Cellular structures for E_n-operads

Clemens Berger

15 july 1998

Introduction

These notes are a detailed account of two lectures I gave during a workshop on operads in Osnabrück (16-19 June 1998). I would like to thank Rainer Vogt for organizing this really stimulating meeting which gave the participants the wonderful chance to exchange their ideas in a very lively atmosphere.

The purpose of my lectures is fourfold :

1. to show "on the nose" that the well known configuration space model for $\Omega^n S^n X$ is homotopy equivalent to Milgram's permutohedral model ;
2. to indicate a "recipe" for constructing cellular decompositions of E_n-operads ;
3. to give a simplicial splitting of $\Omega^n S^n X$ using Jeff Smith's filtration of the "symmetric monoidal" operad ;
4. to outline some interaction between E_n-operads and *immersion theory.*

1 Configuration spaces and permutohedra.

Initially, the interest in iterated loop spaces arose from *homotopy theory*, more precisely from the fact that several important classifying spaces were known to be infinite loop spaces. In Peter May's theory of E_n-operads [20], the *recognition principle* for n-fold iterated loop spaces is based on the *approximation theorem* which (in its crude form) states that (for a connected, well pointed space X) the weak homotopy type of $\Omega^n S^n X$ can be realized as a coend $F(\mathbb{R}^n, -) \otimes_{\mathbf{\Lambda}} \underline{X}$. The ingredients for this coend are the *configuration spaces*

$$F(\mathbb{R}^n, k) = \{(t_1, \dots, t_k) \in (\mathbb{R}^n)^k \mid t_i \neq t_j \text{ for } i \neq j\}.$$

This May-Segal model for $\Omega^n S^n X$ is predated by Milgram's model which can also be written as a coend $J^{(n)} \otimes_{\mathbf{\Lambda}} \underline{X}$, where this time the coend ingredients are constructed out of convex polytopes P_k, nowadays called *permutohedra.* The permutohedron P_k is by definition the *convex hull of the symmetric group* $\mathfrak{S}_k$

"generically" embedded in some affine euclidean space. There are different ways of doing so. Milgram considered the orbit $\mathfrak{S}_k.(1,2,\dots,k)$ in $\mathbb{R}^k$, the symmetric group acting by permutation of the coordinates. Baues [4] considered the $k!$ edge-paths in $[0,1]^k$ joining the vertex $(0,\dots,0)$ to the vertex $(1,\dots,1)$, realizing this way the permutohedron as a *combinatorial path space*. Probability theorists consider $\mathfrak{S}_k$ as the set of permutation matrices embedded in the affine space of $k\times k$-matrices with real coefficients. The permutohedron is then identified with the space of the so called *probabilistic matrices* with column and line sum equal to 1.

The following lemma is crucial to Milgram's construction, cf. also Michael Brinkmeier's lecture [10] :

Lemma 1.1. *The face-poset of the permutohedron P_k is canonically isomorphic to the set of (right) cosets of subgroups $\mathfrak{S}_{i_1}\oplus\cdots\oplus\mathfrak{S}_{i_r}$ of $\mathfrak{S}_k$ with $k=i_1+\cdots+i_r$, partially ordered by inclusion.*

The purpose of this section is to define equivariant cellular decompositions of both "building blocks" $F(\mathbb{R}^n,k)$ and $J_k^{(n)}$ in order to deduce that the May-Segal model and the Milgram model have the same homotopy type. Incidentally, this provides us with a nice compact model $J_k^{(2)}/\mathfrak{S}_k$ for the classifying space of the braid group on k strands $BB_k=F(\mathbb{R}^2,k)/\mathfrak{S}_k$ which is actually known in the literature as the *Salvetti-complex* of the symmetric group $\mathfrak{S}_k$ and which exists for every finite Coxeter group, cf. Charney-Davis [11].

The common structure of the families $F(\mathbb{R}^n,k), J_k^{(n)}, k\geq 1$, is that of a *preoperad* (or coefficient system, cf. [14]).

Definition 1.2. *A (topological) preoperad is a functor $\mathcal{O}:\mathbf{\Lambda}^{op}\to\mathrm{Top}$, where $\mathbf{\Lambda}$ is the category of (non-empty) finite sets $\underline{k}=\{1,\dots,k\}$ and injective maps. Each pointed space $(X,*)$ defines a functor $\underline{X}:\mathbf{\Lambda}\to\mathrm{Top}_*:\underline{k}\mapsto X^k$ so that by the usual coend construction we end up with a bifunctor*

$$\begin{array}{rccl} -\otimes_{\mathbf{\Lambda}}-: & \mathrm{Top}^{\mathbf{\Lambda}^{op}}\times\mathrm{Top}_* & \to & \mathrm{Top}\\ & (\mathcal{O},X) & \mapsto & \mathcal{O}\otimes_{\mathbf{\Lambda}}\underline{X}=(\coprod_{k\geq 1}\mathcal{O}_k\times X^k)/\sim . \end{array}$$

where $(\phi^*(t),x)\sim(t,\phi_*(x))$ for all $t\in\mathcal{O}_l, x\in X^k, \phi:\underline{k}\to\underline{l}$.

In order to make sure that this bifunctor preserves weak equivalences in both arguments, we have two choices : either we look for the "correct" notion of weak equivalence for both arguments, or we restrict to objects for which the "naive" (= pointwise) notion of weak equivalence works. We choose the latter variant which is sufficient for our purpose. Indeed, if we restrict to preoperads $\mathcal{O}$ with *free* $\mathfrak{S}_k$-action on $\mathcal{O}_k$ for all k, and to *well* pointed spaces X (i.e. such that the basepoint inclusion is a cofibration), then the bifunctor $-\otimes_{\mathbf{\Lambda}}-$ preserves (the naive notion of) weak equivalences in both arguments. The proof of this fact relies on the unique factorization property of the category $\mathbf{\Lambda}$ which allows to write $\mathcal{O}\otimes_{\mathbf{\Lambda}}\underline{X}$ as an iterated pushout along cofibrations with values in $\mathcal{O}_k\times_{\mathfrak{S}_k}X^k$, cf. [14].

Notation 1.3. Each morphism of the category $\mathbf{\Lambda}$ factorizes uniquely as a *permutation* followed by an *increasing* map. For $\phi \in \mathbf{\Lambda}(\underline{k}, \underline{l})$, we write $\phi = \phi^{inc} \circ \phi^{\natural}$ with $\phi^{\natural} \in \mathbf{\Lambda}(\underline{k}, \underline{k})$ and $\phi^{inc} \in \mathbf{\Lambda}^{inc}(\underline{k}, \underline{l}) = \{\phi \in \mathbf{\Lambda}(\underline{k}, \underline{l}) | \, \phi(i) < \phi(j) \text{ for } i < j\}$.

Remark 1.4. The increasing maps $\phi : \underline{k} \to \underline{l}$ are generated by elementary *degeneracy* operators $D_i : \underline{k} \to \underline{k+1}$ mapping j to j (resp. $j+1$) if $j < i$ (resp. $j \geq i$). A preoperad $\mathcal{O}$ is thus uniquely determined by a family of $\mathfrak{S}_k$-spaces $\mathcal{O}_k$ together with degeneracy maps $D_i^* : \mathcal{O}_{k+1} \to \mathcal{O}_k$ $(i = 1, \ldots, k+1)$ satisfying the appropriate relations. For an *operad* $\mathcal{O}$, these degeneracy maps are given by *evaluation* :

$$D_i^* : \mathcal{O}_{k+1} \to \mathcal{O}_k : t \mapsto t(1, \ldots, 1, \underset{i}{*}, 1, \ldots, 1)$$

where $1 \in \mathcal{O}_1$ is the unit of the operad, and $* \in \mathcal{O}_0$ is the *unique* constant of the operad. Observe that in our definition of preoperads we discard constants.

Examples 1.5. *(a) The configuration preoperads.*
The collection of configuration spaces $F(\mathbb{R}^n, k)$ defines a topological preoperad $F(\mathbb{R}^n, -) : \mathbf{\Lambda}^{op} \to \text{Top}$ by setting for $\phi \in \mathbf{\Lambda}(\underline{k}, \underline{l})$:

$$\begin{array}{rcl} \phi^* : F(\mathbb{R}^n, l) & \to & F(\mathbb{R}^n, k) \\ (t_1, \ldots, t_l) & \mapsto & (t_{\phi(1)}, \ldots, t_{\phi(k)}). \end{array}$$

(b) The permutation preoperad.
The collection of symmetric groups $\mathfrak{S}_k = \mathbf{\Lambda}(\underline{k}, \underline{k})$ defines a set-valued preoperad $\mathfrak{S} : \mathbf{\Lambda}^{op} \to \text{Sets}$ by setting for $\phi \in \mathbf{\Lambda}(\underline{k}, \underline{l})$:

$$\begin{array}{rcl} \phi^* : \mathfrak{S}_l & \to & \mathfrak{S}_k \\ \sigma & \mapsto & (\sigma \circ \phi)^{\natural}. \end{array}$$

Equivalently, the permutation preoperad can also be defined as the composite functor $\pi_0 \circ F(\mathbb{R}, -) : \mathbf{\Lambda}^{op} \to \text{Sets}$, identifying $\mathfrak{S}_k$ with the path components of $F(\mathbb{R}, k)$.

(c) The permutohedral preoperad. By convex extension of the permutation preoperad we get the permutohedral preoperad $P : \mathbf{\Lambda}^{op} \to \text{Top}$, cf. [5].

Definition 1.6. *Milgram's preoperads* $J^{(n)} : \mathbf{\Lambda}^{op} \to \text{Top}$ *are defined by*

$$J_k^{(n)} = (P_k)^{n-1} \times \mathfrak{S}_k / \sim$$

where the equivalence relation identifies certain boundary cells. Explicitly, for each point $(\tau^*(t_1), \ldots, \tau^*(t_{n-1}); \sigma) \in (P_k)^{n-1} \times \mathfrak{S}_k$ *such that* t_s *belongs to the convex hull of a proper subgroup* $\mathfrak{S}_{i_1} \oplus \cdots \oplus \mathfrak{S}_{i_r} \subset \mathfrak{S}_k$ *and such that* τ^{-1} *is a* $(i_1, \ldots, i_r)$*-shuffle of* $\mathfrak{S}_k$*, we have the relation*

$$\begin{gathered} (\tau^*(t_1), \ldots, \tau^*(t_{n-1}); \sigma) \sim \\ (t_1, \ldots, t_s, D^*_{i_1, \ldots, i_r}(t_{s+1}), \ldots, D^*_{i_1, \ldots, i_r}(t_{n-1}); \tau\sigma) \end{gathered}$$

where $D^*_{i_1, \ldots, i_r}$ *denotes the projection of* P_k *onto the convex hull of* $\mathfrak{S}_{i_1} \oplus \cdots \oplus \mathfrak{S}_{i_r}$*. The* $\mathbf{\Lambda}$*-structure is induced by* $\phi^*(t_1, \ldots, t_{n-1}; \sigma) = (t_1^{(\sigma\phi)^{inc}}, \ldots, t_{n-1}^{(\sigma\phi)^{inc}}; (\sigma\phi)^{\natural})$.

For $n=1$, $J^{(1)} \otimes_{\mathbf{\Lambda}} \underline{X}$ reduces to *James' model* for ΩSX; indeed, $J_k^{(1)} = \mathfrak{S}_k$ and $\mathfrak{S} \otimes \underline{X}$ is nothing but the free monoid generated by the pointed space $(X, *)$.

For $n = 2$, the space $J_k^{(2)}$ is obtained by gluing together $k!$ copies of P_k according to the (right) $\mathfrak{S}_k$-action. This action is only "partially free" and the quotient space $J_k^{(2)}$ can be interpreted as the *free extension* of this partially free action. Indeed, the equivalence relation $(t, \sigma) \sim (t^\sigma, id_k)$ applies precisely when σ acts freely and orientation preserving on the minimal face containing t. The resulting $\mathfrak{S}_k$-action on $J_k^{(2)}$ is free.

Proposition 1.7. *The configuration preoperad $F(\mathbb{R}^n, -)$ and Milgram's preoperad $J^{(n)}$ are equivalent as preoperads. In particular, the $\mathfrak{S}_k$-spaces $F(\mathbb{R}^n, k)$ and $J_k^{(n)}$ have the same equivariant homotopy type and for a well pointed space $(X, *)$, the May-Segal model $F(\mathbb{R}^n, -) \otimes_{\mathbf{\Lambda}} \underline{X}$ and the Milgram model $J^{(n)} \otimes_{\mathbf{\Lambda}} \underline{X}$ are homotopy equivalent.*

Proof. – We shall outline a proof for the case $n = 2$ (cf. [5], [6]). The idea is to define equivariant cellular decompositions for $F(\mathbb{R}^2, k)$ and $J_k^{(2)}$ and to compare the associated cell posets, cf. definition 2.2. For the configuration space $F(\mathbb{R}^2, k)$ this is done by ordering its points antilexicographically, i.e. $(x_1, x_2) < (y_1, y_2)$ iff either $x_2 < y_2$ or $x_2 = y_2 \wedge x_1 < y_1$. In the first case we write $x \leq_1 y$, whereas in the second case we write $x \leq_0 y$. Each labeled linear graph

$$\sigma^{-1}(1) \xrightarrow{\alpha_{12}} \sigma^{-1}(2) \xrightarrow{\alpha_{23}} \cdots \xrightarrow{\alpha_{k-1,k}} \sigma^{-1}(k) \text{ with } \alpha_{i,i+1} \in \{0, 1\}$$

defines then the closed contractible subset

$$F_k^{(\sigma^*\alpha)} = \{(t_1, \ldots, t_k) \in F(\mathbb{R}^2, k) \,|\, t_{\sigma^{-1}(1)} \underset{\alpha_{12}}{\leq} t_{\sigma^{-1}(2)} \underset{\alpha_{23}}{\leq} \cdots \underset{\alpha_{k-1,k}}{\leq} t_{\sigma^{-1}(k)}\}$$

Let us write $\mathcal{K}(F)_k^{(2)}$ for the poset of the so defined "cells". – Similarly, to each labeled linear graph as above, we associate the closed contractible set

$$J_k^{(\sigma^*\alpha)} = Hull(\mathfrak{S}_{i_1} \oplus \mathfrak{S}_{i_2} \oplus \cdots \oplus \mathfrak{S}_{i_r}) \times \{\sigma\} \subset J_k^{(2)}$$

where the "partition" $(i_1, \ldots, i_r)$ of k is defined by the $1'$s in the sequence $(\alpha_{12}, \ldots, \alpha_{k-1,k})$; we write $\mathcal{K}(J)_k^{(2)}$ for the poset of these cells. One then shows that $\mathcal{K}(F)_k^{(2)}$ and $\mathcal{K}(J)_k^{(2)}$ are *antiisomorphic* posets and deduces the statement from the following diagram of $\mathbf{\Lambda}$-equivalences, cf. [6] :

$$F(\mathbb{R}^2, k) \xleftarrow{\sim} \text{hocolim} F_k^{(\sigma^*\alpha)} \xrightarrow{\sim} |\mathcal{K}(F)_k^{(2)}| \cong |\mathcal{K}(J)_k^{(2)}| \xleftarrow{\sim} \text{hocolim} J_k^{(\sigma^*\alpha)} \xrightarrow{\sim} J_k^{(2)}$$

□

Remark 1.8. More generally, there are cellular decompositions of $F(\mathbb{R}^n, k)$ resp. $J_k^{(n)}$ with antiisomorphic cell-posets $\mathcal{K}(F)_k^{(n)}$ resp. $\mathcal{K}(J)_k^{(n)}$. The poset $\mathcal{K}(F)_k^{(n)}$ implicitly appears already in Fadell-Neuwirth's paper [16] and has also been studied by Getzler-Jones [17]. The poset $\mathcal{K}(J)_k^{(n)}$ plays a central role in Balteanu-Fiedorowicz-Schwänzl-Vogt's study of n-fold monoidal categories [2].

Remark 1.9. In [5], [6] I wrongly claimed to be able to endow the Milgram preoperad $J^{(n)} : \mathbf{\Lambda}^{op} \to \mathrm{Top}$ with an operad structure. Although the *permutohedral* preoperad *is* an operad, this operad structure doesn't carry over to $J^{(n)}$, since its multiplication doesn't preserve the boundary. I tried to force the permutohedral multiplication to preserve the boundary by passing from convex to *cubical* extension, but I overlooked that cubical extension commutes with composition *only up to homotopy* so that I really got some kind of lax operad where all relations only hold up to (a uniquely specified) homotopy. Michael Brinkmeier shows in his lecture [10] that just assuming the ordinary (strict) relations between the "cubical" multiplication $J_2^{(2)} \times J_2^{(2)} \times J_1^{(2)} \to J_3^{(2)}$ (resp. $J_2^{(2)} \times J_1^{(2)} \times J_2^{(2)} \to J_3^{(2)}$) and the various degeneracies already implies that a hypothetical $J^{(2)}$-algebra would be a commutative monoid, which is incompatible with the approximation theorem.

Remark 1.10. Jim McClure and Jeff Smith construct an E_2-operad which acts on *topological Hochschild cohomology* and which is closely related to $J^{(2)}$, see Jim McClure's lecture [22]. This E_2-operad is a kind of semi-direct product of the little 1-cubes operad with the permutohedral operad. Its multiplication uses *prismatic* decompositions of the permutohedra P_k (labeled by "formulae") which can be described as follows : The image of $P_2 \times P_1 \times P_{k-1} \to P_k$ is a prism $\Delta^1 \times P_{k-1}$, thus by induction endowed with a prismatic decomposition; it turns out that the (closure of the) complement of the image also admits a prismatic decomposition labeled by the set of proper faces of P_{k-1}. The good thing about this prismatic decomposition of P_k is that all its cells are convex hulls of vertex-sets of P_k and thus allow a very neat combinatorial description.

2 The complete graph operad

While the configuration spaces $F(\mathbb{R}^n, k)$ suffice to approximate $\Omega^n S^n X$ for connected spaces X, they allow us neither to reconstruct the n different loop structures of $\Omega^n S^n X$ nor to recognize general n-fold iterated loop spaces. The appropriate gadget for these tasks is the *little n-cubes operad* of Boardman-Vogt which we shall denote by $\mathcal{C}^{(n)}$ [9]. As a preoperad, the little n-cubes are equivalent to $F(\mathbb{R}^n, -)$, but thanks to the canonical inclusion $\mathcal{C}_k^{(n)} \hookrightarrow Top_*(S^n, \bigvee_{i=1}^k S^n)$ they come equipped with a multiplication (indeed a *substitution product*)

$$\mathcal{C}_k^{(n)} \times \mathcal{C}_{i_1}^{(n)} \times \cdots \times \mathcal{C}_{i_k}^{(n)} \longrightarrow \mathcal{C}_{i_1+\cdots+i_k}^{(n)}$$

turning them into an *operad*. The endofunctor $\mathcal{C}^{(n)} \otimes_{\mathbf{\Lambda}} - : \mathrm{Top}_* \to \mathrm{Top}_*$ is then a *monad* such that the category of n-fold iterated loop spaces fully embeds into the category of $\mathcal{C}^{(n)}$-spaces. Moreover, each connected $\mathcal{C}^{(n)}$-space is $\mathcal{C}^{(n)}$-equivalent to an n-fold loop space [20]. Operads equivalent to the little n-cubes operad are called *E_n-operads.*

The following question then naturally arises : Are there other interesting E_n-operads or is there even an internal characterization of E_n-operads ? For the moment, we are far away from such a characterization if $n \neq 1, 2$ or ∞, however using “cellular techniques”, Zig Fiedorowicz found a method of *constructing* new E_n-operads out of spaces having the equivariant homotopy type of an $(n-1)$-sphere, see his lecture [15].

In this section I shall describe a poset-valued E_n-operad which may serve to define an equivariant cellular decomposition of a topological E_n-operad $\mathcal{O}$. The leading idea is that the k-level space $\mathcal{O}_k$ should be determined by the family of $\binom{k}{2}$ degeneracies $\phi^*_{ij}:\mathcal{O}_k\to\mathcal{O}_2,\ i<j$, whereas the 2-level space $\mathcal{O}_2$ has to have the $\mathfrak{S}_2$-equivariant homotopy type of an $(n-1)$-sphere. Here and later on, $\phi_{ij}:\underline{2}\to\underline{k}$ denotes the map which sends $(1,2)$ to (i,j), and $\sigma_{ij}=\phi^*_{ij}(\sigma)$:

Definition 2.1. *Set* $\mathcal{K}^{(n)}_k=\{0,1,\dots,n-1\}^{\binom{k}{2}}\times\mathfrak{S}_k$ *and denote the elements of* $\mathcal{K}^{(n)}_k$ *by* (μ,σ) *where* $\mu=(\mu_{ij})_{1\le i<j\le k}$*. Define a partial order by*

$$(\mu,\sigma)\le(\nu,\tau)\overset{def}{\Leftrightarrow}\ \textit{either}\ \sigma_{ij}=\tau_{ij}\wedge\mu_{ij}\le\nu_{ij}\ \textit{or}\ \sigma_{ij}\ne\tau_{ij}\wedge\mu_{ij}<\nu_{ij}$$

There is a nice combinatorial description of $\mathcal{K}^{(n)}_k$, due to Fiedorowicz [2] : Each element $(\mu,\sigma)\in\mathcal{K}^{(n)}_k$ corresponds to an *acyclic orientation* of the *complete graph on k vertices* together with a $\{0,1,\dots,n-1\}$-*coloring* of its edges. Explicitly,

$$(\mu,\sigma)\longleftrightarrow\begin{cases}\text{acyclic orientation} & \sigma^{-1}(1)\longrightarrow\sigma^{-1}(2)\longrightarrow\cdots\longrightarrow\sigma^{-1}(k)\\ \text{edge}-\text{coloring} & i\overset{\mu_{ij}}{\longleftrightarrow}j\end{cases}$$

In this setting, $(\mu,\sigma)\le(\nu,\tau)$ iff for all $i<j$, we have $\mu_{ij}\le\nu_{ij}$ with $\mu_{ij}<\nu_{ij}$ whenever the induced orientations on the edge i—j differ.

Observe that throughout these notes I use the convention that $\mathfrak{S}_k$ acts from the *right*, which is the natural choice with respect to the category $\mathbf{\Lambda}$. The reader who prefers left actions has simply to replace every permutation by its inverse, but he has to be aware of the fact that this choice also affects the equivariance relations to be hold by an operad !

Now, let us describe the operad structure of $(\mathcal{K}^{(n)}_k)_{k\ge1}$: Since the edge-colorings of the complete graphs form themselves a preoperad whose degeneracies are given by restriction, we get a preoperad $\mathcal{K}^{(n)}:\mathbf{\Lambda}^{op}\to$ Posets with $\phi^*(\mu,\sigma)=(\phi^*\mu,\phi^*\sigma)$. Like for the little n-cubes, the multiplication is defined as a *substitution product* :

$$\begin{array}{ccc}\mathcal{K}^{(n)}_k\times\mathcal{K}^{(n)}_{i_1}\times\cdots\times\mathcal{K}^{(n)}_{i_k} & \to & \mathcal{K}^{(n)}_{i_1+\cdots+i_k}\\ ((\mu,\sigma);(\mu_1,\sigma_1),\dots,(\mu_k,\sigma_k)) & \mapsto & (\mu(\mu_1,\dots,\mu_k),\sigma(\sigma_1,\dots,\sigma_k)),\end{array}$$

where $(\mu(\mu_1,\dots,\mu_k),\sigma(\sigma_1,\dots,\sigma_k))$ corresponds to the complete (oriented and colored) graph on $i_1+\cdots+i_k$ vertices obtained from (μ,σ) by substituting to each of its k vertices the complete graphs defined by $(\mu_1,\sigma_1),\dots,(\mu_k,\sigma_k)$ respectively. For sake of precision we give the explicit formulae :
$\sigma(\sigma_1,\dots,\sigma_k)=\sigma(i_1,\dots,i_k)\circ(\sigma_1\oplus\cdots\oplus\sigma_k)$, where $\sigma(i_1,\dots,i_k)$ permutes the k

blocks $\{1,\dots,i_1\},\{i_1+1,\dots,i_1+i_2\},\dots,\{i_1+\cdots+i_{k-1}+1,\dots,i_1+\cdots+i_k\}$ according to σ, and:

$$\mu(\mu_1,\dots,\mu_k)_{rs}=\begin{cases}(\mu_j)_{r',s'} & \text{if } r,s \text{ belong to the } j\text{-th block,}\\ \mu_{jj'} & \text{if } r \text{ belongs to the } j\text{-th block and}\\ & \quad s \text{ belongs to the } j'\text{-th block, } j<j'.\end{cases}$$

Definition 2.2. *Let A be a partially ordered set and X a topological space. A collection $(c_\alpha)_{\alpha\in A}$ of closed contractible subspaces (the* "cells"*) of X will be called a* cellular A-decomposition *of X iff the following three conditions hold (where $\check{c}_\alpha=c_\alpha\backslash\bigcup_{\alpha'<\alpha}c_{\alpha'}$) :*

1. *$\alpha\leq\beta\Rightarrow c_\alpha\subseteq c_\beta$ and for all $\alpha,\beta\in A:\check{c}_\alpha\cap\check{c}_\beta=\emptyset$;*
2. *the cell-inclusions are cofibrations ;*
3. *$X=\varinjlim_A c_\alpha$, i.e. X equals the union of its cells and carries the weak topology with respect to its cells.*

Cells c_α with non-empty cell-interiors $\check{c}_\alpha$ will be called *proper*. By the usual homotopy colimit argument, the nerve of the poset A has the same homotopy type as X : $X\xleftarrow{\sim}\mathrm{hocolim}_A c_\alpha\xrightarrow{\sim}|A|$. The same is true for the subposet of proper cells. Moreover, the cell-interiors of the proper cells partition X.

Definition 2.3. *An operad $\mathcal{O}$ is called a* cellular E_n-operad *iff*

1. *$\mathcal{O}_2$ admits an equivariant cellular $\mathcal{K}_2^{(n)}$-decomposition by cells denoted $\mathcal{O}_2^{(\alpha)},\alpha\in\mathcal{K}_2^{(n)}$;*
2. *the closed subsets $\mathcal{O}_k^{(\alpha)}=\{x\in\mathcal{O}_k\,|\,\phi_{ij}^*(x)\in\mathcal{O}_2^{(\phi_{ij}^*\alpha)}\},\alpha\in\mathcal{K}_k^{(n)}$, form a cellular $\mathcal{K}_k^{(n)}$-decomposition of $\mathcal{O}_k$;*
3. *the operad multiplication sends each cell-product $\mathcal{O}_k^{(\alpha)}\times\mathcal{O}_{i_1}^{(\alpha_1)}\times\cdots\times\mathcal{O}_{i_k}^{(\alpha_k)}$ into the cell $\mathcal{O}_{i_1+\cdots+i_k}^{(\alpha(\alpha_1,\dots,\alpha_k))}$ prescribed by the complete graph operad.*

For the second condition of a cellular E_n-operad it is *sufficient* to show :
a) the contractibility of the cells, b) the cofibration property of the cell-inclusions and c) the existence of "ordered" points, i.e. in each $\mathfrak{S}_k$-orbit of $\mathcal{O}_k$ there has to be a point x such that all degeneracies $\phi_{ij}^*(x)$ with $i<j$ belong to "upper hemispheres" $\mathcal{O}_2^{(\cdot,id_2)}$ (this last condition implies that the union of the $\mathcal{O}_k^{(\alpha)}$ covers $\mathcal{O}_k$).

Theorem 2.4. *The little n-cubes operad is a cellular E_n-operad and any two cellular E_n-operads are equivalent as operads.*

The first part of the theorem is due to Fiedorowicz and relies on the analysis of a given configuration of little n-cubes by means of separating hyperplanes perpendicular to the coordinate axis, cf. [6]. The second part (which implies consistency of our terminology) follows from the above homotopy colimit argument, more precisley : for each cellular E_n-operad $\mathcal{O}$ there are operad equivalences

$$\mathcal{O} \stackrel{\sim}{\leftarrow} \mathrm{hocolim}_{\alpha\in\mathcal{K}^{(n)}}\mathcal{O}^{(\alpha)} \stackrel{\sim}{\rightarrow} |\mathcal{K}^{(n)}|.$$

The existence of a cellular structure allows several constructions which would not be available otherwise. For instance, assume that $\mathcal{O}$ is a *cellular E_∞-operad.* The filtration of $\mathcal{K} = \varinjlim \mathcal{K}^{(n)}$ by the E_n-suboperads $\mathcal{K}^{(n)}$ induces an analogous filtration of $\mathcal{O}$ by cellular E_n-suboperads $\mathcal{O}^{(n)}$! As example, consider the simplicial E_∞-operad $\Gamma = E\mathfrak{S}$ where E stands for the universal bundle construction. This operad has a long history : Barratt-Eccles [3] based a general recognition principle for infinite loop spaces on it; Peter May [21] showed that Γ acts canonically on nerves of (strict) symmetric monoidal categories, but it was only Jeff Smith who discovered the existence of a filtration of Γ by suboperads $\Gamma^{(n)}$ [24]. Using the cellular structure of Γ, this filtration comes *for free* as well as the fact that $|\Gamma_k^{(n)}|$ has the $\mathfrak{S}_k$-equivariant homotopy type of $F(\mathbb{R}^n, k)$. The latter was only conjectured by Smith in view of his approximation theorem and later on proved by T. Kashiwabara [18]. For an explicit operad map $|\mathcal{K}^{(n)}| \stackrel{\sim}{\longrightarrow} |\Gamma^{(n)}|$ inducing the equivalence, see [2].

3 A simplicial splitting of $\Omega^n S^n X$

The purpose of this section is to use the Smith filtration $\Gamma^{(n)}$ of the symmetric monoidal operad Γ in order to derive the Snaith-splitting of $\Omega^n S^n X$, cf. [25]. Adapting the method of Cohen-May-Taylor [14] to the simplicial setting I obtain combinatorial estimates for the number of suspensions necessary to split $\Omega^n S^n X$ globally at a given filtration level.

To fix notation, $\Gamma_k = E\mathfrak{S}_k$ is the universal bundle on $\mathfrak{S}_k$: a d-simplex of Γ_k is thus a $(d+1)$-tuple $(\sigma_0, \ldots, \sigma_d)$ of permutations in $\mathfrak{S}_k$ with the usual (homogenous) simplicial operators. The permutation operad $\mathfrak{S} : \mathbf{\Lambda}^{op} \to \mathrm{Sets}$ induces componentwise an operad structure on $\Gamma : \mathbf{\Lambda}^{op} \to \mathrm{SimpSets}$. Let us check that the E_∞-operad Γ is *cellular* : $\Gamma_2 = E\mathfrak{S}_2$ is the "Milnor-sphere" of infinite dimension with one non-degenerate simplex per hemisphere. There exists thus a natural cellular $\mathcal{K}_2$-decomposition of Γ_2. The c(an)onical contraction of Γ_k stays within the formally defined cells $\Gamma_k^{(\alpha)}, \alpha \in \mathcal{K}_k$. Moreover, cell-inclusions are cofibrations and "ordered" points exist. Finally, the multiplication is cellular since induced by the permutation operad.

Theorem 2.4 then implies that the union of the cells labeled by $\mathcal{K}^{(n)}$ defines a cellular E_n-suboperad $\Gamma^{(n)}$ of Γ, explicitly :

$$\Gamma_k^{(n)} = \{\omega \in \Gamma_k \,|\, \phi_{ij}^*(\omega) \in \Gamma_2^{(n)}\} = \{\omega \in \Gamma_k \,|\, \phi_{ij}^*(\omega) \in sk_{n-1}\Gamma_2\}.$$

This is precisely the suboperad defined by J. Smith [24]. Since the geometric realization preserves arbitrary colimits and finite limits, we get for each *connected* simplicial set $(X, *)$:

$$|\Gamma^{(n)} \otimes_{\mathbf{\Lambda}} \underline{X}| \cong |\Gamma^{(n)}| \otimes_{\mathbf{\Lambda}} \underline{|X|} \sim \Omega^n S^n |X|$$

thus the coend $\Gamma^{(n)} \otimes_{\mathbf{\Lambda}} \underline{X}$ is a "simplicial model" for $\Omega^n S^n|X|$. Moreover, the inclusion $\Gamma^{(n)} \hookrightarrow \Gamma^{(n+1)}$ induces the stabilization map $\Omega^n S^n|X| \to \Omega^{n+1}S^{n+1}|X|$ as is the case for the "cellular" filtration of the little cubes by little n-cubes.

Remark 3.1. For $n = 1$, we get the James-construction $\Gamma^{(1)} \otimes_{\mathbf{\Lambda}} \underline{X} = \mathfrak{S} \otimes_{\mathbf{\Lambda}} \underline{X}$ whose group completion is isomorphic to the famous Kan-Milnor model $FX = GSX$ for $\Omega S|X|$ valid also for non-connected simplicial sets $(X, *)$. In general, as $\mathfrak{S} = \Gamma^{(1)}$ is a suboperad of $\Gamma^{(n)}$ the coend $\Gamma^{(n)} \otimes_{\mathbf{\Lambda}} \underline{X}$ is a simplicial monoid which even turns out to be free, cf. [3]; its group completion is thus a free simplicial group model for $\Omega^n S^n|X|$ valid also for non-connected X, cf. [24].

Remark 3.2. Sets with a $\Gamma^{(1)}$-action are precisely monoids ; in [7], it is shown that the nerve of a *braided monoidal category* admits a $\Gamma^{(2)}$-action. Recall that in the terminology of Baez-Dolan [1] braided monoidal categories are 2*-tuply monoidal* 1*-categories*. I conjecture that in general the "nerve" of a "n-tuply monoidal $(n-1)$-category" admits a $\Gamma^{(n)}$-action. There is now at least one additional case, where this conjecture can be verified, the case $n = 3$, i.e. the nerve of a *sylleptic monoidal* 2*-category* should admit a $\Gamma^{(3)}$-action, cf. S. Crans [13].

For simplicity, we shall write $\Gamma^{(n)}(X)$ for the coend $\Gamma^{(n)} \otimes_{\mathbf{\Lambda}} \underline{X}$. The Snaith-splitting can be summarized by the following diagram whose existence we are going to establish (where X is assumed to be *connected*) :

$$\begin{array}{ccccccccccccc}
\Omega S X & \sim & \Gamma^{(1)}(X) & \overset{stably}{\longrightarrow} & D_1^{(1)}(X) & \vee & D_2^{(1)}(X) & \vee & D_3^{(1)}(X) & \vee & \cdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\Omega^2 S^2 X & \sim & \Gamma^{(2)}(X) & \overset{stably}{\longrightarrow} & D_1^{(2)}(X) & \vee & D_2^{(2)}(X) & \vee & D_3^{(2)}(X) & \vee & \cdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\Omega^3 S^3 X & \sim & \Gamma^{(3)}(X) & \overset{stably}{\longrightarrow} & D_1^{(3)}(X) & \vee & D_2^{(3)}(X) & \vee & D_3^{(3)}(X) & \vee & \cdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \| & & \downarrow & & \downarrow & & \vdots \\
\Omega^\infty S^\infty X & \sim & \Gamma(X) & \overset{stably}{\longrightarrow} & D_1(X) & \vee & D_2(X) & \vee & D_3(X) & \vee & \cdots
\end{array}$$

$$D_m^{(n)}(X) = \Gamma_{\leq m}^{(n)}(X)/\Gamma_{\leq m-1}^{(n)}(X) \text{ where } \Gamma_{\leq m}^{(n)}(X) = (\coprod_{k=1}^{m} \Gamma_k^{(n)} \times X^k)/\sim$$

$$\text{In particular, we have : } D_1^{(n)}(X) = X \text{ and } D_m^{(1)}(X) = \underbrace{X \wedge \cdots \wedge X}_{m}$$

$$\text{In general : } D_m^{(n)}(X) = \Gamma_{m,+}^{(n)} \wedge_{\mathfrak{S}_m} \underbrace{X \wedge \cdots \wedge X}_{m}$$

Our purpose here is to show that *James' combinatorial maps*

$$j_m^{(1)} : \Gamma^{(1)}(X) \to \Gamma^{(1)}(D_m^{(1)}(X))$$

naturally extend to maps $j_m^{(n)} : \Gamma^{(n)}(X) \to \Gamma^{(t_{nm})}(D_m^{(n)}(X))$ which *by adjunction* induce "global" Snaith-splittings for each n and m. For $n = t_{nm} = \infty$ this was already observed by Barratt-Eccles [3], and Cohen-May-Taylor [14] used the same idea to define "global" Snaith-splittings in full generality. It is somewhat surprising that one gets purely combinatorial estimates for the number $t = t_{nm}$ of suspensions needed to split $\Omega^n S^n|X|$ at filtration level m :

$$S^t\Omega^n S^n(X) \xrightarrow{\sim} S^t X \vee S^t D_2^{(n)}(X) \vee S^t D_3^{(n)}(X) \vee \cdots \vee S^t D_m^{(n)}(X) \vee R$$

I obtain as estimate $\boxed{t_{nm} = (n-1)(2m-1)+1}$ which gives for $n = 1$

$$S\Omega S(X) \xrightarrow{\sim} SX \vee S(X \wedge X) \vee S(X \wedge X \wedge X) \vee \cdots \qquad \text{(Hilton-Milnor)}$$

and which seems to be quite sharp for $n = 2$, cf. [14]. Snaith's estimates [25] as well as Vogt's estimates [27] are lower for large n,m but they only allow a "non-global" splitting of $S^t\Gamma^{(n)}_{\leq m}(X)$.

Definition 3.3. *For each k and m define (cf. notation 1.3)*

$$\begin{array}{rcl} h_m^k : \mathfrak{S}_k & \to & \mathfrak{S}_{\mathbf{\Lambda}^{inc}(\underline{m},\underline{k})} \overset{def}{=} \mathfrak{S}_{\binom{k}{m}} \\ \sigma & \mapsto & (\phi \mapsto (\sigma \circ \phi)^{inc}). \end{array}$$

Lemma 3.4. *If for each k, the set $\mathbf{\Lambda}^{inc}(\underline{m},\underline{k})$ is antilexicographically ordered, then the following diagram commutes for all $\phi : \underline{k} \to \underline{k}'$ (cf. [14], 4.3 and [3], III.4):*

$$\begin{array}{ccc} \mathfrak{S}_{k'} & \xrightarrow{h_m^{k'}} & \mathfrak{S}_{\binom{k'}{m}} \\ \Big\downarrow \phi^* & & \Big\downarrow \binom{\phi}{m}^* \\ \mathfrak{S}_k & \xrightarrow{h_m^k} & \mathfrak{S}_{\binom{k}{m}} \end{array}$$

Lemma 3.5. *The maps j_m^k below are compatible with the coend relations and induce the* stable *James maps $j_m : \Gamma(X) \to \Gamma(D_m(X))$ (satisfying that $j_m|_{\Gamma_{\leq m}(X)}$ is equal to the projection composed with the unit of $\Gamma(-)$) :*

$$\begin{array}{rcl} j_m^k : \Gamma_k \times X^k & \to & \Gamma_{\binom{k}{m}} \times D_m(X)^{\binom{k}{m}} \\ (\omega; x_1, \ldots, x_k) & \mapsto & \big(h_m^k(\omega); [\phi^*(\omega); x_{\phi(1)}, \ldots, x_{\phi(m)}]_{\phi \in \mathbf{\Lambda}^{inc}(\underline{m},\underline{k})}\big). \end{array}$$

Proposition 3.6. *The stable James maps $j_m : \Gamma(X) \to \Gamma(D_m(X))$ restrict to* unstable *James maps $j_m^{(n)} : \Gamma^{(n)}(X) \to \Gamma^{(t_{nm})}(D_m(X))$ where t_{nm} can be chosen equal to $(n-1)(2m-1)+1$.*

Proof. – We have to show that $\omega \in \Gamma_k^{(n)}$ implies $h_m^k(\omega) \in \Gamma^{(t_{nm})}_{\mathbf{\Lambda}^{inc}(\underline{m},\underline{k})}$ for all k.

Let $\omega = (\sigma_0, \ldots, \sigma_d) \in \Gamma_k^{(n)}$, i.e. for each pair $i < j$, the simplex $\phi_{ij}^*(\omega)$ belongs to the $(n-1)$-skeleton of Γ_2, which means that $\phi_{ij}^*(\omega)$ contains at most $n-1$ changes or equivalently : the sequence

$$(\sigma_c(i) \lesseqgtr \sigma_c(j))_{c=0,\ldots,d} \tag{1}$$

contains at most $n-1$ inversions of the order relation.

Similarly, let $\Omega = h_m^k(\omega) \in \Gamma^{(t)}_{\mathbf{\Lambda}^{inc}(\underline{m},\underline{k})}$, i.e. for each pair $\{i_1, \ldots, i_m\} < \{j_1, \ldots, j_m\}$ in $\mathbf{\Lambda}^{inc}(\underline{m}, \underline{k})$ the simplex $\phi^*_{\{i_1,\ldots,i_m\},\{j_1,\ldots,j_m\}}(\Omega)$ contains at most $t-1$ changes, or equivalently : the sequence

$$(\{\sigma_c(i_1), \ldots, \sigma_c(i_m)\} \lesseqgtr \{\sigma_c(j_1), \ldots, \sigma_c(j_m)\})_{c=0,\ldots d} \tag{2}$$

contains at most $t-1$ inversions of the (antilexicographical) order relation.

Each inversion in (2) induces an inversion in (1) for at least one pair $(i,j) \in \{i_1, \ldots, i_m\} \times \{j_1, \ldots, j_m\}$. The "worst" case occurs when an inversion in (2) induces inversions in (1) for *as few pairs as possible.* From this point of view, the "worst" configuration is $i_1 < j_1 < i_2 < j_2 < \cdots < i_m < j_m$ (thus $k \geq 2m$) with simplex $\omega = (\sigma_0, \ldots, \sigma_{2m-1})$ such that $\sigma_0 = id_k$, σ_1 puts i_m on the top, σ_2 puts j_{m-1} on the top and i_m on the second place, σ_3 puts i_{m-1} on the top, j_{m-1} on the second place and i_m on the third place, and so on, until finally σ_{2m-1} reverses completely the order : $\sigma_{2m-1}(i_1, j_1, \ldots, i_m, j_m) = (j_m, i_m, \ldots, j_1, i_1)$.

For each pair $(i,j) \in \{i_1, \ldots, i_m\} \times \{j_1, \ldots, j_m\}$, the simplex $\phi_{ij}^*(\omega)$ contains then exactly one change, whereas $\phi^*_{\{i_1,\ldots,i_m\},\{j_1,\ldots,j_m\}}(\Omega)$ contains $2m-1$ changes. Repeating the above construction $n-1$ times in alternating directions defines a simplex $\omega \in \Gamma_k$ such that $\phi_{ij}^*(\omega)$ contains $n-1$ changes for each $(i,j) \in \{i_1, \ldots, i_m\} \times \{j_1, \ldots, j_m\}$, whereas $\phi^*_{\{i_1,\ldots,i_m\},\{j_1,\ldots,j_m\}}(\Omega)$ contains $(n-1)(2m-1)$ changes. Since this is presumably the "worst" case, it suffices to put $t_{nm} = (n-1)(2m-1)+1$. □

Let me finish this section by recalling how to associate to the unstable James maps $j_m^{(n)} : \Gamma^{(n)}(X) \to \Gamma^{(t_{nm})}(D_m^{(n)}(X))$ *higher Hopf invariants* for homotopy classes in $[S^n Z, S^n X]$ and state *conjecturally* the properties they should share (extending what is known for $n=1$, see Boardman-Steer [8], and for general n, if X is a "Thom-space", see Koschorke-Sanderson [19]).

Conjecture 3.7. *For each $f : S^n Z \to S^n X$ there is a family of higher Hopf invariants $\gamma_m(f) : S^t Z \to S^t D_m^{(n)}(X)$ $(t = t_{nm})$ satisfying*

1. $\gamma_1(f) = f$;
2. $\gamma_m(S^n g) \sim 0$ *for* $m > 1$;
3. $\gamma_m(f + f') \overset{stably}{\sim} \gamma_m(f) + \gamma_{m-1}(f)\gamma_1(f') + \cdots + \gamma_1(f)\gamma_{m-1}(f') + \gamma_m(f')$;
4. $m!\,\gamma_m(f) \overset{stably}{\sim} \gamma_1(f)^m$ *(if X is an n-fold suspension).*

The Hopf invariant is defined by the formula $\gamma_m(f) = \widehat{j_m^{(n)} \circ \check{f}}$ where $\check{(-)}$ and $\widehat{(-)}$ are given by the S^n-Ω^n-adjunction.

The product of Hopf invariants is given by a *cup product*, compare [8] : For $\alpha : S^t Z \to S^t D_m^{(n)}(X)$ and $\beta : S^{t'} Z \to S^{t'} D_{m'}^{(n)}(X)$ we put

$$\alpha\beta : S^{t+t'} Z \xrightarrow{\nabla} S^t Z \wedge S^{t'} Z \xrightarrow{\alpha\wedge\beta} S^t D_m^{(n)}(X) \wedge S^{t'} D_{m'}^{(n)}(X) \xrightarrow{\times} S^{t+t'} D_{m+m'}^{(n)}(X)$$

where the last map is induced by the monoidal structure of $\Gamma^{(n)}(X)$.

Boardman-Steer's proof of properties 1-4 for the case $n = 1$ should go through in general, since it is combinatorial in nature. Their uniqueness result strongly suggests that properties 1-3 determine the higher Hopf invariants up to some suspensions. I think that the existence of subdivided power operations in the stable homotopy of $\Omega^n S^n X$ (X n-fold suspension) should contain some information about the Hurewicz homomorphism $\pi_*^{st}(\Omega^n S^n X) \to H_*(\Omega^n S^n X)$ and perhaps interact with Fred Cohen's [12] calculations of $H_*(\Omega^n S^n X; \mathbb{Z}/p\mathbb{Z})$ where he completely determines $H_*(D_m^{(n)}(X); \mathbb{Z}/p\mathbb{Z})$ in terms of the reduced homology of X and of suitably weighted *Dyer-Lashof operations*.

4 Decompressible immersions

In this final very "sketchy" section I will indicate how the configuration space model of $\Omega^n S^n X$ for a "Thom space" X appears quite naturally in immersion theory, giving rise to a "more concrete" construction of the higher Hopf invariants. The existence of such a relationship between E_n-operads and immersion theory has been brought to my attention by Jean Lannes, for which I am very grateful to him. In the literature, different approaches have successively been developed by R. Wells [28], P. Vogel [26] and U. Koschorke-B. Sanderson [19]. My presentation heavily relies on the last reference.

Definition 4.1. *An* immersion $j : M \looparrowright V$ *admits a ξ-structure with respect to a vector bundle $\xi : E \to B$ iff there exists a differentiable map $\phi : M \to B$ and a short exact sequence of vector bundles over M :*

$$0 \longrightarrow TM \longrightarrow j^*TV \longrightarrow \phi^*\xi \longrightarrow 0$$

The group of ξ-corbordism classes of closed ξ-embeddings in V will be denoted by $\Omega^\xi(V)$.

Theorem 4.2. *(Thom-Pontryagin)* $\Omega^\xi(V) \cong [\overline{V}, \mathrm{Th}(\xi)]$, *where $\overline{V}$ is the one-point compactification of V and* $\mathrm{Th}(\xi) = D(\xi)/S(\xi)$ *is the Thom-space of ξ.*

Definition 4.3. *The suspension of an immersion $j : M \looparrowright V$ is the immersion $j \times (0 \hookrightarrow \mathbb{R}) : M \looparrowright V \times \mathbb{R}$.*

Lemma 4.4. *The suspension of embeddings corresponds under 4.2 to ordinary suspension:* $[\overline{V}, \mathrm{Th}(\xi)] \longrightarrow [\overline{V} \wedge \overline{\mathbb{R}}, \mathrm{Th}(\xi) \wedge \mathrm{Th}(\epsilon_1)]$ *where we identify* $\overline{V \times \mathbb{R}} \cong \overline{V} \wedge \overline{\mathbb{R}}$ *and* $\mathrm{Th}(\xi \oplus \epsilon_1) \cong \mathrm{Th}(\xi) \wedge \mathrm{Th}(\epsilon_1)$.

Here and later on, I write ϵ_k for the trivial vector bundle $\mathbb{R}^k \to *$.
A *framed* cobordism (resp. immersion) is by definition an ϵ_k-cobordism (resp. ϵ_k-immersion) which in both cases means that the *normal bundle* is endowed with a trivialization. The group

$$\varinjlim_k \Omega^{\epsilon_k}(\mathbb{R}^{m+k}) \cong \varinjlim_k [S^{m+k}, S^k] = \{S^m, S^0\} = \pi_m^{st}$$

then defines simultaneously stable framed cobordism classes of m-dimensional framed manifolds and the m-th stem of the stable homotopy groups of spheres.

Immersions are in some sense *"desuspended embeddings"* since by a sufficient number of suspensions any immersion (of positive codimension) can be made "regularly homotopic" to an embedding.

Definition 4.5. *A* n-decompression *for an immersion* $j : M \looparrowright V$ *is a differentiable map* $\lambda : M \to \mathbb{R}^n$ *such that* $(j, \lambda) : M \to V \times \mathbb{R}^n$ *is an embedding. Immersions for which an* n*-decompression exists are called* n-decompressible.

Theorem 4.6. *(Koschorke-Sanderson) The group of* $(\xi \oplus \epsilon_n)$*-cobordism classses of* n*-decompressed* ξ*-immersions* $(j, \lambda) : M \to V \times \mathbb{R}^n$ *is canonically isomorphic to* $[\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)]$. *Moreover, under* $S^n\Omega^n$*-adjunction, this group is isomorphic to the classical Thom-Pontryagin cobordism group (we assume* $\mathrm{rg}(\xi) > 0$) *:*

$$[\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)] \cong [S^n \overline{V}, S^n \mathrm{Th}(\xi)] = [\overline{V \times \mathbb{R}^n}, \mathrm{Th}(\xi \oplus \epsilon_n)] = \Omega^{\xi \oplus \epsilon_n}(V \times \mathbb{R}^n)$$

Proof. The proof consists essentially in the construction of an "adjoint" representative of (j, λ) inside $[\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)]$ using the configuration space model for the target space as well as the classical Thom-Pontryagin construction. The immersion $j : M \looparrowright V$ defines for a small neighborhood V_y of each $y \in M$ a neighborhood of $j(V_y)$ which by hypothesis can be given the structure of a $D(\xi)$-bundle over V_y. This defines a "fattening" $\overline{j} : M \times D(\xi) \looparrowright V$ of j. The "adjoint" representative is then given by

$$\begin{array}{rcl} \widehat{\tau}_{j,\lambda} : \overline{V} & \to & F(\mathbb{R}^n, -) \otimes_{\mathbf{\Lambda}} \mathrm{Th}(\xi) \\ x & \mapsto & (\lambda(\overline{j}^{-1}(x)); \overline{j}^{-1}(x)) \end{array}$$

□

Corollary 4.7. *The "stable group" of* ξ*-cobordism classes of* n*-decompressible* ξ*-immersions is canonically isomorphic to*

$$\{\overline{V}, \Omega^n S^n \mathrm{Th}(\xi)\} \cong \bigoplus_{k \geq 1} \{\overline{V}, D_k^{(n)}(\mathrm{Th}(\xi))\}.$$

Remark 4.8. The split summand $D_k^{(n)}(\mathrm{Th}(\xi))$ can be identified with the Thom space of the vector bundle $F(\mathbb{R}^n, k) \times_{\mathfrak{S}_k} \xi^k$. For example, let $\xi = \epsilon_1$, $n = 2$, then $D_k^{(2)}(S^1) = D_k^{(2)}(\mathrm{Th}(\epsilon_1)) = \mathrm{Th}(F(\mathbb{R}^2, k) \times_{\mathfrak{S}_k} (\epsilon_1)^k)$ which is thus the Thom-space of the vector bundle $F(\mathbb{R}^2, k) \times_{\mathfrak{S}_k} \mathbb{R}^k \to BB_k$.

Remark 4.9. The projections inducing the above isomorphism are induced by composition with *James' combinatorial maps*

$$j_k^{(n)} : \Omega^n S^n \mathrm{Th}(\xi) \to \Omega^t S^t D_k^{(n)}(Th(\xi)).$$

The remarkable fact here is that there exists a direct construction for immersions performing this composition. Namely, assume that the immersion $j : M \looparrowright V$ is *generic*, i.e. for each k, the natural map $F(M,k) \to V^k$ is *transverse* to the diagonal ; then the subset

$$\tilde{M}(k) = \{(m_1, \dots, m_k) \in F(M,k) \,|\, j(m_1) = \cdots = j(m_k)\}$$

is a *manifold* with free $\mathfrak{S}_k$-action and the quotient manifold $M(k) = \tilde{M}(k)/\mathfrak{S}_k$ is called the *k-tuple manifold* of $j : M \looparrowright V$.

It turns out (cf. [19]) that the immersion $j(k) : M(k) \looparrowright V$ represents precisely the projected class

$$[(j(k), ?)] \in [\overline{V}, \Omega^t S^t D_k^{(n)}(\mathrm{Th}(\xi))].$$

If the estimates of the last section are correct, then $j(k) : M(k) \looparrowright V$ is t-decompressible with $t = t_{nk} = (n-1)(2k-1)+1$, and this universally for every generic n-decompressible immersion $j : M \looparrowright V$!

Remark 4.10. Let us come back to the higher Hopf invariants. Given a class $[f] \in [S^n\overline{V}, S^n\mathrm{Th}(\xi)]$ represented by $(j, \lambda) : M \hookrightarrow V \times \mathbb{R}^n$ with generic immersion j. How can the Hopf invariant

$$[\gamma_k(f)] \in [S^t\overline{V}, S^t D_k^{(n)}(\mathrm{Th}(\xi))] = [S^t\overline{V}, S^t\mathrm{Th}(F(\mathbb{R}^n, k) \times_{\mathfrak{S}_k} \xi^k)]$$

be represented ? According to the preceding remark we have only to find a t-decompression $\lambda(k) : M(k) \to \mathbb{R}^t$ for $j(k) : M(k) \looparrowright V$ and the embedding $(j(k), \lambda(k)) : M(k) \hookrightarrow V \times \mathbb{R}^t$ will be a representative for $[\gamma_k(f)]$. Koschorke-Sanderson [19] prove that for the so defined higher Hopf invariants properties 1-3 of conjecture 3.7 hold, see also Boardman-Steer [8] where the same construction is carried out for $n = 1$.

References

[1] J. C. Baez and M. Neuchl – *Higher dimensional algebra I : Braided monoidal 2-categories,* Adv. in Math. **121** (1996), 196-244.

[2] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. Vogt – *Iterated monoidal categories*, to appear (1999).

[3] M. G. Barratt and P. J. Eccles – Γ^+*-Structures I,II,III,* Topology **13** (1974), 25-45, 113-126, 199-207.

[4] H.-J. Baues – *Geometry of loop spaces and the cobar-construction,* Mem. AMS **230** (1980).

[5] C. Berger – *Opérades cellulaires et espaces de lacets itérés*, Ann. de l'Inst. Fourier **46** (1996), 1125-1157.

[6] C. Berger – *Combinatorial models of real configuration spaces and* E_n*-operads*, Contemp. Math. **202** (1997), 37-52.

[7] C. Berger – *Double loop spaces, braided monoidal categories and algebraic* 3*-type of space*, to appear in Contemp Math. (1998).

[8] J. M. Boardman and B. Steer – *On Hopf invariants,* Comment. Math. Helvet. **42** (1967), 180-221.

[9] J. M. Boardman and R. Vogt – *Homotopy invariant algebraic structures on topological spaces,* Lecture Notes in Math. **347**, Springer Verlag (1973).

[10] M. Brinkmeier – *The Milgram non-operad*, this volume.

[11] R. Charney and M. W. Davis – *Finite* $K(\pi,1)'s$ *for Artin groups,* Annals of Math. Studies **138** (1995), 110-124.

[12] F. R. Cohen – *The homology of* C_{n+1}*-spaces,* Lecture Notes in Math. **533**, Springer Verlag (1976), 207-351.

[13] S. E. Crans – *On braidings, syllepses and symmetries*, to appear in Cah. Top. Geom. Diff. Cat.(1998).

[14] F. R. Cohen, J. P. May and L. R. Taylor – *Splitting of certain spaces* CX*,* Math. Proc. Camb. Phil. Soc. **84** (1978), 465-496.

[15] Z. Fiedorowicz – *Constructions of* E_n*-operads*, this volume.

[16] R. Fox and L. Neuwirth – *The braid groups,* Math. Scand. **10** (1962), 119-126.

[17] E. Getzler and J. D. S. Jones – *Operads, homotopy algebra, and iterated integrals for double loop spaces*, preprint (1995).

[18] T. Kashiwabara – *On the Homotopy Type of Configuration Complexes,* Contemp. Math. **146** (1993), 159-170.

[19] U. Koschorke and B. Sanderson – *Self intersections and higher Hopf invariants,* Topology **17** (1978), 283-290.

[20] J. P. May – *The geometry of iterated loop spaces,* Lecture Notes in Math. **271**, Springer Verlag (1972).

[21] J. P. May – *E_∞-spaces, group completions and permutative categories,* London Math. Soc. Lecture Notes **11** (1974), 61-94.

[22] J. McClure – *Little 2-cubes and Hochschild cohomology,* this volume.

[23] R. J. Milgram – *Iterated loop spaces,* Annals of Math. **84** (1966), 386-403.

[24] J. H. Smith – *Simplicial Group Models for $\Omega^n S^n X$,* Israel J. of Math. **66** (1989), 330-350.

[25] V. P. Snaith – *A stable decomposition of $\Omega^n S^n X$,* J. London Math. Soc. (2) **7** (1974), 577-583.

[26] P. Vogel – *Cobordisme d'immersions,* Ann. scient. Ec. Norm. Sup. (4) **7** (1974), 317-358.

[27] R. Vogt – *Splittings of spaces $C\underline{X}$,* manuscripta math. **38** (1982), 21-39.

[28] R. Wells – *Cobordism groups of immersions,* Topology **5** (1966), 281-294.

Author's address : Université de Nice-Sophia Antipolis, Lab. J.-A. Dieudonné, Parc Valrose, F-06108 Nice, Cedex 2, France.

E-mail : cberger@math.unice.fr