ELSEVIER Journal of Pure and Applied Algebra 131 (1998) 109–123

JOURNAL OF PURE AND APPLIED ALGEBRA

Comparison of the geometric bar and W-constructions

Clemens Berger[a], Johannes Huebschmann[b,*,1]

[a] Université de Nice-Sophia Antipolis, Laboratoire J.A. Dieudonné, UMR 6621, Parc Valrose, F-06 108 Nice Cédex 2, France

[b] Max Planck Institut für Mathematik, Gottfried Claren Str. 26, D-53 225 Bonn, Germany

Communicated by J.D. Stasheff; received 15 September 1995; received in revised form 8 May 1996

Dedicated to the memory of V.K.A.M. Gugenheim

Abstract

For a simplicial group K, the realization of the W-construction $WK \to \overline{W}K$ of K is shown to be naturally homeomorphic to the universal bundle $E|K| \to B|K|$ of its geometric realization $|K|$. The argument involves certain recursive descriptions of the W-construction and classifying bundle and relies on the facts that the realization functor carries an action of a simplicial group to a geometric action of its realization and preserves reduced cones and colimits. © 1998 Elsevier Science B.V. All rights reserved.

AMS Classification: 55Q05; 55P35; 18G30

0. Introduction

Let K be a simplicial group; its realization $|K|$ is a topological group suitably interpreted when K is not countable. The W-construction $WK \to \overline{W}K$ yields a functorial universal simplicial principal K-bundle, and the classifying bundle construction $E|K| \to B|K|$ of its geometric realization $|K|$ yields a functorial universal principal $|K|$-bundle. The realization of the W-construction also yields a universal principal $|K|$-bundle $|WK| \to |\overline{W}K|$, by virtue of the general realization result in [26]. In this note we identify the classifying bundle with the realization of the W-construction. A cryptic remark about the possible coincidence of the two constructions may be found in the

* Correspondence address: Univ. des Sciences & Techn. de Lille, UFR de Mathematiques, 59 655 Villeneuve d'Ascq Cédex, France. E-mail: johannes.huebschmann@univ.lille1.fr.

1 This author carried out this work in the framework of the VBAC research group of EUROPROJ.

0022-4049/98/$19.00 © 1998 Elsevier Science B.V. All rights reserved.
PII: S0022-4049(98)00020-6

introduction to Steenrod's paper [25] but to our knowledge this has never been made explicit in the literature.

Spaces are assumed to be compactly generated, and all constructions on spaces are assumed to be carried out in the compactly generated category. It is in this sense that the realization $|K|$ is always a topological group; in general, the multiplication map will be continuous only in the compactly generated refinement of the product topology on $|K| \times |K|$. For countable K, there is no difference, though. Here is our main result.

Theorem. *There is a canonical homeomorphism of principal $|K|$-bundles between the realization $|WK| \to |\overline{W}K|$ of the W-construction and the classifying bundle $E|K| \to B|K|$ which is natural in K.*

The map from $|WK|$ to $E|K|$ could be viewed as a kind of perturbed geometric Alexander–Whitney map while the map in the other direction is a kind of perturbed geometric shuffle map (often referred to as Eilenberg–Zilber map) but this analogy should not be taken too far.

The classifying space $B|K|$ is the realization of the nerve NK of K as a *bisimplicial* set. The latter is homeomorphic to the realization of its *diagonal* DNK since this is known to be true for an arbitrary bisimplicial set [20]. The diagonal DNK, in turn, does *not* coincide with the reduced W-construction $\overline{W}K$, though, but after realization the two are homeomorphic. We shall spell out the precise relationships in Section 4 below.

Eilenberg–Mac Lane introduced the bar and W-constructions in [6] and showed that, for any (connected) simplicial algebra A, there is a "reduction" of (the normalized chain complex of) the reduced W-construction of A onto the (reduced normalized) bar construction $B|A|$ of the normalized chain algebra $|A|$ of A and raised the question whether this reduction is in fact part of a *contraction*. By means of homological perturbation theory, in his "Diplomarbeit" [27] supervised by the second named author, Wong answered this question by establishing such a contraction. Wong's basic tool is the "perturbation lemma" exploited in [8]; see [10] for details and history.

Our result, apart from being interesting in its own right, provides a step towards a rigorous understanding of lattice gauge theory. See [11] for details. Using the notation K_Y for the Kan group [12] of a reduced simplicial set Y, at this stage, we only spell out the following consequence, relevant for what is said in [11].

Corollary. *For a reduced simplicial set Y, there is a canonical map from its realization $|Y|$ to the classifying space $B|K_Y|$ of the realization of K_Y which is natural in Y and a homotopy equivalence.*

The proof of our main result involves a certain recursive description of the W-construction which mimics Steenrod's elegant description of the classifying bundle [25]. By induction, our argument then reduces to the observation that the realization functor carries an action of a simplicial group to a geometric action of its realization and preserves reduced cones and colimits. It would be interesting to extend the method of the present paper to simplicial groupoids, so that a result of the kind given in the corollary would follow for an arbitrary connected simplicial set, with the Kan group replaced by the Kan groupoid [5]. Such an extension would have to rely on correct descriptions of the requisite monads for groupoid actions and conical contractions in the general non-reduced setting. We hope to return to this issue elsewhere.

We are indebted to Jim Stasheff and to the referee for a number of most helpful comments.

1. The classifying space of a topological group

Let G be a topological group. Its *nerve* NG [2,3,21] is the simplicial space having in degree $k \geq 0$ the constituent $NG_k = G^{\times k}$, with the standard simplicial operations. The usual *lean* realization $BG = |NG|$ of NG is a classifying space for G, cf. [13,21,24]; there is an analoguous construction of a contractible total space EG together with a free G-action and projection ξ onto BG, and this projection is locally trivial provided (G,e) is an NDR (neighborhood deformation retract) [25]. We note, for completeness, that the *fat* realization $||NG||$ yields Milnor's classifying space [16], and the projection from the corresponding total space to $||NG||$ is always locally trivial whether or not (G,e) is an NDR. Below (G,e) will always be a CW-pair and hence an NDR, cf. e.g. the discussion in the appendix to [22], and we shall deal exclusively with the lean realization $BG = |NG|$. To reproduce a description thereof, and to introduce notation, write Δ for the category of finite ordered sets $[q] = (0,1,\ldots,q)$, $q \geq 0$, and monotone maps. We recall the standard *coface* and *codegeneracy* operators

$$\varepsilon^j\colon [q-1] \to [q], \quad (0,1,\ldots,j-1,j,\ldots,q-1) \mapsto (0,1,\ldots,j-1,j+1,\ldots,q),$$
$$\eta^j\colon [q+1] \to [q], \quad (0,1,\ldots,j-1,j,\ldots,q+1) \mapsto (0,1,\ldots,j,j,\ldots,q),$$

respectively. As usual, for a simplicial object, the corresponding face and degeneracy operators will be written d_j and s_j. The assignment to $[q]$ of the standard simplex $\nabla[q] = \Delta_q$ yields a *cosimplicial* space ∇; here we wish to distinguish clearly in notation between the cosimplicial space ∇ and the category Δ. The lean geometric realization $|NG|$ is the *coend* $NG \otimes_\Delta \nabla$, cf. e.g. [14] for details on this notion. Exploiting this observation, Mac Lane observed in [13] that $|NG|$ coincides with the classifying space for G constructed by Stasheff [23] and Milgram [15]; see also Section 1 of Stasheff's survey paper [24] and Segal's paper [21]. Mac Lane actually worked with a variant of the category Δ which enabled him to handle simultaneously the total space EG and the base BG.

Steenrod [25] gave a recursive description of $|NG|$ which we shall subsequently use. For ease of exposition, following [1], we reproduce it briefly in somewhat more categorical language. This will occupy the rest of this section.

For a space X endowed with a G-action $\phi: X \times G \to X$, we write $\eta = \eta_X^G: X \to X \times G$ for the *unit* given by $\eta(x) = (x, e)$. For an arbitrary space Y, right translation of G induces an obvious free G-action μ on $Y \times G$. In categorical language [14], the functor $\times G$ and natural transformations μ and η constitute a *monad* $(\times G, \mu, \eta)$ and a G-action on a space X is an *algebra* structure on X over this monad. Sometimes we shall refer to an action of a topological group on a space as a *geometric* action.

Let D be any space and E a subspace endowed with a G-action $\phi: E \times G \to E$; the inclusion of E into D is written β. Recall that the *enlargement* $\overline{D} \supseteq D$ of the G-action is characterized by the property: if Y is any G-space, and f any map from D to Y whose restriction to E is a G-mapping, then there exists a unique G-mapping $\overline{f}$ from $\overline{D}$ to Y extending f. The space $\overline{D}$ then fits into a push out diagram

$$\begin{array}{ccc} E \times G & \xrightarrow{\phi} & E \\ {\scriptstyle \beta\times \mathrm{Id}}\downarrow & & \downarrow \\ D \times G & \longrightarrow & \overline{D} \end{array} \tag{1.1}$$

and this provides a construction for $\overline{D}$. Moreover, right action of G on $D \times G$ induces an action

$$\overline{\phi}: \overline{D} \times G \to \overline{D} \tag{1.2}$$

of G on $\overline{D}$, and the composite

$$\alpha: D \to \overline{D} \tag{1.3}$$

of the unit $\eta: D \to D \times G$ with the map from $D \times G$ to $\overline{D}$ in (1.1) embeds D into $\overline{D}$. When D is based and E is a based subspace, the products $E \times G$ and $D \times G$ inherit an obvious base point, and the square (1.1) is one in the category of based spaces whence, in particular, the enlargement $\overline{D}$ inherits a base point. This notion of enlargement of G-action is functorial in the appropriate sense. See [25] for details. This kind of universal construction is available whenever one is given an algebra structure over a monad preserving push out diagrams.

The unit interval $I = [0, 1]$ is a topological monoid under ordinary multiplication having 1 as its unit, and hence we can talk about an I-action $X \times I \to X$ on a space X. Such an I-action is plainly a special kind of homotopy which, for $t = 1$, is the identity. In the above categorical spirit, the interval I gives rise to a monad $(\times I, \mu, \eta)$ and an I-action on a space X is an *algebra* structure on X over this monad.

The *base point* of I is defined to be 0. Following [25], for a based space (X, x_0), we shall refer to an I-action $\psi: X \times I \to X$ as a *contraction* of X (*to the base point* $x_0 \in X$) provided ψ sends the base point $(x_0, 0)$ of $X \times I$ to x_0 and factors through the *reduced cone* or *smash product*

$$CX = X \wedge I = X \times I / (X \times \{0\} \cup \{x_0\} \times I)$$

that is to say,

$$\psi(x,0) = x_0 = \psi(x_0,t)$$

for all $x \in X$, $t \in I$; the reduced cone will be endowed with the obvious base point, the image of $X \times \{0\} \cup \{x_0\} \times I$ in CX. Whenever we say "contraction", we mean "contraction to a pre-assigned base point". Abusing notation, the corresponding map from CX to X will as well be denoted by ψ and referred to as a *contraction*. Moreover we write $\eta = \eta_X^C$ for the map, the corresponding *unit*, which embeds X into CX by sending a point x of X to $(x,1) \in CX$. The right action of I on $X \times I$ induces a contraction $\mu_X^C: CCX \to CX$ of CX. Again we can express this in categorical language: the functor C and natural transformations μ and η constitute a *monad* and a contraction of a based space X is an *algebra* structure on X over this monad or, equivalently, a C-*algebra* structure on X. Sometimes we shall refer to a contraction of a space as a *geometric* contraction.

Let (E,x_0) be any based space and (D,x_0) a based subspace endowed with a contraction $\psi: CD \to D$; the inclusion of D into E is written α. The *enlargement* $(\overline{E},x_0) \supseteq (E,x_0)$ of the contraction is characterized by the property: if f is any map from E to a space Y having a contraction to some point y_0 whose restriction to D is an I-mapping, then there exists a unique I-mapping $\overline{f}$ from $\overline{E}$ to Y extending f. The space $\overline{E}$ then fits into a push out diagram

$$\begin{array}{ccc} CD & \xrightarrow{\psi} & D \\ {\scriptstyle C\alpha}\downarrow & & \downarrow \\ CE & \longrightarrow & \overline{E} \end{array} \tag{1.4}$$

which provides a construction for $\overline{E}$. Moreover, the composite

$$\beta: E \to \overline{E} \tag{1.5}$$

of the unit $\eta: E \to CE$ with the map from CE to $\overline{E}$ in (1.4) embeds E into $\overline{E}$ and the right action of I on $E \times I$ induces a contraction of CE which, in turn, induces a contraction

$$\overline{\psi}: C\overline{E} \to \overline{E} \tag{1.6}$$

of $\overline{E}$. This notion of enlargement of contraction is functorial in the appropriate sense. See [25] for details.

Alternating the above constructions, in [25], Steenrod defines based spaces and injections of based spaces

$$D_0 \xrightarrow{\alpha_0} E_0 \xrightarrow{\beta_0} D_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\beta_{n-1}} D_n \xrightarrow{\alpha_n} E_n \xrightarrow{\beta_n} D_{n+1} \xrightarrow{\alpha_{n+1}} \dots \tag{1.7}$$

by induction on n together with contractions $\psi_n: CD_n \to D_n$ (Steenrod writes these contractions as I-actions $D_n \times I \to D_n$) and G-actions $\phi_n: E_n \times G \to E_n$ in the following way: Let D_0 consist of the single point e with the obvious contraction. Let $E_0 = G$, the right action being right translation. Now define (D_1, e) to be the enlargement to (E_0, e), $(\overline{E}_0, e)$, of the contraction of (D_0, e); then D_1 is just the reduced cone on E_0. Define E_1 to be the enlargement to D_1, $\overline{D}_1$, of the G-action on E_0. In general, D_n is the enlargement to (E_{n-1}, e), $(\overline{E}_{n-1}, e)$, of the contraction ψ_{n-1} of (D_{n-1}, e) so that D_n fits into a push out square

$$\begin{array}{ccc} CD_{n-1} & \xrightarrow{\psi_{n-1}} & D_{n-1} \\ {\scriptstyle C\alpha_{n-1}}\downarrow & & \downarrow \\ CE_{n-1} & \longrightarrow & D_n; \end{array} \tag{1.8}$$

the requisite injection $\beta_{n-1} : E_{n-1} \to D_n$ is the map denoted above by β, cf. (1.5); and the requisite contraction $\psi_n : CD_n \to D_n$ of (D_n, e) or, equivalently, I-action $\psi_n : D_n \times I \to D_n$, is the map denoted above by $\overline{\psi}$, cf. (1.6). Likewise, E_n is the enlargement to D_n, $\overline{D}_n$, of the G-action ϕ_{n-1} on E_{n-1}, so that E_n fits into a push out square

$$\begin{array}{ccc} E_{n-1} \times G & \xrightarrow{\phi_{n-1}} & E_{n-1} \\ {\scriptstyle \beta_{n-1} \times \mathrm{Id}}\downarrow & & \downarrow \\ D_n \times G & \longrightarrow & E_n; \end{array} \tag{1.9}$$

the requisite G-action $\phi_n : E_n \times G \to E_n$ and injection $\alpha_n : D_n \to E_n$ are the action denoted above by $\overline{\phi}$, cf. (1.2), and the map denoted above by α, cf. (1.3), respectively. Consider the union

$$E_G = \bigcup_{n=0}^{\infty} E_n = \bigcup_{n=0}^{\infty} D_n,$$

endowed with the weak topology. Since each E_n (and each D_n) carries the compactly generated topology, so does E_G. Furthermore, the compactly generated product being taken, the space E_G inherits a (continuous) G-action $\phi: E_G \times G \to E_G$ and a (continuous) contraction $\psi: CE_G \to E_G$ from the ϕ_n's and ψ_n's, respectively. The G-action is free, and the orbit space $BG = E_G/G$ equals the lean geometric realization $|NG|$ of the nerve of G. This is Steenrod's result in [25].

2. The recursive description of the W-construction

In [1], the first named author observed that the W-construction admits a recursive description of formally the same kind as (1.7) above, except that it is carried out in the category of based simplicial sets. This is among the key points of the paper. We shall explain it now. To elucidate the analogy between the two constructions, we first spell out the simplicial monads for group actions and conical contractions.

Let K be a simplicial group. Let e denote the trivial simplicial group viewed at the same time as the simplicial point. For a simplicial set X endowed with a K-action

$\phi: X \times K \to X$, we write $\eta = \eta_X^K: X \to X \times K$ for the *unit* of the action; in each degree, it is given by $\eta(x) = (x, e)$. Given an arbitrary simplicial set Y, right translation of K induces an obvious action μ of K on $Y \times K$. Much as before, in categorical language, the functor $\times K$ and natural transformations μ and η constitute a *monad* $(\times K, \mu, \eta)$ in the category of simplicial sets and a K-action on a simplicial set X is an *algebra* structure on X over this monad. Moreover realization preserves monad and algebra structures. In other words: the realization of a K-action $\phi: X \times K \to X$ on a simplicial set X is a geometric action $|\phi|: |X| \times |K| \to |X|$ in the usual sense. Notice this involves the standard homeomorphism [17] between the realization $|X \times K|$ of the simplicial set $X \times K$ and the product $|X| \times |K|$ of the realizations (with the compactly generated topology). The homeomorphism between $|X \times K|$ and $|X| \times |K|$ is of course natural and relies on the fact that, for an arbitrary bisimplicial set, the realization of the diagonal is homeomorphic to the realization as a bisimplicial set, cf. [20, Lemma on p. 86]. Note, however, that the simplicial CW-structure of the realization of $X \times K$ arises from the product CW-structure of $|X| \times |K|$ only after suitable subdivision thereof [19, Satz 5, p. 388]. This reflects, of course, precisely the decomposition coming into play in the Eilenberg–Zilber map.

Recall that in the category of simplicial sets there are *two* natural (reduced) cone constructions. The first one is defined by the simplicial smash product with the standard simplicial model $\Delta[1]$ of the unit interval. We shall say more about this in Section 4 below. The recursive description of the W-construction crucially involves the second somewhat more economical cone construction which relies on the observation that an $(n+1)$-simplex serves as a cone on an n-simplex. We reproduce this cone construction briefly; it differs from the one given in [4, p. 113] by the order of face and degeneracy operators; our convention is forced here by our description of the W-construction with structure group acting from the right, cf. what is said in (2.6) below.

Let X be a simplicial set. For $j \geq 0$, we shall need countably many disjoint copies of each X_j which we describe in the following way: For $j \geq 0$, consider the cartesian product $X_j \times \mathbf{N}$ with the natural numbers $\mathbf{N}$. Let o be a point which we formally assign dimension -1 and, given $i \in \mathbf{N}$, write $X_{-1}(i) = \{(o, i)\}$ so that each $X_{-1}(i)$ consists of a single element; next, for $j \geq 0$, let $X_j(i) = X_j \times \{i\}$. The *unreduced simplicial cone* $\widehat{C}X$ on X is given by

$$(\widehat{C}X)_n = X_n(0) \cup \ldots \cup X_0(n) \cup X_{-1}(n+1), \quad n \geq 0,$$

with face and degeneracy operators given by the formulas

$$d_j(x,i) = \begin{cases} (d_j x, i), & j \leq n-i, \\ (x, i-1), & j > n-i, \end{cases}$$

$$s_j(x,i) = \begin{cases} (s_j x, i), & j \leq n-i, \\ (x, i+1), & j > n-i\ . \end{cases}$$

Notice that in these formulas $n - i = \dim x$; in particular,

$$d_j(o, n+1) = (o, n), \qquad s_j(o, n) = (o, n+1), \qquad 0 \leq j \leq n.$$

Let now $(X, *)$ be a *based* simplicial set. The unreduced simplicial cone $\widehat{C}\{*\}$ of the simplicial point $\{*\}$ is the simplicial interval, and the *reduced simplicial cone* CX is simply the quotient

$$CX = \widehat{C}X / \widehat{C}\{*\}.$$

For each $n \geq 0$, its constituent $(CX)_n$ arises from the union $X_n(0) \cup \ldots \cup X_0(n)$ by identifying all $(*, i)$ to a single point written $*$, the *base point* of CX. The non-degenerate simplices of CX different from the base point look like $(x, 0)$ and $(x, 1)$ where x runs through non-degenerate simplices of X. We write $\eta = \eta_X^C : X \to CX$ for the *unit* induced by the assignment to $x \in X_n$ of $(x, 0) \in X_n(0)$. A (simplicial) contraction is, then, a morphism $\psi : CX \to X$ of based simplicial sets satisfying

$$\psi \circ \eta = \mathrm{Id}_X.$$

The cone CX itself admits the obvious contraction

$$\mu = \mu_X^C : CCX \to CX, \quad ((x, i), j) \mapsto (x, i + j).$$

A contraction ψ is called *conical* provided

$$\psi \circ C\psi = \psi \circ \mu.$$

We note that a geometric contraction in the sense of Section 1 above, being defined as an action of the associative monoid I, automatically satisfies the usual associativity law for an action. Under the present circumstances, the property of being conical corresponds to this associativity property. The contraction μ_X^C of CX is conical; in categorical terms, the triple (C, μ, η) is a monad in the category of simplicial sets, and a conical contraction is an algebra structure in the category of simplicial sets over this monad, referred to henceforth as *C-algebra* structure on X.

We now have the machinery in place to reproduce the crucial recursive description the W-construction: Define based simplicial sets and injections of based simplicial sets

$$D_0 \xrightarrow{\alpha_0} E_0 \xrightarrow{\beta_0} D_1 \xrightarrow{\alpha_1} \ldots \xrightarrow{\beta_{n-1}} D_n \xrightarrow{\alpha_n} E_n \xrightarrow{\beta_n} D_{n+1} \xrightarrow{\alpha_{n+1}} \ldots \tag{2.1}$$

by induction on n together with conical contractions $\psi_n : CD_n \to D_n$ and K-actions $\phi_n : E_n \times K \to E_n$ on each E_n from the *right* in the following way: Let $D_0 = e$, with the obvious conical contraction ψ_0, let $E_0 = K$, viewed as a based simplicial set in the obvious way, the right action ϕ_0 being translation, and let α_0 be the obvious morphism of based simplicial sets from D_0 to E_0. For $n \geq 1$, define (D_n, e) to be the *enlargement* to (E_{n-1}, e) of the contraction $\psi_{n-1} : CD_{n-1} \to D_{n-1}$, that is, D_n is characterized by the requirement that the diagram

$$\begin{array}{ccc} CD_{n-1} & \xrightarrow{\psi_{n-1}} & D_{n-1} \\ {\scriptstyle C\alpha_{n-1}}\downarrow & & \downarrow \\ CE_{n-1} & \longrightarrow & D_n \end{array} \tag{2.2}$$

be a push out square of (based) simplicial sets; the composite of the unit η from E_{n-1} to CE_{n-1} with the morphism $CE_{n-1} \to D_n$ of simplicial sets in (2.2) yields the requisite injection $\beta_{n-1}: E_{n-1} \to D_n$, and the contraction ψ_{n-1} and the conical contraction of CE_{n-1} induce a conical contraction $\psi_n: CD_n \to D_n$. Likewise, E_n is the *enlargement* to D_n of the K-action ϕ_{n-1} on E_{n-1}, that is, E_n is characterized by a push out square of based simplicial sets of the kind

$$\begin{array}{ccc} E_{n-1} \times K & \xrightarrow{\phi_{n-1}} & E_{n-1} \\ {\scriptstyle \beta_{n-1}\times \mathrm{Id}}\downarrow & & \downarrow \\ D_n \times K & \longrightarrow & E_n; \end{array} \tag{2.3}$$

the requisite K-action $\phi_n: E_n \times K \to E_n$ is induced by ϕ_{n-1} and the obvious K-action on $D_n \times K$, and the requisite injection $\alpha_n: D_n \to E_n$ is the composite of the unit with the morphism $D_n \times K \to E_n$ of simplicial sets in (2.3). The limit

$$WK = \varinjlim E_n = \varinjlim D_n$$

inherits a K-action $\phi: WK \times K \to WK$ and conical contraction $\psi: CWK \to WK$ from the ϕ_n's and ψ_n's, respectively. The K-action is free, and the projection map to the quotient $\overline{W}K = WK/K$ yields the universal simplicial K-bundle

$$WK \to \overline{W}K$$

or W-construction of K, cf. [1], with action of K from the *right*.

For intelligibility, we explain some of the requisite details: A straightforward induction establishes the following descriptions of the simplicial sets D_k and E_k:

$$\begin{aligned}(D_k)_n = {} & \{(i_0, k_0, i_1, k_1, \ldots, k_{\ell-1}, i_\ell) \mid 0 \le \ell \le k,\ i_s \ge 0, \\ & n = i_0 + \ldots + i_\ell,\ k_s \in K_{i_0+\ldots+i_s},\ 0 \le s < \ell\} / \sim, \\ (E_k)_n = {} & \{(i_0, k_0, i_1, k_1, \ldots, k_{\ell-1}, i_\ell, k_\ell) \mid 0 \le \ell \le k,\ i_s \ge 0, \\ & n = i_0 + \ldots + i_\ell,\ k_s \in K_{i_0+\ldots+i_s},\ 0 \le s \le \ell\} / \sim,\end{aligned}$$

where

$$(\ldots, i_s, e, i_{s+1}, \ldots) \sim (\ldots, i_s + i_{s+1}, \ldots), \quad (\ldots, k_s, 0, k_{s+1}, \ldots) \sim (\ldots, k_s k_{s+1}, \ldots).$$

Thus, for $n \ge 0$,

$$\begin{aligned}(WK)_n = {} & \{(k_{j_0}, k_{j_1}, \ldots, k_{j_\ell}) \mid 0 \le j_0 < \ldots < j_\ell = n \quad \text{and} \\ & k_{j_s} \in K_{j_s} \setminus e_{j_s},\ 0 \le s < \ell,\ k_{j_\ell} \in K_{j_\ell}\}.\end{aligned}$$

From this, adding the requisite neutral elements wherever appropriate, we deduce the following more common explicit description: For $n \ge 0$,

$$(WK)_n = K_0 \times \ldots \times K_n,$$

with face and degeneracy operators given by the formulas

$$\begin{aligned} d_0(x_0,\dots,x_n) &= (d_0x_1,\dots,d_0x_n),\\ d_j(x_0,\dots,x_n) &= (x_0,\dots,x_{j-2},x_{j-1}d_jx_j,d_jx_{j+1},\dots,d_jx_n), \quad 1\le j\le n,\\ s_j(x_0,\dots,x_n) &= (x_0,\dots,x_{j-1},e,s_jx_j,s_jx_{j+1},\dots,s_jx_n), \quad 0\le j\le n; \end{aligned} \tag{2.4}$$

further, $(\overline{W}K)_0 = \{e\}$ and, for $n \geq 1$,

$$(\overline{W}K)_n = K_0 \times \cdots \times K_{n-1},$$

with face and degeneracy operators given by the formulas

$$\begin{aligned} d_0(x_0,\dots,x_{n-1}) &= (d_0x_1,\dots,d_0x_{n-1}),\\ d_j(x_0,\dots,x_{n-1}) &= (x_0,\dots,x_{j-2},x_{j-1}d_jx_j,d_jx_{j+1},\dots,d_jx_{n-1}), \quad 1\le j\le n-1,\\ d_n(x_0,\dots,x_{n-1}) &= (x_0,\dots,x_{n-2}),\\ s_0(e) &= e \in K_0,\\ s_j(x_0,\dots,x_{n-1}) &= (x_0,\dots,x_{j-1},e,s_jx_j,s_jx_{j+1},\dots,s_jx_{n-1}), \quad 0\le j\le n. \end{aligned} \tag{2.5}$$

Remark 2.6. Here preferred treatment is given to the *last* face operator, as is done in [9, 12]. This turns out to be the appropriate thing to do for principal bundles with structure group acting on the total space from the *right* and simplifies comparison with the bar construction. See for example what is said on p. 75 of [9]. The formulas (2.4) and (2.5) arise from those given in (A.14) of [9] for a simplicial algebra by the obvious translation to the corresponding formulas for a simplicial monoid; they differ from those in [4, pp. 136 and 161] where the constructions are carried out with structure group acting from the *left*.

3. The proof of the theorem

The realization of a conical contraction $\psi: CX \to X$ of a based simplicial set (X, x_0) is a geometric contraction $|\psi|: C|X| \to |X|$ in the sense reproduced in Section 1 above. In fact, the association

$$(|x|(t_0,\dots,t_n),t) \longmapsto (|(x,1)|(tt_0,\dots,tt_n,1-t), \qquad x\in X_n,\ n\geq 0,$$

yields a homeomorphism from the reduced cone $C|X|$ on the realization $|X|$ to the realization $|CX|$ of the cone and, furthermore, the realizations of the unit η and C-algebra structure $\mu_X^C: CCX \to CX$ yield the geometric unit $|X| \to C|X|$ and geometric C-algebra structure $\mu_{|X|}^C: CC|X| \to C|X|$, that is, the realization preserves monad- and C-algebra structures.

The proof of the theorem is now merely an elaboration of the observation that the realization functor $|\cdot|$ carries an action of a simplicial group to a geometric action of its realization, preserves reduced cones and, having a right adjoint (the singular complex functor), also preserves colimits. In fact, denote the corresponding sequence (1.7) of based topological spaces for the realization $|K|$ by

$$D_0|K| \xrightarrow{\alpha_0|K|} E_0|K| \xrightarrow{\beta_0|K|} \dots \xrightarrow{\alpha_n|K|} E_n|K| \xrightarrow{\beta_n|K|} D_{n+1}|K| \xrightarrow{\alpha_{n+1}|K|} \dots \tag{3.1}$$

and, likewise, write

$$D_0K \xrightarrow{\alpha_0 K} E_0K \xrightarrow{\beta_0 K} \dots \xrightarrow{\alpha_n K} E_nK \xrightarrow{\beta_n K} D_{n+1}K \xrightarrow{\alpha_{n+1} K} \dots \tag{3.2}$$

for the corresponding sequence (2.1) in the category of based simplicial sets. Realization carries the sequence (3.2) to the sequence

$$|D_0K| \xrightarrow{|\alpha_0 K|} |E_0K| \xrightarrow{|\beta_0 K|} \dots \xrightarrow{|\alpha_n K|} |E_nK| \xrightarrow{|\beta_n K|} |D_{n+1}K| \xrightarrow{|\alpha_{n+1} K|} \dots \tag{3.3}$$

of based topological spaces. Now

$$D_0|K| = e = |D_0K|, \; E_0|K| = |K| = |E_0K|,$$

and the map $\alpha_0|K| = |\alpha_0 K|$ is the canonical inclusion. Let

$$\tau_0 \colon D_0|K| \to |D_0K| \quad \text{and} \quad \rho_0 \colon E_0|K| \to |E_0K|$$

be the identity mappings. Let $n \geq 1$ and suppose by induction that homeomorphisms

$$\tau_j \colon D_j|K| \to |D_jK| \quad \text{and} \quad \rho_j \colon E_j|K| \to |E_jK|,$$

each ρ_j being $|K|$-equivariant, have been constructed for $j < n$, having the following properties:

(1) The diagrams

$$\begin{array}{ccc} E_{j-1}|K| & \xrightarrow{\beta_{j-1}|K|} & D_j|K| \\ \downarrow{\scriptstyle \rho_{j-1}} & & \downarrow{\scriptstyle \tau_j} \\ |E_{j-1}K| & \xrightarrow{|\beta_{j-1}K|} & |D_jK| \end{array} \qquad \begin{array}{ccc} D_j|K| & \xrightarrow{\alpha_j|K|} & E_j|K| \\ \downarrow{\scriptstyle \tau_j} & & \downarrow{\scriptstyle \rho_j} \\ |D_jK| & \xrightarrow{|\alpha_j K|} & |E_jK| \end{array} \tag{3.4}$$

are commutative;

(2) each τ_j identifies the realization $|\psi_j K| \colon |CD_jK| \to |D_jK|$ of the conical contraction $\psi_j K \colon CD_jK \to D_jK$ of simplicial sets with the geometric contraction $\psi_j|K| \colon CD_j|K| \to D_j|K|$;

(3) each ρ_j identifies the realization $|\phi_j K| \colon |(E_jK) \times K| \to |E_jK|$ of the simplicial K-action $\phi_j K \colon (E_jK) \times K \to E_jK$ with the topological $|K|$-action $\phi_j|K| \colon (E_j|K|) \times |K| \to E_j|K|$.

Consider the realization of (2.2); it is a push out square of topological spaces. Hence the maps τ_{n-1} and ρ_{n-1} induce a map τ_n from $D_n|K|$ to $|D_nK|$, necessarily a homeomorphism, so that $C|\tau_{n-1}|, |\tau_{n-1}|, C|\rho_{n-1}|$ and $|\tau_n|$ yield a homeomorphism of squares between the realization of (2.2) and (1.8), where (1.8) is taken with reference to $G = |K|$. Moreover, the homeomorphism τ_n identifies the realization $|\psi_n K| \colon |CD_nK| \to |D_nK|$ of the conical contraction $\psi_n K \colon CD_nK \to D_nK$ of simplicial sets with the contraction $\psi_n|K|: CD_n|K| \to D_n|K|$. Likewise the maps ρ_{n-1} and τ_n induce a map ρ_n from $E_n|K|$ to $|E_nK|$, necessarily a $|K|$-equivariant homeomorphism, so that $|\rho_{n-1} \times \mathrm{Id}_K|, |\rho_{n-1}|, |\tau_n \times \mathrm{Id}_K|$ and $|\rho_n|$ yield a homeomorphism of squares between the realization of (2.3) and (1.9), where (1.9) is understood with reference to $G = |K|$. Moreover, the homeomorphism ρ_n is $|K|$-equivariant and identifies the realization $|\phi_n K|: |(E_nK) \times K| \to |E_nK|$ of the simplicial K-action $\phi_n K: (E_nK) \times K \to E_nK$ with the topological $|K|$-action $\phi_n|K|: (E_n|K|) \times |K| \to E_n|K|$. The requisite diagrams (3.4) for $j = n$ are manifestly commutative. This completes the inductive step.

The limit

$$\rho = \lim \rho_n = \lim \tau_n : E|K| \to |WK|$$

is a $|K|$-equivariant homeomorphism; it identifies the principal $|K|$-bundles $E|K| \to B|K|$ and $|WK| \to |\overline{W}K|$ as asserted and is plainly natural in K. This proves the theorem. □

4. The other cone construction

The classifying space $B|K|$ is homeomorphic to the realization of the nerve NK of K as a *bisimplicial* set. (With reference to their obvious CW-structures, the two spaces are not isomorphic as CW-complexes, though.) On the other hand, the diagonal DNK is a simplicial set which does *not* coincide with the reduced W-construction $\overline{W}K$, but its realization is homeomorphic to the realization of the nerve NK of K as a bisimplicial set since this is known to be true for an arbitrary bisimplicial set [20]. The purpose of this section is to clarify the relationships between the various spaces and constructions.

As already pointed out, the construction (2.1) can be carried out with the simplicial smash product $(\cdot) \wedge \Delta[1]$ instead of the reduced cone: The simplicial interval $\Delta[1]$ carries a (unique) structure of a simplicial monoid having (1) as its unit, and hence we can talk about an action $X \times \Delta[1] \to X$ of $\Delta[1]$ on a simplicial set X; such an action is a special kind of simplicial homotopy which "ends" at the identity morphism of X. The fact that the naive notion of homotopy of morphisms of simplicial sets is not an equivalence relation is not of significance here. Much as before, the simplicial interval $\Delta[1]$ gives rise to a monad $(\times\Delta[1], \mu, \eta)$ in the category of simplicial sets and an action of $\Delta[1]$ on a simplicial set X is an *algebra* structure on X over this monad.

The *base point* of $\Delta[1]$ is defined to be (0). For a based simplicial set (X, x_0), we shall refer to an action $\psi: X \times \Delta[1] \to X$ as a $\Delta[1]$-*contraction* of X provided ψ sends the base point $(x_0, 0)$ of $X \times \Delta[1]$ to x_0 and factors through the *simplicial smash product*

$$X \wedge \Delta[1] = X \times \Delta[1] / (X \times \{0\} \cup \{x_0\} \times \Delta[1]).$$

The latter is viewed endowed with the obvious base point, the image of $X \times \{0\} \cup \{x_0\} \times \Delta[1]$ in $X \wedge \Delta[1]$. Abusing notation, the corresponding map from $X \wedge \Delta[1]$ to X will as well be denoted by ψ and referred to as a $\Delta[1]$-*contraction*. Moreover we write $\eta = \eta_X^{\Delta[1]}$ for the map, the corresponding *unit*, which embeds X into $X \wedge \Delta[1]$ by sending a simplex x of X to $(x, 1) \in X \wedge \Delta[1]$. The right action of $\Delta[1]$ on $X \times \Delta[1]$ induces a $\Delta[1]$-contraction

$$\mu_X^{\Delta[1]}: X \wedge \Delta[1] \wedge \Delta[1] \to X \wedge \Delta[1]$$

of $X \wedge \Delta[1]$. In categorical language, the functor $(\cdot) \wedge \Delta[1]$ and natural transformations μ and η constitute a *monad* in the category of simplicial sets, and a $\Delta[1]$-contraction of a based simplicial set X is an *algebra* structure on X over this monad.

Formally carrying out the construction (2.1) with the simplicial smash product $(\cdot) \wedge \Delta[1]$ instead of the reduced cone we obtain the diagram

$$D'_0 \xrightarrow{\alpha'_0} E'_0 \xrightarrow{\beta'_0} D'_1 \xrightarrow{\alpha'_1} \ldots \xrightarrow{\beta'_{n-1}} D'_n \xrightarrow{\alpha'_n} E'_n \xrightarrow{\beta'_n} D'_{n+1} \xrightarrow{\alpha'_{n+1}} \ldots \tag{4.1}$$

of based simplicial sets and injections of based simplicial sets together with $\Delta[1]$-contractions $\psi'_n: D'_n \wedge \Delta[1] \to D'_n$ and free K-actions $\phi'_n: E'_n \times K \to E'_n$. Its limit

$$D = \varinjlim E'_n = \varinjlim D'_n$$

inherits a $\Delta[1]$-contraction $\psi': D \wedge \Delta[1] \to D$ and a free K-action $\phi': D \times K \to D$. To explain the significance thereof, recall that the nerve construction yields a simplicial object

$$K \to ENK \to NK \tag{4.2}$$

in the category of principal simplicial K-bundles which is natural for morphisms of simplicial groups. Here ENK and NK inherit structures of bisimplicial sets, one from the nerve construction and the other one from the simplicial structure of K, and the projection from ENK to NK is a morphism of bisimplicial sets; further, for each simplicial degree $q \geq 0$ coming from the nerve construction, (4.2) amounts to a principal K-bundle

$$K_* \to (ENK)_{*,q} \to (NK)_{*,q}$$

while for each simplicial degree $p \geq 0$ of $K = \{K_p\}$ itself, (4.2) comes down to the universal simplicial principal K_p-bundle

$$K_p \to (ENK)_{p,*} \to (NK)_{p,*};$$

in particular, each $(ENK)_{p,*}$ is contractible in the usual sense. The diagonal bundle

$$\delta: DENK \to DNK$$

is manifestly a principal K-bundle having $DENK$ contractible, and we have

$$DENK = \varinjlim E'_n = \varinjlim D'_n$$

as (right) K-set; moreover, the above morphism $\psi': DENK \wedge \Delta[1] \to DENK$ induces a simplicial contraction of $DENK$.

Theorem 4.3. *There is a canonical homeomorphism of principal $|K|$-bundles between the realization $|DENK| \to |DNK|$ of the diagonal bundle and the realization $|WK| \to |\overline{W}K|$ of the W-construction which is natural in K.*

Proof. The classifying space $B|K|$ is the realization of NK as a bisimplicial set, and the same kind of remark applies to $E|K|$ and the projection to $B|K|$. The already cited fact that, for an arbitrary bisimplicial set, the realization of the diagonal is homeomorphic to the realization as a bisimplicial set [20] implies the following statement.

Theorem 4.4. *There is a canonical $|K|$-equivariant homeomorphism between $|DENK|$ and $E|K|$ and hence a canonical homeomorphism between $|DNK|$ and $B|K|$. These homeomorphisms are natural in K.*

We conclude from this that *the statement of the Theorem* (in the Introduction) *is formally equivalent to the statement of* (4.3). In fact, the Theorem identifies the realization of the W-construction with the realization of the nerve as a *bisimplicial set* whereas (4.3) identifies the realization of the W-construction with the realization of the *diagonal* of the nerve.

Remark 1. While the statement of (4.4) is obtained for free, the identifications just mentioned, in turn, are *not* obtained for free, as we have shown in this paper.

Remark 2. For a based simplicial set $(X,*)$, the realization $|CX|$ of the cone CX is naturally homeomorphic to the realization $|X \wedge \Delta[1]|$ of $X \wedge \Delta[1]$. In fact, a suitable subdivision of $|CX|$ yields a realization of $X \wedge \Delta[1]$. It is tempting to try to construct a homeomorphism between $|DENK|$ and $|WK|$ in a combinatorial way by inductively constructing the requisite maps between the realizations of the constituents of (4.1) and of the corresponding terms in (2.1) but we did not succeed in so doing. The problem is that the realization of the simplicial monoid $\Delta[1]$ does not yield the geometric monoid structure on the interval I coming into play in Section 1 above, whence the realization of an action $X \times \Delta[1] \to X$ of $\Delta[1]$ on a simplicial set X is *not* an I-action on the realization of X in the sense of Section 1. Rather, the realization of the simplicial monoid structure on $\Delta[1]$ yields the function from $I \times I$ to I which sends (a,b) to $\min(a,b)$. A suitable homeomorphism identifies this monoid structure with the more usual one considered in Section 1 above. For example, as pointed out by the referee, one could take the function which assigns $(a\max(a,b), b\max(a,b)) \in I^2$ to $(a,b) \in I^2$. Further, the monoid structure arising from the function min also gives rise to a monad in the category of spaces and with reference to it, the construction (1.7) can still be carried out; formally the same argument as that for the proof of our main result then identifies the limit (say) LK of the resulting sequence of spaces with the realization $|DNK|$ of DNK and, by virtue of (4.4), LK is naturally homeomorphic to $B|K|$. However we do not see how this homeomorphism may be obtained directly since we are unable to identify the monad in the category of spaces arising from the unit interval having the usual multiplication as monoid structure with the other monad arising from the function min as monoid structure.

References

[1] C. Berger, Une version effective du théorème de Hurewicz, Thèse de doctorat, Université de Grenoble, 1991.

[2] R. Bott, On the Chern–Weil homomorphism and the continuous cohomology of Lie groups, Adv. in: Math. 11 (1973) 289–303.

[3] R. Bott, H. Shulman, J.D. Stasheff, On the de Rham theory of certain classifying spaces, Adv. in: Math. 20 (1976) 43–56.

[4] E.B. Curtis, Simplicial homotopy theory, Adv. in: Math. 6 (1971) 107–209.

[5] W.G. Dwyer, D.M. Kan, Homotopy theory and simplicial groupoids, Indag. Math. 46 (1984) 379–385.

[6] S. Eilenberg, S. Mac Lane, On the groups $H(\pi, n)$. I., Ann. Math. 58 (1953) 55–106.

[7] S. Eilenberg, S. Mac Lane, On the groups $H(\pi, n)$. II. Methods of computation, Ann. Math. 60 (1954) 49–139.

[8] V.K.A.M. Gugenheim, On the chain complex of a fibration, Illinois J. Math. 16 (1972) 398–414.

[9] V.K.A.M. Gugenheim, J.P. May, On the theory and applications of differential torsion products, Mem. Amer. Math. Soc. 142 (1974).

[10] J. Huebschmann, T. Kadeishvili, Small models for chain algebras, Math. Z. 207 (1991) 245–280.

[11] J. Huebschmann, Extended moduli spaces, Kan construction, and lattice gauge theory, Topology, to appear.

[12] D.M. Kan, On homotopy theory and c.s.s. groups, Ann. Math. 68 (1958) 38–53.

[13] S. Mac Lane, Milgram's classifying space as a tensor product of functors, in: F.P. Peterson (Ed.), The Steenrod Algebra and its Applications, Lecture Notes in Mathematics, vol. 168, Springer, Berlin, 1970, pp. 135–152.

[14] S. Mac Lane, Categories for the Working Mathematician, Graduate Texts in Mathematics, vol. 5, Springer, Berlin, 1971.

[15] J. Milgram, The bar construction and abelian H-spaces, Illinois J. Math. 11 (1967) 242–250.

[16] J. Milnor, Construction of universal bundles. I, II, Ann. Math. 63 (1956) 272–284, 430–436.

[17] J. Milnor, The realization of a semi-simplicial complex, Ann. Math. 65 (1957) 357–362.

[18] J. Moore, Comparison de la bar construction à la construction W et aux complexes $K(\pi, n)$, Exposé 13, Séminaire H. Cartan (1954/1955) 242–250.

[19] D. Puppe, Homotopie und Homologie in abelschen Gruppen und Monoidkomplexen. I. II, Math. Z. 68 (1958) 367–406, 407–421.

[20] D. Quillen, Higher algebraic K-theory, I in: H. Bass (Ed.), Algebraic K-theory I, Higher K-theories, Lecture Notes in Mathematics, vol. 341, Springer, Berlin, 1973, pp. 85–147.

[21] G.B. Segal, Classifying spaces and spectral sequences, Publ. Math. I.H.E.S. 34 (1968) 105–112.

[22] G.B. Segal, Categories and cohomology theories, Topology 13 (1974) 293–312.

[23] J.D. Stasheff, Homotopy associativity of H-spaces. I, II, Trans. Amer. Math. Soc. 108 (1963) 275–292, 293–312.

[24] J.D. Stasheff, H-spaces and classifying spaces: foundations and recent developments, in: Proc. Symp. Pure Math., vol. 22 American Math. Soc., Providence, RI, 1971, pp. 247–272.

[25] N.E. Steenrod, Milgram's classifying space of a topological group, Topology 7 (1968) 349–368.

[26] S. Weingram, The realization of a semisimplicial bundle map is a k-bundle map, Trans. Amer. Math. Soc. 127 (1967) 495–514.

[27] S.-C. Wong, Comparison between the reduced bar construction and the reduced W-construction, Diplomarbeit, Math. Institut der Universität Heidelberg, 1985.