product of unital moment categories $\mathcal{A}, \mathcal{B}$)

$\mathrm{Ob}(\mathcal{A} \wr \mathcal{B}) = \{(A, B_{e_\alpha}) \mid A \in \mathrm{Ob}(\mathcal{A}), e_\alpha \in el_A, B_{e_\alpha} \in \mathrm{Ob}(\mathcal{B})\}$

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Examples (cf. Haugseng-Gepner, Lurie, Barwick)

Δ-operads=nonsymmetric operads; Γ-operads=symmetric operads

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Δ-operads=nonsymmetric operads; Γ-operads=symmetric operads
Θ_n-operads=n-operads (cf. Batanin) where $\Theta_n = \Delta^{\wr n}$ (cf. Joyal).

Definition ($\mathcal{M}$-operads for unital moment categories $\mathcal{M}$)

An $\mathcal{M}$-operad $\mathcal{O}$ in a symmetric monoidal category $(\mathcal{E}, \otimes, I)$ assigns to each object A of $\mathcal{M}$ an object $\mathcal{O}(A)$ of $\mathcal{E}$, equipped with

- a unit $I \to \mathcal{O}(U)$ in $\mathcal{E}$ for each unit U in $\mathcal{M}$;
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Definition (wreath product of unital moment categories $\mathcal{A}, \mathcal{B}$)

$\mathrm{Ob}(\mathcal{A} \wr \mathcal{B}) = \{(A, B_{e_\alpha}) \mid A \in \mathrm{Ob}(\mathcal{A}), e_\alpha \in el_A, B_{e_\alpha} \in \mathrm{Ob}(\mathcal{B})\}$

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