Combinatorial models for real configuration spaces and E_n-operads

Clemens Berger

Abstract. We define several partially ordered sets with the equivariant homotopy type of real configuration spaces $F(\mathbb{R}^n, p)$. The main tool is a general method for constructing E_n-suboperads of a given E_∞-operad by appropriate cellular subdivision.

Introduction

The configuration space $F(\mathbb{R}^\infty, p)$ of p-tuples of pairwise distinct points of $\mathbb{R}^\infty$ can serve as universal $\mathfrak{S}_p$-bundle, the symmetric group acting freely by permutation of the p points. The main result of this paper is a combinatorial construction of the natural filtration of $F(\mathbb{R}^\infty, p)$ induced by the finite-dimensional configuration spaces.

More generally, an E_∞-operad with some extra cell structure has a combinatorially defined filtration by E_n-suboperads. As a byproduct, we obtain several partially ordered sets with the equivariant homotopy type of $F(\mathbb{R}^n, p)$. In particular, we rediscover the Smith-filtration [**19**] of Barratt-Eccles' Γ-functor [**4**] and also Milgram's permutohedral models of $F(\mathbb{R}^n, p)$ [**17**], [**3**].

We have tried to concentrate here on the combinatorial aspects of E_n-operads and to trace connections to other similar developments (cf. [**1**], [**11**]) when we were aware of them. We completely left out the application of E_n-operads to n-fold iterated loop spaces and refer the interested reader to [**15**], [**8**], [**5**].

The combinatorial aspects of the theory of E_n-operads have perhaps been underestimated for some time. This is quite surprising, if one considers F. Cohen's already classical computation [**8**] of the homology and cohomology of $F(\mathbb{R}^{n+1}, p)$ which among others identifies (in modern language) the cohomology ring with the Orlik-Solomon n-algebra of the complete graph on p vertices and the homology with the multilinear part of the free Poisson n-algebra on p generators (cf. [**11**]). It would be nice to have a purely combinatorial proof of this result (possibly along these lines) relating it to some surprising combinatorial work (cf. [**2**]).

We have divided our exposition into two parts :

1991 *Mathematics Subject Classification.* 18B35 06A07 20B30.

Key words and phrases. Configuration spaces, cellular E_n-operads, symmetric groups, poset-models, permutohedra.

Part One introduces the language of cellular E_∞-(pre)operads, defines their combinatorial filtration and relates this filtration to the equivariant homotopy type of the configuration spaces $F(\mathbb{R}^n, p)$.

Part Two discusses three basic examples : the configuration preoperad F itself, the simplicial operad Γ and the permutohedral operad J.

I would like to thank Takuji Kashiwabara for helpful remarks and suggestions and Zig Fiedorowicz for his careful and prompt reading of the first draft, which led to an important comparison theorem.

1. Homotopy uniqueness of cellular E_n-operads.

Following Cohen, May and Taylor [**9**] we neglect at first the multiplicative structure of an *operad* and retain only the "functorial" part of the structure which will be sufficient to determine the homotopy types we are interested in.

DEFINITION 1.1. Define $\mathbf{\Lambda}$ to be the category whose objects are the finite (non empty) sets $\mathbf{p} = \{1, 2, \ldots, p\}$ and whose morphisms are the injective maps.

A *preoperad* with values in the category $\mathcal{C}$ is a contravariant functor $\mathcal{O} : \mathbf{\Lambda} \to \mathcal{C}$, written $(\mathcal{O}_p)_{p>0}$ on objects and $\phi^* : \mathcal{O}_q \to \mathcal{O}_p$ on morphisms $\phi \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{q})$.

A *map of preoperads* is a natural transformation of functors. If there is a notion of (weak) equivalence in $\mathcal{C}$, we shall call *(weak)* $\mathbf{\Lambda}$*-equivalence* any map of preoperads $f : \mathcal{O} \to \mathcal{O}'$ such that for each $p > 0$, the induced map $f_p : \mathcal{O}_p \to \mathcal{O}'_p$ is a (weak) equivalence. Two preoperads will then be called (weakly) $\mathbf{\Lambda}$-equivalent if they can be joined by a chain of not necessarily composable (weak) $\mathbf{\Lambda}$-equivalences.

These notions apply in particular to partially ordered, simplicial and topological preoperads. The *nerve* functor $\mathcal{N} : \mathbf{Poset} \to \mathbf{Set}^{\mathbf{\Delta}}$ transforms by composition partially ordered preoperads into simplicial preoperads and the *realization* functor $|-| : \mathbf{Set}^{\mathbf{\Delta}} \to \mathbf{Top}$ transforms simplicial into topological preoperads. Both transformations preserve weak equivalences.

NOTATION 1.2. Each morphism of the category $\mathbf{\Lambda}$ decomposes uniquely in a *bijection* followed by an *increasing* map. For $\phi \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{q})$, we write

$$\phi = \phi^{inc} \circ \phi^{\natural},$$

with $\phi^{\natural} \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{p})$ and $\phi^{inc} \in \mathbf{\Lambda}^{inc}(\mathbf{p}, \mathbf{q}) = \{\phi \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{q}) |\, \phi(i) < \phi(j) \text{ for } i < j\}$.

For p distinct integers $i_1, \ldots, i_p$ in $\mathbf{q}$ we shall denote

$$\phi_{i_1,\ldots,i_p} : \mathbf{p} \to \mathbf{q}$$

the morphism which maps $(1, \ldots, p)$ onto $(i_1, \ldots, i_p)$.

EXAMPLES 1.3. *(a) The symmetric groups and their universal bundles.*
The collection of symmetric groups $\mathfrak{S}_p = \mathbf{\Lambda}(\mathbf{p}, \mathbf{p})$ defines a set-valued preoperad $\mathfrak{S} : \mathbf{\Lambda} \to \mathbf{Set}$ by setting for $\phi \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{q})$:

$$\begin{array}{rcl} \phi^* : \mathfrak{S}_q & \to & \mathfrak{S}_p \\ \sigma & \mapsto & (\sigma \circ \phi)^{\natural}. \end{array}$$

Composing $\mathfrak{S}$ with the universal bundle construction $W : \mathbf{Set} \to \mathbf{Set}^{\mathbf{\Delta}}$ one gets a simplicial preoperad $\Gamma = W \circ \mathfrak{S}$ whose rich combinatorial structure has been studied by Barratt-Eccles [**4**] and Smith [**19**].

(b) The configuration preoperad.

The collection of configuration spaces $F_p = F(\mathbb{R}^\infty, p)$ defines a topological preoperad $F : \mathbf{\Lambda} \to \mathbf{Top}$ by setting for $\phi \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{q})$:

$$\begin{array}{rcl} \phi^* : F_q & \to & F_p \\ (x_1, \dots, x_q) & \mapsto & (x_{\phi(1)}, \dots, x_{\phi(p)}). \end{array}$$

Like Γ_p each F_p is a universal $\mathfrak{S}_p$-bundle. This suggests some relationship between the two. Indeed Smith [**19**] constructed a filtration $\Gamma_p^{(n)}$ of the simplicial set Γ_p which was shown by Kashiwabara [**14**] to be homotopy equivalent to the geometric filtration $F_p^{(n)} = F(\mathbb{R}^n, p) \times \mathbb{R}^\infty$ of the configuration preoperad F. This result was the starting point of our investigation, and we shall see below that this filtered homotopy equivalence is based on some functorially constructed cell decompositions of both preoperads.

(c) The complete graph preoperad.

Let $\mathbb{N}^{\binom{p}{2}}$ denote the cartesian product of $\binom{p}{2}$ copies of the set $\mathbb{N}$ of natural numbers. An element $\mu \in \mathbb{N}^{\binom{p}{2}}$ will be written with a *double index* : $\mu = (\mu_{ij})_{1 \le i < j \le p}$. Such an element is most naturally interpreted as an *edge-labeling* (by natural numbers) of the complete graph on p vertices. The collection of these labeling sets $\mathbb{N}^{\binom{p}{2}}$ extends to a (set-valued) preoperad. For $\phi \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{q})$ the induced map $\phi^* : \mathbb{N}^{\binom{q}{2}} \to \mathbb{N}^{\binom{p}{2}}$ is given by the evident formula :

$$\phi^*(\mu)_{ij} = \begin{cases} \mu_{\phi(i),\phi(j)} & \text{if } \phi(i) < \phi(j); \\ \mu_{\phi(j),\phi(i)} & \text{if } \phi(j) < \phi(i). \end{cases}$$

Definition 1.4. Let $\mathcal{K} : \mathbf{\Lambda} \to \mathbf{Poset}$ be the partially ordered preoperad defined by $\mathcal{K}_p = \mathbb{N}^{\binom{p}{2}} \times \mathfrak{S}_p$ and for $\phi \in \mathbf{\Lambda}(\mathbf{p}, \mathbf{q})$:

$$\begin{array}{rcl} \phi^* : \mathcal{K}_q & \to & \mathcal{K}_p \\ (\mu, \sigma) & \mapsto & (\phi^*(\mu), \phi^*(\sigma)), \end{array}$$

where the partial order on $\mathcal{K}_p$ is given by

$$(\mu, \sigma) \le (\nu, \tau) \Leftrightarrow \forall i < j \text{ either } \phi_{ij}^*(\mu, \sigma) = \phi_{ij}^*(\nu, \tau) \text{ or } \mu_{ij} < \nu_{ij}.$$

Remark 1.5. The universal $\mathfrak{S}_2$-bundle can be realized as the unit-sphere S^∞ in $\mathbb{R}^\infty$, the non trivial element of $\mathfrak{S}_2$ acting as antipodal map. The minimal CW-structure of S^∞ compatible with this action is given by the hemispheres of each dimension. The set of these cells, ordered by inclusion, is canonically isomorphic to $\mathcal{K}_2$. The partially ordered sets $\mathcal{K}_p$ serve to define analogous cell decompositions of the universal $\mathfrak{S}_p$-bundles. Observe in particular that the partial order on $\mathcal{K}_p$ is the least fine partial order such that all maps $\phi^* : \mathcal{K}_p \to \mathcal{K}_2$ are monotone. Formally,

$$(\mu, \sigma) \le (\nu, \tau) \Leftrightarrow \phi_{ij}^*(\mu, \sigma) \le \phi_{ij}^*(\nu, \tau) \text{ for all } i, j.$$

Definition 1.6. Let A be a partially ordered set and X a topological space. A collection $(c_\alpha)_{\alpha \in A}$ of closed contractible subspaces (the *"cells"*) of X will be called a *cellular A-decomposition* of X if the following three conditions hold :

(1) $c_\alpha \subseteq c_\beta \Leftrightarrow \alpha \le \beta$;
(2) the cell inclusions are (closed) cofibrations ;
(3) $X = \varinjlim_A c_\alpha$, so X equals the union of its cells and has the weak topology with respect to its cells.

LEMMA 1.7. *If a topological space X admits a cellular A-decomposition, then there is a cellular homotopy equivalence from X to $|\mathcal{N}A|$.*

PROOF. – Since cell inclusions are closed cofibrations, the homotopy colimit $\text{h-}\varinjlim_A c_\alpha$ contains the ordinary colimit $\varinjlim_A c_\alpha$ as a deformation retract. On the other hand, contracting the cells to a point defines a homotopy equivalence from the homotopy colimit to the realization of the nerve $|\mathcal{N}A| = \text{h-}\varinjlim_A |*|$. □

REMARK 1.8. Condition (1) of a cellular A-decomposition can be replaced by a weaker condition without losing property (1.7). To this purpose let us formally define the *cell-interior* $\breve{c}_\alpha$ to be the difference

$$\breve{c}_\alpha = \quad c_\alpha \setminus (\bigcup_{\beta<\alpha} c_\beta).$$

Suppose now that instead of (1) we have only

$$(1') \qquad \alpha \leq \beta \Rightarrow c_\alpha \subseteq c_\beta, \text{ while equivalence holds if } \breve{c}_\alpha \neq \emptyset.$$

By (1′) and (3), each cell is the union of cells with nonempty interior. Thus, X is the colimit over A as well as the colimit over the partially ordered set A' of cells with nonempty interior. But for A', condition (1) holds, so by Lemma 1.7, there is a homotopy equivalence from X to $|\mathcal{N}A'|$.

On the other hand, by Quillen's Theorem A, the poset inclusion of A' into A induces a homotopy equivalence $|\mathcal{N}A'| \xrightarrow{\sim} |\mathcal{N}A|$ since the "homotopy fibers" $i_\alpha = \{\beta \in A' | \, \beta \leq \alpha\}$ are all contractible, again by Lemma 1.7 : $c_\alpha \xrightarrow{\sim} |\mathcal{N}i_\alpha|, \alpha \in A$. Hence, Lemma 1.7 remains valid for cellular A-decompositions which satisfy only (1′), (2), (3).

To facilitate language we shall call cells with nonempty cell-interior *proper* cells and those with empty cell-interior *improper* cells. What we have shown reads as follows : only the poset of proper cells forms a cell-decomposition in the strict sense, but adjoining improper cells does not modify the homotopy type of the poset as long as the improper cells are contractible.

DEFINITION 1.9. A topological preoperad $\mathcal{O}$ is called a *cellular E_∞-preoperad* if the $\mathfrak{S}_2$-space $\mathcal{O}_2$ admits a cellular $\mathcal{K}_2$-decomposition $(\mathcal{O}_2^{(\alpha)})_{\alpha\in\mathcal{K}_2}$, compatible with the action of $\mathfrak{S}_2$, such that

(1) for each $p > 0$ and each $\alpha \in \mathcal{K}_p$ the formally defined "cell"

$$\mathcal{O}_p^{(\alpha)} = \bigcap_{1\leq i<j\leq p} (\phi_{ij}^*)^{-1}(\mathcal{O}_2^{\phi_{ij}^*(\alpha)})$$

is contractible, and for each $\alpha, \beta \in \mathcal{K}_p$ with $\alpha \leq \beta$ the natural "cell-inclusion" $\mathcal{O}_p^{(\alpha)} \subseteq \mathcal{O}_p^{(\beta)}$ is a cofibration ;

(2) each $\mathfrak{S}_p$-orbit of $\mathcal{O}_p$ contains an *ordered* point, i.e. a point $x \in \mathcal{O}_p$ whose projections $\phi_{ij}^*(x)$ belong to cell-interiors of the form $\breve{\mathcal{O}}_2^{(\mu,id_{\mathbf{2}})}$.

NOTATION 1.10. A cellular E_∞-preoperad $\mathcal{O}$ induces the partially ordered preoperad $\mathcal{K}(\mathcal{O})$ consisting of those elements of the complete graph preoperad $\mathcal{K}$ which index the *proper* cells of $\mathcal{O}$ (cf. 1.8). The formally defined cell-interior $\breve{\mathcal{O}}_p^{(\alpha)}$ can also be defined as an intersection of inverse images of cell-interiors :

$$\breve{\mathcal{O}}_p^{(\alpha)} = \bigcap_{1\le i<j\le p} (\phi_{ij}^*)^{-1}(\breve{\mathcal{O}}_2^{\phi_{ij}^*(\alpha)}).$$

This shows that the partially ordered sets $\mathcal{K}(\mathcal{O})_p = \{\alpha \in \mathcal{K}_p \,|\, \breve{\mathcal{O}}_p^{(\alpha)} \neq \emptyset\}$ are indeed subposets of $\mathcal{K}_p$ closed under the operations of the category $\mathbf{\Lambda}$.

On the other hand, the natural filtration of the complete graph preoperad $\mathcal{K}$ by $\mathcal{K}^{(n)} = \{(\mu,\sigma) \in \mathcal{K} |\, \mu_{ij} < n \text{ for } i<j\}$ induces a filtration $\mathcal{O}^{(n)}$ of the cellular E_∞-preoperad $\mathcal{O}$; explicitly :

$$\mathcal{O}_p^{(n)} = \bigcup_{\alpha\in\mathcal{K}_p^{(n)}} \mathcal{O}_p^{(\alpha)}.$$

It follows at once that for each p, the filtration of $\mathcal{O}_p$ is induced from the canonical filtration of $\mathcal{O}_2$ through the projections ϕ_{ij}^*.

Note the dimensional shift : $\mathcal{O}_2^{(n)}$ has the equivariant homotopy type of an $(n-1)$-dimensional sphere.

Topological preoperads of the form $\mathcal{O}^{(n)}$ (for a cellular E_∞-preoperad $\mathcal{O}$) will be called *cellular E_n-preoperads.*

THEOREM 1.11. *Let $\mathcal{O}$ be a cellular E_∞-preoperad.*

(a) For each p, the set of proper cells $(\mathcal{O}_p^{(\alpha)})_{\alpha\in\mathcal{K}(\mathcal{O})_p}$ defines a cellular $\mathcal{K}(\mathcal{O})_p$-decomposition of $\mathcal{O}_p$.

(b) The inclusion of $\mathcal{K}(\mathcal{O})$ in $\mathcal{K}$ is a filtered $\mathbf{\Lambda}$-equivalence of partially ordered preoperads. In particular, there is a $\mathbf{\Lambda}$-equivalence $\mathcal{O}^{(n)} \xrightarrow{\sim} |\mathcal{N}\mathcal{K}^{(n)}|$. Hence, any two cellular E_n-preoperads are $\mathbf{\Lambda}$-equivalent ($1 \le n \le \infty$).

PROOF. – In view of Remark 1.8, it remains to show that the cells $\mathcal{O}_p^{(\alpha)}, \alpha \in \mathcal{K}_p$, satisfy the weak form of a cell-decomposition, i.e. conditions (1′), (2), (3) of (1.6-1.8). For this, let $\mathcal{O}_p^{(\alpha)}$ be a *proper* cell such that $\mathcal{O}_p^{(\alpha)} \subseteq \mathcal{O}_p^{(\beta)}$ and suppose (by contraposition) that α does not precede β in $\mathcal{K}_p$. Then there is a map $\phi \in \mathbf{\Lambda}(\mathbf{2},\mathbf{p})$ such that $\phi^*(\alpha)$ does not precede $\phi^*(\beta)$ in $\mathcal{K}_2$; we thus have empty intersections $\breve{\mathcal{O}}_2^{\phi^*(\alpha)} \cap \mathcal{O}_2^{\phi^*(\beta)}$ and $\breve{\mathcal{O}}_p^{(\alpha)} \cap \mathcal{O}_p^{(\beta)}$, in contradiction with the hypothesis.

Furthermore, as each cell is the union of proper cells, the colimit condition (3) is equivalent to the statement that the space $\mathcal{O}_p$ decomposes into a disjoint union of cell-interiors $\breve{\mathcal{O}}_p^{(\alpha)}, \alpha \in \mathcal{K}(\mathcal{O})_p$. Now, given a point $x \in \mathcal{O}_p$, there are unique indices $(\mu_{ij}^x, \sigma_{ij}^x) \in \mathcal{K}_2$ such that $\phi_{ij}^*(x)$ belongs to the cell-interior $\breve{\mathcal{O}}_2^{(\mu_{ij}^x,\sigma_{ij}^x)}$. By condition (1.9.2) there is also a permutation $\sigma \in \mathfrak{S}_p$ such that $(\sigma^{-1})^*(x)$ is an *ordered* point, which is unique because of the relation $\sigma_{ij}^x = \phi_{ij}^*(\sigma)$ for all i,j. The cell-interior $\breve{\mathcal{O}}_p^{(\alpha)}$ containing x has thus index $\alpha = (\mu,\sigma)$ with $\mu = (\mu_{ij}^x)_{1\le i<j\le p}$ and $\sigma = (\sigma_{ij}^x)_{1\le i<j\le p}$. □

REMARK 1.12. The preceding theorem suggests a slight generalization of the concept of a cellular E_∞-preoperad $\mathcal{O}$. All we need for the comparison with the complete graph preoperad is the contractibility of the *proper* cells and the filtered Λ-equivalence $\mathcal{K}(\mathcal{O}) \xrightarrow{\sim} \mathcal{K}$. The contractibility of the improper cells is not the only way of obtaining the latter equivalence, cf. Quillen's Theorem B [**18**].

The configuration preoperad F is actually a cellular preoperad with contractible proper cells but some noncontractible improper cells, yet the inclusion of $\mathcal{K}(F)$ into

$\mathcal{K}$ is a filtered Λ-equivalence. Moreover, the cell-structure of F gives $\mathfrak{S}_p$-equivariant homeomorphisms $F_p^{(n)} \cong F(\mathbb{R}^n, p) \times \mathbb{R}^\infty$ relating very naturally the filtration of the complete graph preoperad to the finite-dimensional configuration spaces $F(\mathbb{R}^n, p)$.

On the other hand, the *Smith-filtration* [**19**] of $\Gamma_p = W\mathfrak{S}_p$ coincides with the filtration $\Gamma_p^{(n)}$ formally derived from the cellular E_∞-structure of Γ. The comparison theorem thus gives the following corollary which was conjectured by Smith and proved by Kashiwabara [**14**] :

COROLLARY 1.13. *The realization of the simplicial set* $\Gamma_p^{(n)}$ *has the same* $\mathfrak{S}_p$-*equivariant homotopy type as the real configuration space* $F(\mathbb{R}^n, p)$.

The above corollary is true for each cellular E_n-preoperad $\mathcal{O}^{(n)}$. In the next chapter we shall examine several cellular E_n-preoperads from a combinatorial viewpoint. Often, they come equipped with a combinatorially defined multiplication

$$\begin{array}{rcl} m^{\mathcal{O}^{(n)}}_{i_1\dots i_p} : \mathcal{O}^{(n)}_p \times \mathcal{O}^{(n)}_{i_1} \times \cdots \times \mathcal{O}^{(n)}_{i_p} & \to & \mathcal{O}^{(n)}_{i_1+\cdots+i_p} \\ (z, z_1, \dots, z_p) & \mapsto & z(z_1, \dots, z_p) \end{array}$$

turning them into an E_n-operad. We refer the reader to [**16**], [**15**] or [**5**] for the exact definition of an operad and more specifically for the relationship between E_n-operads and n-fold iterated loop spaces. The *degeneracy* operators of a (unital) operad define the actions by increasing maps of the category $\mathbf{\Lambda}$ so that each (unital) operad has an underlying preoperad structure, see [**16**], Variant 4(iii).

DEFINITION 1.14. An operad $\mathcal{O}$ is called a *cellular* E_∞*-operad* if the underlying preoperad is a cellular E_∞-preoperad such that the multiplication $m^{\mathcal{O}}_{i_1\dots i_p}$ preserves the cellular structure in the sense specified below (1.15b).

We then call the suboperads $\mathcal{O}^{(n)}$ *cellular* E_n*-operads.*

EXAMPLES 1.15. (a) *The permutation operad.*
The set-valued preoperad $\mathfrak{S}$ is in fact an operad with the obvious unit $1 \in \mathfrak{S}_1$ and multiplication given by

$$\begin{array}{rcl} m^{\mathfrak{S}}_{i_1\dots i_p} : \mathfrak{S}_p \times \mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_p} & \to & \mathfrak{S}_{i_1+\cdots+i_p} \\ (\sigma; \sigma_1, \dots, \sigma_p) & \mapsto & \sigma(i_1, \dots, i_p) \circ (\sigma_1 \oplus \cdots \oplus \sigma_p), \end{array}$$

where $\sigma(i_1, \dots, i_p)$ permutes the p subsets $\mathbf{i_k} \hookrightarrow \mathbf{i_1} + \cdots + \mathbf{i_p}$ according to σ.

(b) *The complete graph operad* $\mathcal{K}$.
The preoperad $\mathcal{K}$ is an operad with obvious unit $1 \in \mathcal{K}_1$ and multiplication

$$\begin{array}{rcl} m^{\mathcal{K}}_{i_1\dots i_p} : \mathcal{K}_p \times \mathcal{K}_{i_1} \times \cdots \times \mathcal{K}_{i_p} & \to & \mathcal{K}_{i_1+\cdots+i_p} \\ ((\mu, \sigma); (\mu_1, \sigma_1), \dots, (\mu_p, \sigma_p)) & \mapsto & (\mu(\mu_1, \dots, \mu_p), \sigma(\sigma_1, \dots, \sigma_p)), \end{array}$$

where $\sigma(\sigma_1, \dots, \sigma_p) = \sigma(i_1, \dots, i_p) \circ (\sigma_1 \oplus \cdots \oplus \sigma_p)$ as above, and where the edge-labeling $\mu(\mu_1, \dots, \mu_p)$ of the complete graph on $i_1 + \cdots + i_p$ vertices is defined by the following formula (for sake of precision $\psi_r : \mathbf{i_r} \hookrightarrow \mathbf{i_1} + \cdots + \mathbf{i_p}$ denotes the canonical inclusion) :

$$\mu(\mu_1, \dots, \mu_p)_{jk} = \begin{cases} (\mu_r)_{\psi_r^{-1}(j), \psi_r^{-1}(k)} & \text{if } j, k \in \psi_r(\mathbf{i_r}), \\ \mu_{rs} & \text{if } j \in \psi_r(\mathbf{i_r}) \text{ and } k \in \psi_s(\mathbf{i_s}), r < s. \end{cases}$$

In other words, on edges of the complete subgraph spanned by $\psi_r(\mathbf{i_r})$ the labeling $\mu(\mu_1, \dots, \mu_p)$ coincides with μ_r, whereas on edges joining vertices of different subsets $\psi_r(\mathbf{i_r})$ and $\psi_s(\mathbf{i_s})$ the labeling $\mu(\mu_1, \dots, \mu_p)$ is induced by μ.

The complete graph operad $\mathcal{K}$ is thus a cellular E_∞-operad, naturally filtered by suboperads $\mathcal{K}^{(n)}$. They will serve as *universal models* for cellular E_n-operads. Indeed, given an arbitrary cellular E_∞-operad $\mathcal{O}$, we assume that the multiplication $m^{\mathcal{O}}_{i_1\dots i_p}$ sends each cell-product into the cell prescribed by the complete graph operad:

$$m^{\mathcal{O}}_{i_1\dots i_p}(\mathcal{O}_p^{(\mu,\sigma)}\times\mathcal{O}_{i_1}^{(\mu_1,\sigma_1)}\times\cdots\times\mathcal{O}_{i_p}^{(\mu_p,\sigma_p)})\subseteq\mathcal{O}_{i_1+\cdots+i_p}^{(\mu(\mu_1,\dots,\mu_p),\sigma(\sigma_1,\dots,\sigma_p))}$$

This implies that multiplication is filtration-preserving, making our definition of a cellular E_n-operad meaningful. Both parts of the following theorem are due to Zig Fiedorowicz and improve considerably an earlier "up to homotopy" version. In particular, cellular E_n-operads are actually E_n-operads in May's [**15**] sense endowed with some extra cell-structure.

THEOREM 1.16. *(Fiedorowicz). Any two cellular E_n-operads are multiplicatively $\mathbf{\Lambda}$-equivalent (i.e. equivalent as operads). Moreover, the little n-cubes operad of Boardman-Vogt has the structure of a cellular E_n-operad.* $(1\le n\le\infty)$

PROOF. – The main lemma of Section 5 of [**1**] shows that the collection of homotopy colimits $(\text{h-}\varinjlim_{\alpha\in\mathcal{K}_p}\mathcal{O}_p^{(\alpha)})_{p>0}$ defines a topological operad. This operad retracts by multiplicative $\mathbf{\Lambda}$-equivalences onto the given E_∞-operad $\mathcal{O}$ as well as onto the complete graph operad $|\mathcal{N}\mathcal{K}|$, cf. 1.7-1.8. It follows that any cellular E_n-operad is multiplicatively $\mathbf{\Lambda}$-equivalent to the n-th filtration of the complete graph operad, whence the first part of the theorem.

Let $\mathcal{C}([0,1]^n,p)$ denote the space of p-fold configurations of (open) "little n-cubes" (cf. [**7**], [**15**]) and $\mathcal{C}_p$ the inductive limit $\varinjlim_n\mathcal{C}([0,1]^n,p)$, where $\mathcal{C}([0,1]^n,p)$ is embedded in $\mathcal{C}([0,1]^{n+1},p)$ as the space of those little $(n+1)$-cubes having last coordinate equal to $id_{]0,1[}$.

For $(c_1,c_2)\in\mathcal{C}_2$ we write $c_1\square_\mu c_2$ if c_1 and c_2 are separated by a hyperplane H_i perpendicular to the i-th coordinate axis for some $i\le\mu+1$ such that, whenever there is no separating hyperplane H_i for $i<\mu+1$, the left cube c_1 lies on the negative side of $H_{\mu+1}$ and the right cube c_2 on the positive side of $H_{\mu+1}$. In the latter case we write more precisely $c_1\boxdot_\mu c_2$.

For $(\mu,\sigma)\in\mathcal{K}_p$ we then define the associated cell by

$$\mathcal{C}_p^{(\mu,\sigma)}=\{(c_1,c_2,\dots,c_p)\in C_p\,|\,c_i\underset{\mu_{ij}}{\square}c_j\text{ if }\sigma(i)<\sigma(j),\text{ and }c_j\underset{\mu_{ij}}{\square}c_i\text{ if }\sigma(j)<\sigma(i)\}.$$

These cells endow the little cubes operad $(\mathcal{C}_p)_{p>0}$ with the structure of a cellular E_∞-operad. In particular, the little cubes multiplication plainly preserves this cellular structure in the aforementioned sense (1.15b). The only subtle point is the *contractibility* of the cells; we shall sketch a proof.

Suppose first that the cell $\mathcal{C}_p^{(\mu,\sigma)}$ contains an interior point $(c_1,\dots,c_p)\in\check{\mathcal{C}}_p^{(\mu,\sigma)}$. By the definition of the cell-interior (1.8), this means that $c_i\boxdot_{\mu_{ij}}c_j$ if $\sigma(i)<\sigma(j)$, and $c_j\boxdot_{\mu_{ij}}c_i$ if $\sigma(j)<\sigma(i)$. Furthermore, if $(\mu,\sigma)\in\mathcal{K}_p^{(n)}$, the cell $\mathcal{C}_p^{(\mu,\sigma)}$ projects onto $\mathcal{C}_p^{(\mu,\sigma)}\cap\mathcal{C}([0,1]^n,p)$ by a fibration with contractible fibers; it will thus be sufficient to show the base is contractible. Indeed, the projected cell contracts to the projected configuration $(\bar{c}_1,\dots,\bar{c}_p)\in\check{\mathcal{C}}_p^{(\mu,\sigma)}\cap\mathcal{C}([0,1]^n,p)$ by an n-step *affine* contraction which deforms the little n-cubes *coordinatewise*, beginning with the last coordinate and ending with the first. The descending order guarantees that the contraction stays within $\mathcal{C}_p^{(\mu,\sigma)} \cap \mathcal{C}([0,1]^n, p)$ since the appropriate separating hyperplanes exist at each moment of the deformation.

It remains to show that *improper* cells are contractible. In our case, even more is true ; each cell $\mathcal{C}_p^{(\mu,\sigma)}$ contains a unique maximal proper cell $\mathcal{C}_p^{(\hat{\mu},\hat{\sigma})}$ with which it can be identified, or equivalently : the inclusion of $\mathcal{K}(\mathcal{C})_p$ into $\mathcal{K}_p$ has a right adjoint $(\mu,\sigma) \mapsto (\hat{\mu},\hat{\sigma})$. The explicit formula for this right adjoint is given by $\hat{\sigma} = \sigma$ and $\hat{\alpha}_{ij} = \min\{a \mid \alpha_{i,i_1} = \alpha_{i_1,i_2} = \cdots = \alpha_{i_s,j} = a \text{ for } i < i_1 < \cdots < i_s < j\}$, where we have used the indexing 2.2 for $\alpha = (\mu,\sigma)$ and $\hat{\alpha} = (\hat{\mu},\hat{\sigma})$.

The suboperad $\mathcal{C}^{(n)}$ of the little cubes operad is thus a cellular E_n-operad which projects, as above, onto the little n-cubes operad by a multiplicative $\mathbf{\Lambda}$-equivalence. □

2. On the combinatorial structure of cellular E_n-operads.

As Theorem 1.11 suggests there is some fruitful interplay between the combinatorial structure and the geometry of a cellular E_n-operad $\mathcal{O}^{(n)}$ because the subposet $\mathcal{K}(\mathcal{O})_p^{(n)}$ of $\mathcal{K}_p^{(n)}$ defined by the proper cells represents in its own right a combinatorial model of the equivariant homotopy type of the configuration space $F(\mathbb{R}^n, p)$. So, there might be cellular E_n-operads which are "combinatorially smaller" than others.

We shall compare here three cellular E_∞-(pre)operads : the *configuration* preoperad F with its natural filtration by "dimension", the *simplicial* operad Γ of Barratt-Eccles with its Smith-filtration and the *permutohedral* operad J which is based on Milgram's combinatorial models of iterated loop spaces.

Example 2.1. *The cell structure of the configuration preoperad F.*
The points of $\mathbb{R}^\infty$ will be written as real number series $(x^{(i)})_{i\geq 0}$ satisfying $x^{(i)} = 0$ for large i. For $x, y \in \mathbb{R}^\infty$ we introduce different semi-order relations by

$$x \underset{i}{\leq} y \text{ iff } x^{(i)} \leq y^{(i)} \text{ and } x^{(k)} = y^{(k)} \text{ for } k > i,$$

and similarly for $x \underset{i}{<} y$.

By the very definition of a cellular E_∞-preoperad, the cell decomposition of F_p relies on the cell decomposition of F_2. It is often possible and convenient to give explicit cell-decompositions for each p and to verify a posteriori that these cell-decompositions satisfy the necessary properties and relations.

For $(\mu,\sigma) \in \mathcal{K}_p$ we find :

$$F_p^{(\mu,\sigma)} = \{(x_1,\ldots,x_p) \in F_p \mid x_i \underset{\mu_{ij}}{\leq} x_j \text{ if } \sigma(i) < \sigma(j), \text{ and } x_j \underset{\mu_{ij}}{\leq} x_i \text{ if } \sigma(j) < \sigma(i)\}.$$

It is easy to check that for $p = 2$ this defines a cellular $\mathcal{K}_2$-decomposition of F_2 which retracts by a $\mathfrak{S}_2$-deformation onto the canonical $\mathcal{K}_2$-decomposition of the unit sphere in $\mathbb{R}^\infty$. Note that the (formal) cell-interior $\breve{F}_p^{(\sigma,\mu)}$ is obtained by replacing everywhere $\leq$ by $<$; in fact, this is true for $p = 2$ by direct verification and follows for the general case from the definition of the cell-interiors (cf. 1.10).

In view of 2.3c and Remark 1.12, it is sufficient to show that *proper* cells are contractible; there are actually improper cells with several components, for example $\mu_{12} = \mu_{13} = 0$, $\mu_{23} = 1$ defines an improper cell $F_3^{(\mu,id_3)}$ with two components. Now, using 2.3b and the affine structure of F_p, each proper cell contracts conically to any of its interior points. Moreover, proper cells are defined by inequalities of coordinates, so they are (up to a factor $\mathbb{R}^\infty$) polyhedral cones in some finite dimensional euclidean space whence cell-inclusions are cofibrations. Finally, condition 1.9.2 is also satisfied, since a point $(x_1, x_2, \dots, x_p) \in F_p$ is ordered iff $x_1 \underset{\mu_{12}}{<} x_2 \underset{\mu_{23}}{<} \cdots \underset{\mu_{p-1,p}}{<} x_p$ and 2.3b shows that each point of F_p is of this form up to permutation.

Notation 2.2. For $\alpha = (\mu, \sigma) \in \mathcal{K}_p$ it is often convenient to use the following indexing : $\alpha_{ij} = (\sigma^{-1})^*(\mu)_{ij}$, which is equivalent to $\alpha = \sigma^*((\alpha_{ij})_{1\leq i<j\leq p}, id_{\mathbf{p}})$.

Proposition 2.3. *Let F be the configuration preoperad.*
(a) The posets of proper cells are given by

$$\begin{aligned}\mathcal{K}(F)_p &= \{\alpha \in \mathcal{K}_p \mid \alpha_{ik} = \max(\alpha_{ij}, \alpha_{jk}) \text{ for } i<j<k\} \\ &= \{\alpha \in \mathcal{K}_p \mid \alpha_{ij} = \max_{i\leq k<j} \alpha_{kk+1}\}.\end{aligned}$$

(b) The cell-interior $\breve{F}_p^{(\alpha)}$ for $\alpha = (\mu, \sigma) \in \mathcal{K}(F)_p$ is given by :

$$\breve{F}_p^{(\alpha)} = \{(x_1, \dots, x_p) \in F_p \mid x_{\sigma^{-1}(1)} \underset{\alpha_{12}}{<} x_{\sigma^{-1}(2)} \underset{\alpha_{23}}{<} \cdots \underset{\alpha_{p-1,p}}{<} x_{\sigma^{-1}(p)}\}$$

(c) The inclusion of $\mathcal{K}(F)$ into $\mathcal{K}$ is a filtered Λ-equivalence of partially ordered preoperads. Moreover, each intermediate preoperad $\mathcal{K}'$ breaks the latter inclusion into two filtered Λ-equivalences $\mathcal{K}(F) \overset{\sim}{\hookrightarrow} \mathcal{K}' \overset{\sim}{\hookrightarrow} \mathcal{K}$.

Proof. – The existence of an interior point in $F_p^{(\mu,\sigma)}$ implies that for each triple index (ijk) such that $\phi_{ij}^*(\sigma) = \phi_{jk}^*(\sigma)$ we have either

$$x_i \underset{\mu_{ij}}{<} x_j \underset{\mu_{jk}}{<} x_k \text{ and } x_i \underset{\mu_{ik}}{<} x_k$$

or the opposite inequalities, thus in both cases $\mu_{ik} = \max(\mu_{ij}, \mu_{jk})$. This shows that proper cells $F_p^{(\alpha)}$ satisfy $\alpha_{ik} = \max(\alpha_{ij}, \alpha_{jk})$ for all $i<j<k$. Conversely, if the latter property holds for an index $\alpha \in \mathcal{K}_p$ then the cell-interior $\breve{F}_p^{(\alpha)}$ contains all points of the form $x_{\sigma^{-1}(1)} \underset{\alpha_{12}}{<} x_{\sigma^{-1}(2)} \underset{\alpha_{23}}{<} \cdots \underset{\alpha_{p-1,p}}{<} x_{\sigma^{-1}(p)}$, so it is clearly nonempty, proving (a) and (b).

For (c), it will be sufficient to show that the induced inclusion of quotient categories $\mathcal{K}(F)_p/\mathfrak{S}_p \to \mathcal{K}_p/\mathfrak{S}_p$ admits a filtration-preserving right adjoint. Indeed, since the action of the symmetric group is free, the nerve of the quotient map is a Kan fibration, whence (by the five lemma) the inclusion of $\mathcal{K}(F)_p$ into $\mathcal{K}_p$ is an equivalence iff the quotient inclusion is. Furthermore, restriction of the right adjoint to $\mathcal{K}'_p/\mathfrak{S}_p$ yields the second part of (c).

The $\mathfrak{S}_p$-invariance of the indexing $(\alpha_{ij})_{1\leq i<j\leq p}$ for $\alpha = (\mu, \sigma) \in \mathcal{K}_p$ defines a canonical bijection between "labelings" and $\mathfrak{S}_p$-orbits. Under this bijection the morphisms $\alpha \overset{\rho}{\to} \beta$ of the quotient category $\mathcal{K}_p/\mathfrak{S}_p$ correspond to permutations $\rho \in \mathfrak{S}_p$ such that $(\alpha, id_{\mathbf{p}}) \leq \rho^*(\beta, id_{\mathbf{p}})$ in $\mathcal{K}_p$.

We now define the right adjoint $\mathcal{K}_p/\mathfrak{S}_p \to \mathcal{K}(F)_p/\mathfrak{S}_p : \alpha \mapsto \hat{\alpha}$ by the components of the counit of the adjunction, i.e. by a family of universal maps

$$\hat{\alpha} \overset{id}{\to} \alpha, \text{ where } \hat{\alpha}_{ij} = \max_{i\leq k<j} \min_{r\leq k<s} \alpha_{rs}.$$

It follows from (a) that $\hat{\alpha}$ belongs to $\mathcal{K}(F)_p/\mathfrak{S}_p$ and that for each $\beta \overset{\rho}{\to} \alpha$ such that $\beta \in \mathcal{K}(F)_p/\mathfrak{S}_p$ we get $(\beta, id_{\mathbf{p}}) \leq \rho^*(\hat{\alpha}, id_{\mathbf{p}}) \leq \rho^*(\alpha, id_{\mathbf{p}})$ which gives the desired universal property. □

REMARK 2.4. The poset $\mathcal{K}(F)_p^{(n)}$ can be found in Getzler-Jones' article [**11**] under the name *lexicographical cell-decomposition* of $F(\mathbb{R}^n,p)$. As their description is sligthly different from ours we shall recall it here, especially since this second description will reappear quite naturally when dealing with the permutohedral operad.

DEFINITION 2.5. An *ordered partition* of an integer $p>0$ is an ordered decomposition $p=i_1+\cdots+i_r$ into a sum of integers $i_k>0$.

We associate to an ordered partition three combinatorially equivalent objects:

(1) the *direct sum decomposition* $\mathbf{i_1}+\cdots+\mathbf{i_r}\cong\mathbf{p}$;
(2) the *subgroup* $\mathfrak{S}_{(i_1,\dots,i_r)}$ of $\mathfrak{S}_p$ consisting of all permutations of the form $m^{\mathfrak{S}}_{i_1,\dots,i_r}(1;\sigma_1,\dots,\sigma_r)$, which is canonically isomorphic to $\mathfrak{S}_{i_1}\times\cdots\times\mathfrak{S}_{i_r}$;
(3) the *bar code* $(\epsilon_i)_{1\le i<p}\in[1]^{p-1}$ where ϵ_i is 1 (resp. 0) iff the ordered partition separates (resp. does not separate) the integers i and $i+1$.

We partially order the set of ordered partitions by *refinement*. This order is opposite to *subgroup-inclusion* but equals the *product order* on the bar codes $(\epsilon_i)_{1\le i<p}\in[1]^{p-1}$, where $[1]=\{0<1\}$. So we have

$$(i_1,\dots,i_r)\preceq(j_1,\dots,j_s)$$

iff one of the following three equivalent conditions is satisfied :

(1) there is an ordered partition $s=k_1+\cdots+k_r$ such that

$$\begin{aligned}i_1&=j_1+\cdots+j_{k_1},\\ i_2&=j_{k_1+1}+\cdots+j_{k_1+k_2},\\ &\dots\\ i_r&=j_{k_1+\cdots+k_{r-1}+1}+\cdots+j_{k_1+\cdots+k_r};\end{aligned}$$

(2) $\mathfrak{S}_{(i_1,\dots,i_r)}\supseteq\mathfrak{S}_{(j_1,\dots,j_s)}$
(3) the associated bar codes $(\epsilon_i)_{1\le i<p},(\zeta_i)_{1\le i<p}$ verify $\epsilon_i\le\zeta_i$ for all i.

The correspondence between ordered partitions and bar codes extends naturally to a correspondence

$$\{\text{ascending chains of ordered partitions}\}\leftrightarrow\{\textit{multiple bar codes}\}$$

$$part_1\preceq part_2\preceq\cdots\preceq part_l\leftrightarrow(\epsilon_i)_{1\le i<p}\in\mathbb{N}^{p-1}$$

where the *multiple bar code* is simply obtained by summing up the bar codes of the chain-elements; conversely, $part_k$ is the least fine ordered partition separating all couples $i,i+1$ such that $\epsilon_i>l-k$.

PROPOSITION 2.6. *The natural bijection between $\mathcal{K}(F)_p^{(n)}$ and $[n-1]^{p-1}\times\mathfrak{S}_p$ identifies $\mathcal{K}(F)_p^{(n)}$ with the covering category of a $\mathfrak{S}_p$-valued "shuffle" functor $Sh_p^{(n)}:\underline{[n-1]^{p-1}}\to\mathbf{Set}$.*

PROOF. – By 2.3a, each index $\alpha=(\mu,\sigma)\in\mathcal{K}(F)_p^{(n)}$ is uniquely determined by the permutation σ and the integer-family $(\alpha_{k,k+1})_{1\le k<p}\in[n-1]^{p-1}$. It remains to determine the order relation induced by $\mathcal{K}(F)_p^{(n)}$. The partial order on $\mathcal{K}_p^{(n)}$ can be characterized as follows, cf. 2.3c :

$$\alpha=(\mu,\sigma)\le\beta=(\nu,\tau)\Leftrightarrow\begin{cases}\alpha_{ij}\le\beta_{\rho(i),\rho(j)} & \text{if }\rho(i)<\rho(j),\\ \alpha_{ij}<\beta_{\rho(j),\rho(i)} & \text{if }\rho(j)<\rho(i),\end{cases}$$

where the permutation ρ is determined by $\tau = \rho \circ \sigma$.

This leads to the following category structure on $[n-1]^{p-1}$:

$$\underline{[n-1]^{p-1}}((\epsilon_k)_{1\leq k<p},(\zeta_k)_{1\leq k<p}) =$$
$$\{\rho \in \mathfrak{S}_p \mid \max_{i\leq k<j} \epsilon_k \leq \max_{\rho(i)\leq k<\rho(j)} \zeta_k \text{ if } \rho(i)<\rho(j), \text{ and}$$
$$\max_{i\leq k<j} \epsilon_k < \max_{\rho(j)\leq k<\rho(i)} \zeta_k \text{ if } \rho(j)<\rho(i)\}$$

Indeed, the above characterization of the order relation identifies $\mathcal{K}(F)^{(n)}_p$ with the covering category (see [**18**]) of the functor

$$\begin{array}{rcl} Sh^{(n)}_p : \underline{[n-1]^{p-1}} & \to & \mathbf{Set} \\ (\epsilon_k)_{1\leq k<p} & \mapsto & \mathfrak{S}_p \\ \rho & \mapsto & \rho_*. \end{array}$$

. □

REMARK 2.7. The name "shuffle" functor is chosen because, for $n=2$, *adjacent* elements of the poset $[1]^{p-1}$ define a *morphism-set* in $\underline{[1]^{p-1}}$ containing only shuffle-orderings, i.e. inverses of shuffle-permutations.

The categories $\underline{[n-1]^{p-1}}$ are models for the quotient spaces $F(\mathbb{R}^n,p)/\mathfrak{S}_p$. In particular, for $n=2$, the nerve of $\underline{[1]^{p-1}}$ is a classifying space for the braid group B_p on p strands; this model appears already in a paper of Greenberg [**12**], see also Fox and Neuwirth's combinatorial deduction of Artin's presentation of the braid groups B_p [**10**].

As an illustration, let us have a look at the category $\underline{[1]^2}$, whose nerve is thus a classifying space for B_3 (we use the bar code for the objects) :

[1|2|3]

$\mathfrak{S}_1\times\mathfrak{S}_2$ ⇉ ⇇ $\mathfrak{S}_2\times\mathfrak{S}_1$

[1|2 3] [1 2|3]

$(1,2)-shuffles^{-1}$ ⇶ ⇶ $(2,1)-shuffles^{-1}$

[1 2 3]

Balteanu, Fiedorowicz, Schwänzl and Vogt [**1**] embed the covering category $\mathcal{K}(F)^{(n)}_p$ in the "multilinear part" $\mathcal{M}_n(p)$ of the *free n-fold monoidal category* generated by p objects. The collection of categories $(\mathcal{M}_n(p))_{p>0}$ defines a cellular E_n-operad. In particular, nerves of connected n-fold monoidal categories are n-fold iterated loop spaces. A central role in their proof is played by the so-called *Coherence Theorem*, which roughly states that the category $\mathcal{M}_n(p)$ underlies a poset. Fiedorowicz pointed out that there is a natural poset-inclusion of $\mathcal{M}_n(p)$ into $\mathcal{K}^{(n)}_p$ compatible with the operad structure and generalizing the $K^{(n)}_2$-decomposition of the "octahedral" $(n-1)$-sphere $\mathcal{M}_n(2)$. There is actually a chain of Λ-equivariant poset-inclusions

$$\mathcal{K}(F)_p^{(n)} \subset \mathcal{M}_n(p) \subset \mathcal{K}(\mathcal{C})_p^{(n)} \subset \mathcal{K}_p^{(n)}$$

proving geometrically (1.16, [**1**]) as well as combinatorially (2.3c) that the n-fold monoidal operad $(\mathcal{M}_n(p))_{p>0}$ is a cellular E_n-operad.

Getzler-Jones [**11**] also embed the poset $\mathcal{K}(F)_p^{(n)}$ in a larger poset which corresponds to the cell structure of Fulton-MacPherson's compactification of $F(\mathbb{R}^n,p)$. It turns out that the latter cell structure is in some precise sense "freely generated" by the former via the formalism of planary trees (see also [**7**]). The underlying combinatorics are intimately related to Stasheff's *associahedra*.

There is a similar relationship between $\mathcal{K}(F)_p^{(n)}$ and $\mathcal{M}_n(p)$ in the form of a canonical surjective map $\mathcal{K}(F)_p^{(n)} \times \mathcal{A}_p \twoheadrightarrow \mathcal{M}_n(p)$, where $\mathcal{A}_p$ denotes the set of all bracketings of a p-element set. The inclusion of $\mathcal{K}(F)_p^{(n)}$ into $\mathcal{M}_n(p)$ endows each element $(\mu, id_p) \in \mathcal{K}(F)_p^{(n)}$ with a natural bracketing such as

$$((1 \underset{\mu_{12}}{\Box} 2) \underset{\mu_{23}}{\Box} (3 \underset{\mu_{34}}{\Box} \cdots (\cdots \underset{\mu_{p-2,p-1}}{\Box} p-1) \underset{\mu_{p-1,p}}{\Box} p)).$$

In this setting, Property 2.3a becomes equivalent to the condition that the indices of the composition laws increase "from inside to outside" in the bracketing. If all $p-1$ composition indices are distinct, the permutation of the composition indices defines a map $\mathfrak{S}_{p-1} \to \mathcal{A}_p$ studied in Tonk's paper in this volume [**20**].

EXAMPLE 2.8. *The cell structure of the simplicial operad* Γ.
As the universal bundle functor W commutes with cartesian products, unit and multiplication of the permutation operad (1.15a) induce a unit and a multiplication of the composite functor $\Gamma = W\mathfrak{S}$, turning it into a cellular E_∞-operad, as we shall see.

The cells of $\Gamma_p = W\mathfrak{S}_p$ are realized by certain simplicial subsets $\Gamma_p^{(\mu,\sigma)}$ of Γ_p. We recall that a k-simplex of Γ_p is written as a $(k+1)$-tuple of elements of $\mathfrak{S}_p$. We shall write σ_x for the *last* component of the simplex $x \in \Gamma_p$. The n-skeleton of Γ_p will be denoted by $sk_n\Gamma_p$. For $(\mu,\sigma) \in \mathcal{K}_p$ we find :

$$\Gamma_p^{(\mu,\sigma)} = \{x \in \Gamma_p \mid \phi_{ij}^*(x) \in sk_{\mu_{ij}}\Gamma_2 \text{ and } \phi_{ij}^*(\sigma_x) = \phi_{ij}^*(\sigma) \text{ if } \phi_{ij}^*(x) \notin sk_{\mu_{ij}-1}\Gamma_2\}$$

Condition (1.9.1) is satisfied, since there are simplicial contractions

$$\begin{array}{rcl} \gamma : \Gamma_p^{(\mu,\sigma)} & \to & \Gamma_p^{(\mu,\sigma)} \\ (\sigma_0,\ldots,\sigma_k) & \mapsto & (\sigma_0,\ldots,\sigma_k,\sigma) \end{array}$$

and geometric realization transforms cell-inclusions into closed cofibrations. Condition (1.9.2) is also satisfied since a point in $|\Gamma_p|$ is ordered iff it is contained in the interior of a simplex of Γ_p whose last component is the neutral element of $\mathfrak{S}_p$.

The multiplication $m^\Gamma_{i_1\ldots i_p} : \Gamma_p \times \Gamma_{i_1} \times \cdots \times \Gamma_{i_p} \to \Gamma_{i_1+\cdots+i_p}$ preserves the cellular structure, since we deduce from $m^\Gamma_{i_1\ldots i_p} = Wm^{\mathfrak{S}}_{i_1\ldots i_p}$ the relations (cf. 1.15):

$$\phi^*_{\psi_r(i),\psi_r(j)}(m^\Gamma_{i_1\ldots i_p}(x;x_1,\ldots,x_p)) = \phi^*_{ij}(x_r) \text{ for } i,j \in \mathbf{i_r}, \text{ and}$$
$$\phi^*_{\psi_r(i),\psi_s(j)}(m^\Gamma_{i_1\ldots i_p}(x;x_1,\ldots,x_p)) = \phi^*_{rs}(x) \text{ for } i \in \mathbf{i_r}, j \in \mathbf{i_s}, r<s.$$

The cell-interior of $|\Gamma_p^{(\mu,\sigma)}|$ is spanned by the interiors of the simplices of the following subset (which is not a simplicial subset) :

$$\breve{\Gamma}_p^{(\mu,\sigma)} = \{x \in \Gamma_p \,|\, \sigma_x = \sigma \text{ and } \phi_{ij}^*(x) \in sk_{\mu_{ij}}\Gamma_2 \backslash sk_{\mu_{ij}-1}\Gamma_2\}.$$

So, a cell $|\Gamma_p^{(\mu,\sigma)}|$ is proper iff there exists a simplex $(\sigma_0, \sigma_1, \ldots, \sigma_k) \in \Gamma_p$ such that $\sigma = \sigma_k$ and such that for each $i < j$, the image-sequence

$$(\phi_{ij}^*(\sigma_0), \ldots, \phi_{ij}^*(\sigma_k)) \in \Gamma_2$$

contains exactly μ_{ij} changes. This leads to the following proposition, for which a proof can be found in [**5**].

PROPOSITION 2.9. *Let Γ be the simplicial E_∞-operad of Barratt-Eccles. Then the posets of proper cells are given by :*

$$\begin{aligned}\mathcal{K}(\Gamma)_p = \{\alpha \in \mathcal{K}_p \,|\, & \textit{there exists a descending chain of labelings } (\alpha_{ij}^{(r)}) \in \mathbb{N}^{\binom{p}{2}} \\ & \textit{beginning with } (\alpha_{ij}^{(0)}) = (\alpha_{ij}) \textit{ and ending at } (0) \textit{ such that} \\ & \alpha_{ij}^{(r)} \equiv \alpha_{jk}^{(r)} \bmod 2 \textit{ implies } \alpha_{ij}^{(r)} \equiv \alpha_{jk}^{(r)} \equiv \alpha_{ik}^{(r)} \bmod 2 \textit{ and} \\ & 0 \leq \alpha_{ij}^{(r)} - \alpha_{ij}^{(r+1)} \leq 1 \textit{ for all } i,j,r\}.\end{aligned}$$

REMARK 2.10. The posets $\mathcal{K}(F)_p$ and $\mathcal{K}(\Gamma)_p$ are quite different, which explains the difficulty in showing directly that $F_p^{(n)}$ and $|\Gamma_p^{(n)}|$ are homotopy equivalent (cf. [**14**]). The importance of the Γ-construction comes from the fact that the Smith-filtration

$$\Gamma_p^{(n)} = \{x \in \Gamma_p \,|\, \phi_{ij}^*(x) \in sk_{n-1}\Gamma_2 \text{ for all } i < j\}$$

defines a family of cellular E_n-operads in the category of simplicial sets so that, by Fiedorowicz's comparison theorem, May's entire theory of E_n-operads can be applied in this simplicial context, including the approximation and detection theorems, cf. [**15**], [**19**] or [**5**]. The case $n = \infty$ was the initial motivation of Barratt-Eccles [**4**]. Moreover, the operad structure of Γ is purely group-theoretic, so that the cellular E_n-operads $\Gamma^{(n)}$ relate the homology of the symmetric groups rather directly to universal phenomena occuring in the theory of n-fold iterated loop spaces.

NOTATION 2.11. As a last example of a cellular E_∞-operad we present here Milgram's permutohedral models [**17**]. Like Kapranov [**13**] we shall write P_n for the permutohedron embedded in $\mathbb{R}^n$, i.e. for the convex hull of the point set

$$\{(\sigma(1), \ldots, \sigma(n)) \in \mathbb{R}^n \,|\, \sigma \in \mathfrak{S}_n\}.$$

The permutohedron P_n is a *convex polytope* whose *face-poset* is canonically isomorphic to the poset $\mathcal{P}(\mathfrak{S}_n)$ formed by all (right) cosets

$$\mathfrak{S}_{(i_1,\ldots,i_r)}\sigma \in \mathfrak{S}_{(i_1,\ldots,i_r)}\backslash\mathfrak{S}_n,$$

with respect to subgroups of $\mathfrak{S}_n$ of the form $\mathfrak{S}_{(i_1,\ldots,i_r)}$, where $(i_1, \ldots, i_r)$ is an *ordered partition* of n, cf. 2.5. Indeed, each coset $\mathfrak{S}_{(i_1,\ldots,i_r)}\sigma$ corresponds to the convex hull of the *vertices* $(\tau(1), \ldots, \tau(n)) \in \mathbb{R}^n$, as τ runs through $\mathfrak{S}_{(i_1,\ldots,i_r)}\sigma$.

In the literature ([**17**], [**3**], [**13**]), the faces of the permutohedra are labelled by *left* cosets instead of right cosets and the symmetric group acts by left multiplication instead of right multiplication. Our convention follows the definition of the category Λ where the permutations act on themselves by right multiplication. There is however an easy way to switch between the two conventions by means of the involution $inv : \sigma \mapsto \sigma^{-1}$ which associates to a permutation its *"ordering"* and vice versa (cf. [**19**], [**14**], [**13**]). In particular, our multiplication 1.15a of the permutation operad also reflects this notational convention, so that a reader who prefers left notation, has to change $m^{\mathfrak{S}}_{i_1\dots i_n}$ into $inv \circ m^{\mathfrak{S}}_{i_1\dots i_n} \circ (inv \times \cdots \times inv)$ which corresponds to "place-permutation" rather than "element-permutation".

The definition of the Milgram operads relies formally on the existence of an operad structure on the collection of permutohedra $(P_n)_{n>0}$ which

(a) restricts to the permutation operad on vertices, and

(b) satisfies the *boundary condition*, i.e. the multiplication $m^P_{i_1\dots i_n}$ sends the boundary of $P_n \times P_{i_1} \times \cdots \times P_{i_n}$ to the boundary of $P_{i_1+\cdots+i_n}$.

The affine extension of the permutation operad does *not* satisfy the boundary condition, so that some additional combinatorial properties of the permutohedra have to be used. I am indebted to Fiedorowicz for insisting on this point and for sending me some helpful pictures.

There is actually a natural *cubical* subdivision of the permutohedron P_n induced by simplicial stars with respect to the barycentric subdivision of P_n. This cubical subdivision admits the following geometric description : Each vertex $\sigma \in P_n$ carries a natural $(n-1)$*-frame* f_σ defined by the union of all σ-incident edges in the barycentrically subdivided 1-skeleton of P_n. The *simplicial hull* of f_σ (i.e. the co-cell of f_σ in the nerve of $\mathcal{P}(\mathfrak{S}_n)$) yields a standard simplicial $(n-1)$-cube bipointed by σ and the barycenter of P_n. The cubical decomposition of P_n thus corresponds to a *frame-decomposition* of the barycentrically subdivided 1-skeleton of P_n.

We shall show below that the permutation operad is naturally "framed": for each vertex $(\sigma; \sigma_1, \dots \sigma_n) \in P_n \times P_{i_1} \times \cdots \times P_{i_n}$ the image of the product-frame $f_\sigma \times f_{\sigma_1} \times \cdots \times f_{\sigma_n}$ under $m^{\mathfrak{S}}_{i_1\dots i_n}$ defines a frame $f_{(\sigma;\sigma_1,\dots,\sigma_n)}$ in $P_{i_1+\cdots+i_n}$ whose simplicial hull is a $(i_1+\cdots+i_n-1)$-cube. We then define the *permutohedral operad* to be the *cubical extension of this "framed" permutation operad*. The underlying *pre*operad coincides with the affine extension of the permutation preoperad, since the image-frame induced by a Λ-action is the natural one.

The definition of the image-frames uses an alternative description of the face-poset $\mathcal{P}(\mathfrak{S}_n)$ based on the beautiful theorem of Blind and Mani [**6**] that the face-poset of a *simple* polytope is uniquely determined by its 1-skeleton. The permutohedron is simple and its (oriented) 1-skeleton coincides with the (left) *weak Bruhat order* on the symmetric group $\mathfrak{S}_n$. To be more precise : each cell of the permutohedron P_n has a canonical *initial* (resp. *final*) vertex given by the unique permutation $\tau \in \mathfrak{S}_{(i_1,\dots,i_r)}\sigma$ such that τ^{-1} is increasing (resp. decreasing) on subsets $\mathbf{i_k}$ of $\mathbf{i_1} + \cdots + \mathbf{i_r}$, cf. [**3**]. This orientation of the 1-skeleton of P_n defines precisely the weak Bruhat order on $\mathfrak{S}_n$, and each coset in $\mathcal{P}(\mathfrak{S}_n)$ is an *interval* for the weak Bruhat order. In other words, cells of the permutohedron correspond bijectively to "admissible" intervals $[\tau_1, \tau_2]$ of the weak Bruhat order on $\mathfrak{S}_n$, where admissible means that the lower and upper bounds are the initial and final vertices of some coset in $\mathcal{P}(\mathfrak{S}_n)$.

The permutation operad *preserves* the weak Bruhat order, i.e. the multiplication $m^{\mathfrak{S}}_{i_1\dots i_n}$ embeds $\mathfrak{S}_n \times \mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_n}$ in $\mathfrak{S}_{i_1+\cdots+i_n}$ as a *subposet*. Furthermore, we define the *geodesic* between two comparable vertices of the permutohedron P_n

to be the barycenter of all oriented edge-paths between them, and the *barycenter* of an arbitrary interval $[\tau_1, \tau_2]$ to be the middle of the geodesic between τ_1 and τ_2. The barycenter of an admissible interval coincides with the barycenter of the associated cell. The extremal vertices of the $(n-1)$-frame f_σ at $\sigma \in P_n$ are now precisely the barycenters of the 2-element intervals of $\mathfrak{S}_n$ containing σ. More generally, the extremal vertices of the product-frame $f_\sigma \times f_{\sigma_1} \times \cdots \times f_{\sigma_n}$ are precisely the barycenters of the 2-element intervals of $\mathfrak{S}_n \times \mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_n}$ containing $(\sigma; \sigma_1, \ldots, \sigma_n)$. The multiplication $m^{\mathfrak{S}}_{i_1 \cdots i_n}$ sends these intervals to well defined intervals of $\mathfrak{S}_{i_1 + \cdots + i_n}$ containing $\sigma(\sigma_1, \ldots, \sigma_n)$. The *image-frame* $f_{(\sigma;\sigma_1,\ldots,\sigma_n)}$ is then by definition the union of the geodesics between $\sigma(\sigma_1, \ldots, \sigma_n)$ and the barycenters of these image-intervals. The simplicial hull of $f_{(\sigma;\sigma_1,\ldots,\sigma_n)}$ is an $(i_1 + \cdots + i_n - 1)$-cube bipointed by $\sigma(\sigma_1, \ldots, \sigma_n)$ and the barycenter of $P_{i_1+\cdots+i_n}$, actually isomorphic to a well defined subdivision of the standard simplicial $(i_1 + \cdots + i_n - 1)$-cube, see [**5**] for more details.

Finally we need the *convex projectors*

$$D_{(i_1,\ldots,i_r)} = \psi_1^* \times \cdots \times \psi_r^* : P_n \to P_{i_1} \times \cdots \times P_{i_r} \cong c_{(i_1,\ldots,i_r)},$$

where ψ_k is the canonical inclusion of $\mathbf{i_k}$ in $\mathbf{n} = \mathbf{i_1} + \cdots + \mathbf{i_r}$, and where the convex hull $c_{(i_1,\ldots,i_r)}$ of the subgroup $\mathfrak{S}_{(i_1,\ldots,i_r)}$ is identified with the cartesian product of the corresponding permutohedra by affine extension of the canonical isomorphism

$$\mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_r} \cong \mathfrak{S}_{(i_1,\ldots,i_r)}.$$

For each coset $\mathfrak{S}_{(i_1,\ldots,i_r)}\tau \in \mathcal{P}(\mathfrak{S}_n)$ such that τ is initial, this defines a convex projector $D^\tau_{(i_1,\ldots,i_r)} = \tau^* D_{(i_1,\ldots,i_r)}(\tau^*)^{-1}$ onto the corresponding cell of the permutohedron. The map which associates to a coset its convex projector "transforms" the permutation operad into the permutohedral operad.

Definition 2.12. Milgram's E_k-operads are defined as quotient spaces

$$J_n^{(k)} = (P_n)^{k-1} \times \mathfrak{S}_n / \sim$$

where the equivalence relation identifies certain boundary cells of the cartesian product. Explicitly, for each point $(\tau^*(x_1), \ldots, \tau^*(x_{k-1}); \sigma) \in (P_n)^{k-1} \times \mathfrak{S}_n$ such that x_s belongs to the convex hull of a proper subgroup $\mathfrak{S}_{(i_1,\ldots,i_r)}$ of $\mathfrak{S}_n$ and such that τ is the initial vertex of the coset $\mathfrak{S}_{(i_1,\ldots,i_r)}\tau$, we have the relation

$$(\tau^*(x_1), \ldots, \tau^*(x_{k-1}); \sigma) \sim$$
$$(x_1, \ldots, x_s, D_{(i_1,\ldots,i_r)}(x_{s+1}), \ldots, D_{(i_1,\ldots,i_r)}(x_{k-1}); \tau\sigma).$$

The action of $\phi \in \mathbf{\Lambda}(\mathbf{m}, \mathbf{n})$ is induced by

$$\begin{array}{rcl} \phi^* : (P_n)^{k-1} \times \mathfrak{S}_n & \to & (P_m)^{k-1} \times \mathfrak{S}_m \\ (x_1, \ldots, x_{k-1}; \tau) & \mapsto & (((\tau\phi)^{inc})^*(x_1), \ldots, ((\tau\phi)^{inc})^*(x_{k-1}); (\tau\phi)^\natural). \end{array}$$

The space $J_n^{(k)}$ embeds in $J_n^{(k+1)}$ by identifying $(P_n)^{k-1}$ with the subset of $(P_n)^k$ formed by the points whose first component is the barycenter of the permutohedron, i.e. the fixed point under the action of $\mathfrak{S}_n$.

The previously defined Λ-structure as well as the diagonal multiplication on $(P_n)^{k-1} \times \mathfrak{S}_n$ are compatible with the equivalence relation and induce thus a natural operad structure on the spaces $(J_n^{(k)})_{n>0}$. The boundary condition of the permutohedral operad is crucial at this point, since it implies that the gluing of the cells is preserved under multiplication. The associated monad can be identified with Milgram's construction J_k which models for connected CW-spaces the functor

$\Omega^k S^k$ [**17**]. The cellular E_k-structure of $(J_n^{(k)})_{n>0}$ is based on the following lemma, which relates our equivalence relation to that of Baues [**3**], [**5**] :

LEMMA 2.13. *Each point $x \in J_n^{(k)}$ has a unique representative*

$$x_{can} = (x_1, \dots, x_{k-1}; \sigma) \in (P_n)^{k-1} \times \mathfrak{S}_n,$$

such that the minimal cells c_i whose cell-interiors contain x_i are convex hulls of a decreasing chain of subgroups of $\mathfrak{S}_n$, in particular $c_1 \supset c_2 \supset \cdots \supset c_{k-1}$.

The collection of these cells $c_1 \times \cdots \times c_{k-1} \times \sigma$ induces a cell-decomposition of $J_n^{(k)}$ whose poset is antiisomorphic *to $\mathcal{K}(F)_n^{(k)}$, cf. 2.5.*

THEOREM 2.14. *Milgram's operads $J^{(k)}$ form the natural filtration of a cellular E_∞-operad. In particular, the space $J_n^{(k)}$ has the $\mathfrak{S}_n$-equivariant homotopy type of the real configuration space $F(\mathbb{R}^k, n)$. The posets of proper cells are given by $\mathcal{K}(J)_n^{(k)} = \mathcal{K}(F)_n^{(k)}$.*

PROOF. – By the comparison theorem it remains to define the underlying cell structure beginning with a cellular $\mathcal{K}_2$-decomposition of the inductive limit $J_2 = \varinjlim_k J_2^{(k)}$. But the previous lemma shows that we have only to *dualize* the cell-decompositions defined by the cartesian products, which is possible because of the compactness of the $J_2^{(k)}$ and the underlying affine structure. This dualization process works $\mathbf{\Lambda}$-equivariantly for all n and gives thus the asserted posets of proper cells. □

References

[1] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt – *Iterated monoidal categories*, preprint (1995).

[2] H. Barcelo and N. Bergeron – *The Orlik-Solomon Algebra on the Partition Lattice and the Free Lie Algebra*, J. of Comb. Theory (A) **55** (1990), 80-92.

[3] H.-J. Baues – *Geometry of loop spaces and the cobar-construction,* Mem. AMS **230** (1980).

[4] M. G. Barratt and P. J. Eccles – *Γ^+-Structures,* Topology **13** (1974), 25-45, 113-126, 199-207.

[5] C. Berger – *Opérades cellulaires et espaces de lacets itérés*, Ann. de l'Inst. Fourier **46** (1996).

[6] R. Blind and P. Mani – *On puzzles and polytope isomorphism*, Aequationes Math. **34** (1987), 287-297.

[7] J. M. Boardman and R. M. Vogt – *Homotopy invariant algebraic structures on topological spaces,* Lecture Notes in Math. **347**, Springer Verlag (1973).

[8] F. R. Cohen – *The homology of C_{n+1}-spaces,* Lecture Notes in Math. **533**, Springer Verlag (1976), 207-351.

[9] F. R. Cohen, J. P. May and L. R. Taylor – *Splitting of certain spaces CX,* Math. Proc. Camb. Phil. Soc. **84** (1978), 465-496.

[10] R. Fox and L. Neuwirth – *The braid groups,* Math. Scand. **10** (1962), 119-126.

[11] E. Getzler and J. D. S. Jones – *Operads, homotopy algebra, and iterated integrals for double loop spaces*, preprint (1995).

[12] P. Greenberg – *Triangulating groups ; two examples*, preprint (1992).

[13] M. M. Kapranov – *The permutoassociahedron, MacLane's coherence theorem and asymptotic zones for KZ equation,* J. of Pure and Applied Algebra **85** (1993), 119-142.

[14] T. Kashiwabara – *On the Homotopy Type of Configuration Complexes,* Contemp. Math. **146** (1993), 159-170.

[15] J. P. May – *The Geometry of iterated loop spaces,* Lecture Notes in Math. 277, Springer Verlag (1972).

[16] J. P. May – *Definitions : Operads, algebras and modules,* this volume.

[17] R. J. Milgram – *Iterated loop spaces,* Annals of Math. **84** (1966), 386-403.

[18] D. Quillen – *Higher algebraic K-theory I,* Lecture Notes in Math. **341**, Springer Verlag (1973), 85-147.

[19] J. H. Smith – *Simplicial Group Models for $\Omega^n S^n X$,* Israel J. of Math. **66** (1989), 330-350.
[20] A. Tonk – *Relating the permutohedra and associahedra*, this volume.

Université de Nice-Sophia Antipolis, Laboratoire J. A. Dieudonné, URA 168, Parc Valrose, F-06108 Nice Cedex 2, France.
E-mail address: cberger@math.unice.fr