# Goodwillie's cubical cross-effects & nilpotency in semiabelian categories

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based on joint work with Dominique Bourn

Topos Institute Workshop, 14 - 18 March, 2022

## Definition (polynomials)

- $k[X] \ni F(X) = \sum_{i \geq 0} \alpha_i X^i$
- $\deg(F) \leq n$ iff $\alpha_i = 0$ for $i > n$
- $F : k \to k$ is linear iff $F(0) = 0$ and $\deg(F) \leq 1$.

## Definition (polynomial functors of Eilenberg-MacLane)

- A functor between abelian categories is of degree $\leq n$ iff $cr^F_{n+1}(X_1, \ldots, X_{n+1}) = 0$ for all $X_1, \ldots, X_{n+1}$ of the domain.
- $F$ is linear iff $F(0) = 0$ and $\deg(F) \leq 1$.

## Purpose of the talk

- Degree for functors between non-additive categories
- Goodwillie's cubical cross-effects
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**Definition (additive/abelian)**

- $(\mathbb{E}, \star_{\mathbb{E}})$ *additive* iff $\theta_{X,Y} : X + Y \to X \times Y$ is invertible, and every identity has an additive inverse.
- An *abelian* category is an additive category with kernels and cokernels sth. every mono/epi is a kernel/cokernel.

**Definition (idempotent-complete)**

An additive category is *idempotent-complete* if every idempotent endomorphism has kernel/cokernel.

**Lemma (idempotent-complete $\Longrightarrow$ protomodular (Bourn '96))**

In an idempotent-complete additive category, every *split epi* $f : X \overset{\curvearrowleft}{\twoheadrightarrow} Y$ is protomodular: $f$ has a kernel and $Y + \ker(f) \twoheadrightarrow X$.

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**Definition (semiadditive)**

A pointed category is *semiadditive* iff it has binary sums, pullbacks of split epis, and every split epi is protomodular.

**Lemma**

In a semiadditive category $\theta_{X,Y} : X + Y \to X \times Y$ is a strong epi.
$\mathbb{E}$ additive & idempotent-complete $\iff$ $\mathbb{E}$ and $\mathbb{E}^{op}$ semi-additive.

**Theorem (Tierney)**

$\mathbb{E}$ abelian $\iff$ $\mathbb{E}$ additive and exact.

**Definition (Janelidze-Márki-Tholen '01)**

$\mathbb{E}$ *semiabelian* iff $\mathbb{E}$ semiadditive and exact[a].

[a] finitely complete, stable strong epi/mono fact, effective equ. relations

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**Examples (semiabelian categories)**

- groups
- Lie algebras
- cocommutative Hopf algebras (Gran-Sterck-Vercruysse '19)

**Proposition (abelian core)**

Each semiabelian category $\mathbb{E}$ has an abelian core $\mathrm{Ab}(\mathbb{E})$ spanned by those objects $X$ for which $[X, X] = \star_{\mathbb{E}}$.

**Definition (commutator subobject)**

The commutator subobject $[X, X]$ is the image of $\ker(\theta_{X,X})$ along the folding map $\nabla_X : X + X \to X$.

**Remark**

In an abelian category the commutator subobjects are trivial.

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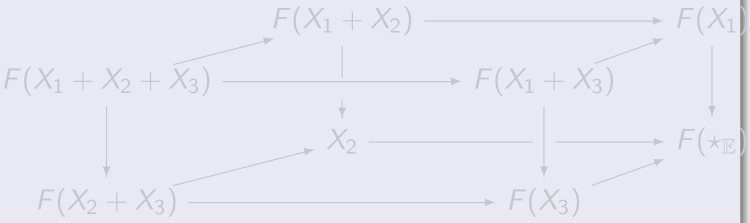
## Definition (Goodwillie cubes for pointed $F : \mathbb{E} \to \mathbb{E}'$)

$$\Xi^F_{X_1,X_2} \qquad \begin{array}{ccc} F(X_1+X_2) & \longrightarrow & F(X_1) \\ \downarrow & & \downarrow \\ F(X_2) & \longrightarrow & F(\star_{\mathbb{E}}) \end{array}$$

$\Xi^F_{X_1,X_2,X_3}$



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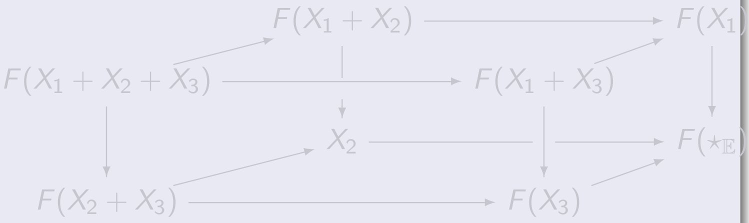

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## Definition (cubical cross-effects)

- $P^F_{X_1,\ldots,X_n}$ = limit of the punctured cube
- $\theta^F_{X_1,\ldots,X_n} : F(X_1 + \cdots + X_n) \to P^F_{X_1,\ldots,X_n}$
- $cr^F_n(X_1,\ldots,X_n) = \ker(\theta^F_{X_1,\ldots,X_n})$ = "total" kernel of the cube
- pointed $F : \mathbb{E} \to \mathbb{E}'$ is *of degree* $\leq n$
  iff $\Xi^F_{X_1,\ldots,X_{n+1}}$ is a limit-cube $\forall X_1,\ldots,X_{n+1}$
  iff $\theta^F_{X_1,\ldots,X_{n+1}}$ is invertible $\forall X_1,\ldots,X_{n+1}$ ($\theta^F$ is strong epi !)
  iff $cr^F_{n+1}(X_1,\ldots,X_{n+1}) = \star_{\mathbb{E}'}$ $\forall X_1,\ldots,X_{n+1}$

## Example (functors of degree $\leq 1$)

- $\theta^F_{X_1,X_2} : F(X_1 + X_2) \to F(X_1) \times F(X_2)$
- $F$ is of degree $\leq 1$ iff $F$ takes sums to products
- $Id_{\mathbb{E}}$ is of degree $\leq 1$ iff $\mathbb{E} = \mathrm{Ab}(\mathbb{E})$.

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$X$ is *$n$-folded* iff $\nabla_X^{n+1}$ factors through $\theta_{X,\ldots,X}$.

## Theorem (BB '17)

TFAE for a semiabelian category $\mathbb{E}$:

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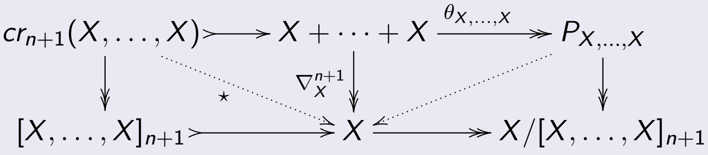


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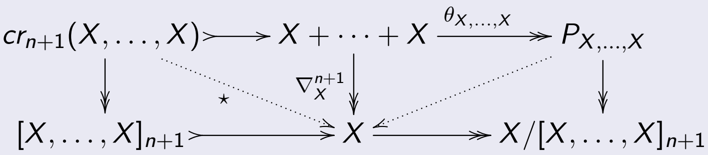


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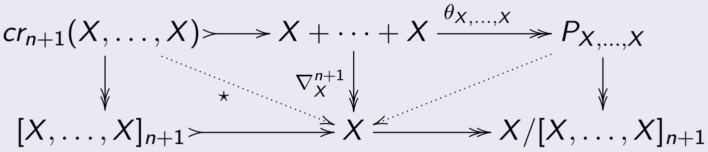


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### Definition (central extensions)

*Central extensions* are strong epis $X \overset{f}{\twoheadrightarrow} Y$ sth. $[X, \ker(f)] = \star_{\mathbb{E}}$.

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An object $X$ is $n$-nilpotent iff it is an *n-fold central extension* of the trivial object, i.e. $X \overset{f_n}{\twoheadrightarrow} X_{n-1} \overset{f_{n-1}}{\twoheadrightarrow} \cdots \twoheadrightarrow X_2 \overset{f_2}{\twoheadrightarrow} X_1 \overset{f_1}{\twoheadrightarrow} \star_{\mathbb{E}}$.

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$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

- $\{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$ and $\{\pm 1, \pm i\} = \mathbb{Z}/4\mathbb{Z}$
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## Definition (Nilpotency)

$\mathrm{Nil}^n(\mathbb{E})$ is the subcategory spanned by the $n$-nilpotent objects.
A category is *n-nilpotent* iff $\mathbb{E} = \mathrm{Nil}^n(\mathbb{E})$.
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A category is *n-nilpotent* iff $\mathbb{E} = \mathrm{Nil}^n(\mathbb{E})$.
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The subcategory $\mathrm{Nil}^n(\mathbb{E})$ is a reflective Birkhoff subcategory of $\mathbb{E}$.

**Example ($n$-folded $\neq$ $n$-nilpotent for $n \geq 2$)**

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O}$$

- $\{\pm 1\} = \mathbb{Z}/2\mathbb{Z}$ and $\{\pm 1, \pm i\} = \mathbb{Z}/4\mathbb{Z}$
- $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ 2-nilpotent and 2-folded group
- $O_{16} = \{\pm 1, \pm e_2, \cdots, \pm e_8\}$ 2-nilpotent, but not 2-folded loop.

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### Proposition (BB '17)

$\mathbb{E}$ is $n$-nilpotent iff for all $X, Y$ the map $\theta_{X,Y} : X + Y \to X \times Y$ exhibits $X + Y$ as an $(n-1)$-fold central extension of $X \times Y$.

### Lemma (nilpotency tower)

The first Birkhoff reflection $I^1 : \mathbb{E} \to \mathrm{Nil}^1(\mathbb{E}) = \mathrm{Ab}(\mathbb{E})$ is *abelianization*.
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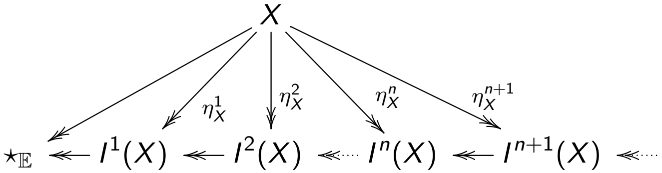

## Corollary

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For a group $X$, $L(X) = \bigoplus_{n\geq 1} L_n(X)$ is a *Lie ring* which is free if $X$ is free. This shows that the properties above hold for groups.

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A Quillen model structure on a bicomplete $\mathbb{E}$ consists of three composable classes of morphisms $\mathrm{cof}_{\mathbb{E}}, \mathrm{we}_{\mathbb{E}}, \mathrm{fib}_{\mathbb{E}}$ such that

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**Proposition (Carboni-Kelly-Pedicchio '93)**

A variety $V_T$ of $T$-algebras is a Mal'cev variety if and only if $U_T : sV_T \to s\mathrm{Sets}$ takes values in fibrant simplicial sets.

**Proposition (Bourn '96)**

Every semiabelian category is a Mal'cev category.

**Corollary**

The simplical objects of a semiabelian variety $V_T$ carry a model structure sth

- we's are the maps inducing a quasi-iso on *Moore complexes*;
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For cofibrant $X$ in $sV_T$ one has $\mathrm{nil}_1^T(X) \leq \mathrm{nil}_2^T(X) \leq \mathrm{nil}_3^T(X)$

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**Corollary (Berstein-Ganea '61, Hovey '93, Biedermann-Dwyer '10)**

For a reduced simplical set $X$ one has

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