Double loop spaces, braided monoidal categories and algebraic 3-type of space

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15 march 1997

Abstract

We show that the nerve of a braided monoidal category carries a natural action of a simplicial E_2-operad and is thus up to group completion a double loop space. Shifting up dimension twice associates to each braided monoidal category a 1-reduced lax 3-category whose nerve realizes an explicit double delooping whenever all cells are invertible. We deduce that lax 3-groupoids are algebraic models for homotopy 3-types.

Introduction

The concept of *braiding* as a refinement of symmetry is the starting point of a rich interplay between geometry (knot theory) and algebra (representation theory). The underlying structure of a *braided monoidal category* reveals an interest of its own in that it encompasses two at first sight different geometrical objects : *double loop spaces* and *homotopy* 3*-types*. The link to double loop spaces was pointed out by J. Stasheff [38] and made precise by Z. Fiedorowicz [15], who proves that double loop spaces may be characterized (up to group completion) as algebras over a contractible free braided operad. The link to homotopy 3-types goes back to A. Grothendieck's pursuit of stacks [20] and was taken up by O. Leroy [29], who shows that a weak form of 3-groupoid, we call here *lax* 3*-groupoid*, models homotopy 3-types; the laxness stems precisely from "relaxing" a strict commutativity constraint to an interchange (i.e. braiding) cell. Lax 3-groupoids are called semi-strict 3-groupoids by Baez-Neuchl [2], and Gray groupoids by Gordon-Power-Street [17].

Our main concern here is to "tie together" these two aspects of the same structure. In particular, we hope this might help to construct a general scheme relating iterated loop spaces to homotopy n-types. The point is that the combinatorial structure of iterated loop spaces is by now quite well understood (cf. [8], [3]), whereas the same is not true for homotopy n-types when $n \geq 4$.

The text is divided into three parts :

• Part One proves the existence of a double delooping for braided monoidal categories in the "realm" of simplicial E_2-operads. In some precise sense, the braid groups are contained in the combinatorial structure of the symmetric groups by means of the *weak Bruhat order*.

• Part Two fully embeds the category of braided monoidal categories into the category of *lax* 3-*categories* thus providing an explicit double delooping for braided monoidal *groupoids*. The main tool is a cosimplicial lax 3-categorical object.

• Part Three proves the equivalence of the homotopy categories of *simplicial* 3-*types* and lax 3-groupoids on the basis of the following observation : the fundamental groupoid of the double loops on a simply connected 3-type is braided monoidal.

I would like to take the opportunity to thank the members of Sydney's Category Seminar for their hospitality during my visit of the southern hemisphere. The following text owes quite a lot to all of them.

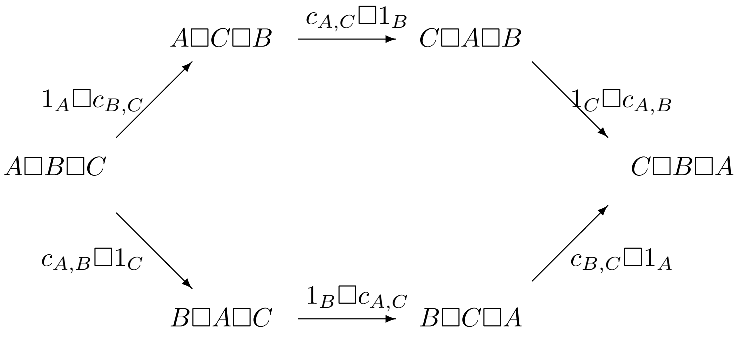
1 Braided monoidal categories

Throughout, we shall adopt the following conventions and notations:
'Monoidal' always means 'strict monoidal'.– A braiding is *not* assumed to be invertible.– The class of n-cells of a (multiple) category $\mathcal{C}$ is written $\mathcal{C}_n$.– The symmetric group on a set I is denoted by $\mathfrak{S}_I$. For $\mathfrak{S}_{\{1,\ldots,n\}}$ we write $\mathfrak{S}_n$ and the permutation which maps $(1,\ldots,n)$ to $(a_1,\ldots,a_n)$ is represented by $[a_1,\ldots,a_n]$.

Definition 1.1. *A* braided monoidal category *is a monoidal category* $(\mathcal{C},\Box,U)$ *endowed with a binatural family of morphisms* $c_{A,B} : A\Box B \to B\Box A$, *called* braidings*, such that for all* $A,B,C \in \mathcal{C}_0$ *we have*

i) $c_{A,U} = c_{U,A} = 1_A$ *(unitarity),*
ii) $c_{A\Box B,C} = (c_{A,C}\Box 1_B)\circ(1_A\Box c_{B,C})$
$c_{A,B\Box C} = (1_B\Box c_{A,C})\circ(c_{A,B}\Box 1_C)$ *(transitivity).*

Naturality and transitivity of the braidings imply the commutativity of the so called *Yang-Baxter hexagon* :



The purpose of this chapter is a new proof of the following theorem :

Theorem 1.2. (Stasheff [39], Fiedorowicz [15])
The nerve of a braided monoidal category is up to group completion a double loop space.

By virtue of P. May's theory of E_n-operads [33],[12], it suffices to construct an action of a *simplicial* E_2*-operad* upon the nerve of a braided monoidal category, which then implies the existence of a *double delooping* of its geometric realization.

There are several convenient choices for such simplicial E_2-actions. In [15], Fiedorowicz introduces the notion of a braided operad as a braid analog of May's $\mathfrak{S}$-operads, and characterizes E_2-operads (up to equivalence) as contractible free braided operads. In [3], Balteanu, Fiedorowicz, Schwänzl and Vogt construct categorical E_n-operads acting on n-fold monoidal categories and show that braided monoidal categories are special 2-fold monoidal categories.

Here, we choose yet another approach, based on the *weak Bruhat order* of the symmetric groups. On the categorical level, we get a non-$\mathfrak{S}$-operad action which is merely a restatement of Joyal-Street's *Coherence Theorem* for braided monoidal categories [22]. On the simplicial level, this action induces a $\mathfrak{S}$-operad action of the simplicial E_2-operad $E\mathfrak{S}^{(2)}$ which is the second term of Smith's filtration of the simplicial E_∞-operad $E\mathfrak{S}$ [37],[4]. In the case of a *symmetric* monoidal category (i.e. with invertible braidings satisfying $c_{A,B}^{-1} = c_{B,A}$), the action extends canonically to the entire E_∞-operad $E\mathfrak{S}$ by MacLane's classical Coherence Theorem [30], and we recover P. May's original line of argument proving that the nerve of a symmetric monoidal category admits an infinite delooping [34].

The *Smith-filtration* of $E\mathfrak{S}$ was shown in [8] and [3] to define simplicial E_n-operads $E\mathfrak{S}^{(n)}$ for all n. If $n = 1, 2, \infty$, an action of $E\mathfrak{S}^{(n)}$ detects nerves of monoidal, braided monoidal resp. symmetric monoidal categories. It seems likely that for $2 < n < \infty$, the categorical structure detected by an $E\mathfrak{S}^{(n)}$-action (and thus corresponding to an n-fold loop space) involves some higher order analogue of braiding related to Manin-Schechtman's $(n-1)$-categorical structure of the symmetric group $\mathfrak{S}_n$ (cf. [32], [25]).

Let us fix some notation and recall the definition of the Smith-filtration.

Notation 1.3. We define $E\mathfrak{S}_k$ as the *nerve* of the translation category on the symmetric group $\mathfrak{S}_k$. A d-simplex in $E\mathfrak{S}_k$ is thus a $(d+1)$-tuple of permutations in $\mathfrak{S}_k$. There is a (diagonal) free $\mathfrak{S}_k$-action on $E\mathfrak{S}_k$. The operad-structure on the family $(\mathfrak{S}_k)_{k\geq 1}$ defines a simplicial operad-structure on the family $E\mathfrak{S} = (E\mathfrak{S}_k)_{k\geq 1}$. In particular, the multiplication of the permutation operad

$$\begin{array}{rcl} m^{\mathfrak{S}}_{i_1\dots i_p} : \mathfrak{S}_p \times \mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_p} & \to & \mathfrak{S}_{i_1+\cdots+i_p} \\ (\sigma;\sigma_1,\dots,\sigma_p) & \mapsto & \sigma(i_1,\dots,i_p)\circ(\sigma_1\oplus\cdots\oplus\sigma_p), \end{array}$$

(where $\sigma(i_1,\dots,i_p)$ permutes the p "blocks" according to σ) defines diagonally the multiplication of the simplicial E_∞-operad $E\mathfrak{S} = (E\mathfrak{S}_k)_{k\geq 1}$.

For each couple of integers (i,j) such that $1 \leq i < j \leq k$ there is a mapping

$$
\begin{aligned}
\phi_{ij}^*:\mathfrak{S}_k &\to \mathfrak{S}_2\\
\sigma &\mapsto \begin{cases}[1,2] \text{ if } \sigma(i)<\sigma(j),\\ [2,1] \text{ if } \sigma(j)<\sigma(i).\end{cases}
\end{aligned}
$$

These mappings induce *injective* product mappings

$$\prod_{1\leq i<j\leq k} \phi_{ij}^*:\mathfrak{S}_k \to (\mathfrak{S}_2)^{\binom{k}{2}} \tag{1}$$

$$\prod_{1\leq i<j\leq k} E\phi_{ij}^*:E\mathfrak{S}_k \to (E\mathfrak{S}_2)^{\binom{k}{2}} \tag{2}$$

Let $\mathfrak{S}_2$ be ordered according to $[1,2]<[2,1]$ and let $E\mathfrak{S}_2$ be filtered by the $(n-1)$-skeleta $E\mathfrak{S}_2^{(n)}=sk_{n-1}E\mathfrak{S}_2$.

Definition 1.4. *The (left)* weak Bruhat order $(\mathfrak{S}_k,\leq)$ *is the partial order on* $\mathfrak{S}_k$ *induced by the injective mapping (1) and the product order on* $(\mathfrak{S}_2)^{\binom{k}{2}}$*, i.e.*

$$\sigma\leq\tau \Leftrightarrow \phi_{ij}^*(\sigma)\leq\phi_{ij}^*(\tau) \textit{ for all } i<j.$$

The Smith-filtration $E\mathfrak{S}^{(n)}$ *of* $E\mathfrak{S}$ *is the filtration induced by the injective mapping (2) and the diagonal filtration on* $(E\mathfrak{S}_2)^{\binom{k}{2}}$*, i.e.*

$$E\mathfrak{S}_k^{(n)}=\bigcap_{1\leq i<j\leq k}\phi_{ij}^{*-1}(E\mathfrak{S}_2^{(n)}).$$

Remark 1.5. The (left resp. right) weak Bruhat order on the symmetric group $\mathfrak{S}_k$ is usually defined as the partial order which underlies the Cayley graph on $\mathfrak{S}_k$ for (left resp. right) translation by the family of elementary transpositions $(i,i+1), i=1,\dots,k-1$. We leave it to the reader to check that this is equivalent to our definition.

Lemma 1.6. *A simplex* $(\sigma_0,\dots,\sigma_d)\in E\mathfrak{S}_k$ *ending at* $\sigma_d=1_k$ *belongs to* $E\mathfrak{S}_k^{(2)}$ *if and only if* $\sigma_0\geq\sigma_1\geq\cdots\geq\sigma_{d-1}\geq 1_k$.

Proof. – By definition of the Smith-filtration, the simplex $(\sigma_0,\dots,\sigma_d)$ belongs to $E\mathfrak{S}_k^{(2)}$ if and only if $(\phi_{ij}^*(\sigma_0),\dots,\phi_{ij}^*(\sigma_d))$ belongs to $E\mathfrak{S}_2^{(2)}$ for each $i<j$, which means that there is at most one change in each projected sequence of permutations, necessarily of the form $[2,1]>[1,2]$ as $\phi_{ij}^*(1_k)=1_2$. This in turn is equivalent to the ordering $\sigma_0\geq\sigma_1\geq\cdots\geq\sigma_{d-1}\geq 1_k$ by our definition of the weak Bruhat order. □

Proposition 1.7. *The nerve of a braided monoidal category carries a canonical action of the simplicial* E_2*-operad* $E\mathfrak{S}^{(2)}$.

Proof. – The existence of a monoidal structure on the category $\mathcal{C}$ is equivalent to the existence of an action by the permutation operad :

$$
\begin{aligned}
\mathfrak{S}_k\times\mathcal{C}^k &\to \mathcal{C}^k\\
(\sigma;A_1,\dots,A_k) &\mapsto A_{\sigma^{-1}(1)}\square\cdots\square A_{\sigma^{-1}(k)}.
\end{aligned}
$$

The existence of braidings $c_{A,B}$ on the monoidal category $(\mathcal{C}, \Box, U)$ amounts then to a uniquely determined extension of the above action to the weak Bruhat orders $(\mathfrak{S}_k, \leq)_{k\geq 1}$ with braiding $c_{A_1,A_2} : A_1 \Box A_2 \to A_2 \Box A_1$ given by the image of the *relator arrow* $[1,2] \to [2,1]$. The transitivity of the partial order corresponds to the transitivity of the braiding. Indeed, in this strict setting, Joyal-Street's *Coherence Theorem* for braided monoidal categories [22] may be restated as follows : On the set of k-fold products $A_{\sigma^{-1}(1)} \Box \cdots \Box A_{\sigma^{-1}(k)}, \sigma \in \mathfrak{S}_k$ the set of braidings forms exactly the set of relators of a partial order, in fact the weak Bruhat order (cf. remark 1.5).

The family of weak Bruhat orders $(\mathfrak{S}_k, \leq)_{k\geq 1}$ actually defines a categorical non-$\mathfrak{S}$-operad since the multiplication of the permutation operad preserves the weak Bruhat orders, whereas right translation does not (it only preserves adjacency, but not order). Taking nerves, we get a canonical non-$\mathfrak{S}$-operad action of $\mathcal{N}(\mathfrak{S}_k, \leq)_{k\geq 1}$ on $\mathcal{NC}$. Furthermore, there is a *partial action* of $\mathfrak{S}_k$ upon $\mathcal{N}(\mathfrak{S}_k)$ which allows one to canonically extend the action to the following simplicial set:

$$E_k^{(2)} = (\mathcal{N}(\mathfrak{S}_k, \leq) \times \mathfrak{S}_k)/(x, \sigma) \sim (x^\sigma, 1_k),$$

where the relation holds whenever right translation of $x \in \mathcal{N}(\mathfrak{S}_k, \leq)$ by σ is well defined. Now, by lemma 1.6, there is a canonical simplicial isomorphism $E\mathfrak{S}_k^{(2)} \cong E_k^{(2)}$ and the extended action on $\mathcal{NC}$ is compatible with the operad structure of $E\mathfrak{S}^{(2)}$. $\Box$

Theorem 1.2 follows now from the fact that $E\mathfrak{S}^{(2)}$ is a simplicial E_2-operad ([9],[3]) by an immediate application of May's theory of E_n-operads ([34],[12]). The double delooping construction is provided by a generalized bar-construction and only takes place after replacement of $|\mathcal{NC}|$ by a homotopy equivalent space upon which the *little squares* operad of Boardman-Vogt acts [10]. This replacement space depends on an explicit equivalence of operads between $|E\mathfrak{S}^{(2)}|$ and the little squares operad.

Remark 1.8. According to the preceeding proof, the simplicial set $E\mathfrak{S}_k^{(2)}$ is built up out of $k!$ copies of the nerve $\mathcal{N}(\mathfrak{S}_k, \leq)$ by means of gluing relations determined through the partial $\mathfrak{S}_k$-action on $\mathcal{N}(\mathfrak{S}_k, \leq)$. These gluing relations are reminiscent of Milgram's permutohedral model $(P_k \times \mathfrak{S}_k)/\sim$ of the configuration space $F(\mathbb{R}^2, k)$ of ordered k-tuples of distinct points in the plane [35]. The formal analogy between the two can be made precise and relies on the structure of a *cellular E_2-preoperad* (cf. [9]) shared as well by the family $|E\mathfrak{S}_k^{(2)}|_{k\geq 1}$ as by the family $F(\mathbb{R}^2, k)_{k\geq 1}$. We obtain as corollary a homotopy equivalence between $|E\mathfrak{S}_k^{(2)}|$ and $F(\mathbb{R}^2, k)$.

The (algebraically defined) fundamental group of $E\mathfrak{S}_k^{(2)}$ is thus isomorphic to the fundamental group of $F(\mathbb{R}^2, k)$ which (nearly by definition) is the *pure braid group* on k strands. This induces a canonical *representation* of the pure braid group in the automorphism group of each k-fold product $A_1 \Box \cdots \Box A_k$ of a braided monoidal category (cf. [26]).

Remark 1.9. The *free braided monoidal category* $\mathfrak{B}$ on one generating object 1 (with invertible braidings) contains Artin's braid groups $B_k = \pi_1(F(\mathbb{R}^2, k)/\mathfrak{S}_k)$

in the form of the automorphism groups $\mathrm{Aut}_{\mathfrak{B}}(1^{\square k})$, as shown by Joyal-Street [21]. For example, if $k = 3$ and $1 = A = B = C$, then $s_1 = c_{A,B}\square 1_C$ and $s_2 = 1_A\square c_{B,C}$ form the generators for the "algebraic" braid group B_3 and the Yang-Baxter hexagon realizes the defining relation $s_2s_1s_2 = s_1s_2s_1$. The preceeding remark can serve as a combinatorial proof that the geometric and algebraic definitions of B_k coincide, which was one of E. Artin's main initial concerns (cf. [1]).

The nerve of $\mathfrak{B}$ is a disjoint union of classifying spaces

$$\mathcal{N}\mathfrak{B} = \bigsqcup_{k\geq 0} BB_k$$

and is homotopy equivalent to the *free* $|E\mathfrak{S}^{(2)}|$*-space* generated by a one-point space. The group completion of the monad associated to $|E\mathfrak{S}^{(2)}|$ is equivalent to the double looping of the double suspension functor so that the group completion $\overline{\mathcal{N}\mathfrak{B}}$ of $\mathcal{N}\mathfrak{B}$ is homotopy equivalent to Ω^2S^2, the double loop space of the 2-sphere.

The group completion process can be realized as a homological completion followed by Quillen's plus-construction, which yields

$$\overline{\mathcal{N}\mathfrak{B}} \sim \mathbb{Z} \times (BB_\infty)^+ \sim \Omega^2S^2,$$

a classical result of F. Cohen [12] and G. Segal [36].

2 Lax 3-categories

The purpose of this chapter is to fully embed the category of braided monoidal categories into the category $3\text{-Cat}_\otimes$ of lax 3-categories by means of a categorical double delooping. The subtle point is then the definition of a 3-nerve, which was done by O. Leroy in [29]. We interpret his nerve through the construction of a *standard cosimplicial object* in $3\text{-Cat}_\otimes$ inducing a natural adjunction between the category of simplicial sets and $3\text{-Cat}_\otimes$.

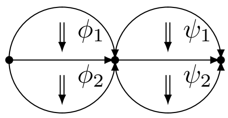
Notation 2.1. A 2-category $\mathcal{C}$ is a category endowed with categorical hom-sets $\mathcal{C}(A,B)$ in such a way that composition is a functor of categories

$$\mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C).$$

The objects of $\mathcal{C}$ are also called 0-*cells*, the morphisms of $\mathcal{C}$ (identified with objects of $\mathcal{C}(A,B)$) are called 1-*cells* and the morphisms of $\mathcal{C}(A,B)$ 2-*cells*. We represent 1-cells by arrows $f : A \to B$ and 2-cells by double arrows $\phi : f \Rightarrow g$.

Composition internal to hom-sets $\mathcal{C}(A,B)$ is called *"vertical" composition* of 2-cells and denoted as usual by $\circ$. Composition internal to $\mathcal{C}$ is called *"horizontal" composition* and denoted by $\star$ in the case of 2-cells and by juxtaposition in the case of 1-cells (both from right to left).

The fact that horizontal composition is a *functor* amounts (beside the usual unity and associativity constraints) to the following interchange relation : For each diagram of the form



we have

$$(\psi_2 \star \phi_2) \circ (\psi_1 \star \phi_1) = (\psi_2 \circ \psi_1) \star (\phi_2 \circ \phi_1). \tag{int}$$

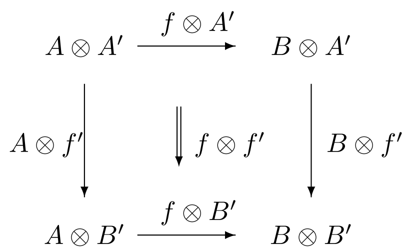
In particular, for each 2-cell $\phi : f_0 \Rightarrow f_1$ in $\mathcal{C}(A, B)$ and each 2-cell $\psi : g_0 \Rightarrow g_1$ in $\mathcal{C}(B, C)$ we get for the horizontal composition $\psi \star \phi$ the two following expressions

$$(\psi \star 1_{f_1}) \circ (1_{g_0} \star \phi) = (1_{g_1} \star \phi) \circ (\psi \star 1_{f_0}). \tag{cmt}$$

Conversely, if the commutativity relation (cmt) holds for every horizontally composable pair of 2-cells $(\phi, \psi) \in \mathcal{C}(A, B) \times \mathcal{C}(B, C)$ then the interchange relation (int) follows from the unity and associativity constraints in $\mathcal{C}(A, B)$ and $\mathcal{C}(B, C)$.

Definition 2.2. ([18], [28]) *Given two* 2*-categories* $\mathcal{C}, \mathcal{D}$ Gray's tensor product $\mathcal{C} \otimes \mathcal{D}$ *is defined to be the* 2*-category*

- *with* 0*-cells given by (formal) products* $A \otimes A'$ *for all pairs* $(A, A') \in \mathcal{C}_0 \times \mathcal{D}_0$*,*
- *with* 1*-cells given by products* $A \otimes f'$ *and* $f \otimes A'$ *for all pairs* $(A, f') \in \mathcal{C}_0 \times \mathcal{D}_1$ *and* $(f, A') \in \mathcal{C}_1 \times \mathcal{D}_0$*,*
- *with* 2*-cells generated by products* $A \otimes \phi', f \otimes f', \phi \otimes A'$ *for all pairs* $(A, \phi') \in \mathcal{C}_0 \times \mathcal{D}_2, (f, f') \in \mathcal{C}_1 \times \mathcal{D}_1, (\phi, A') \in \mathcal{C}_2 \times \mathcal{D}_0$*, where* $f \otimes f'$ *denotes the following* 2*-cell :*



These cells are subject to the following three types of relations :

- *The assignments* $(A \otimes -) : \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ *and* $(- \otimes A') : \mathcal{C} \to \mathcal{C} \otimes \mathcal{D}$ *are* 2*-functors for all* $(A, A') \in \mathcal{C}_0 \times \mathcal{D}_0$ *(*0*-functoriality),*

- $gf \otimes f' = (g \otimes f') \circ (f \otimes f')$
  $f \otimes g'f' = (f \otimes g') \circ (f \otimes f')$
  $f \otimes 1_{A'} = 1_{f \otimes A'} = 1_f \otimes A'$
  $1_A \otimes f' = 1_{A \otimes f'} = A \otimes 1_{f'}$
  for all composable 1*-cells* $f, g \in \mathcal{C}_1$ *and* $f', g' \in \mathcal{D}_1$ *(*1*-functoriality),*

- $(g \otimes f') \circ (\phi \otimes A') = (\phi \otimes B') \circ (f \otimes f')$
  $(f \otimes g') \circ (A \otimes \phi') = (B \otimes \phi') \circ (f \otimes f')$
  for all $\phi : f \Rightarrow g$ *in* $\mathcal{C}$ *and* $\phi' : f' \Rightarrow g'$ *in* $\mathcal{D}$ *(*2*-functoriality).*

The Cartesian product $\mathcal{C} \times \mathcal{D}$ carries a natural (diagonal) 2-categorical structure with respect to which there is a (strictifying) 2-functor $\mathcal{C} \otimes \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ sending $(f \otimes f')$ to $1_{(f,f')}$.

Definition 2.3. *A* lax 3-category $\mathcal{C}$ *is a category endowed with* 2*-categorical hom-sets such that composition is a* 2*-functor* $\mathcal{C}(A, B) \otimes \mathcal{C}(B, C) \to \mathcal{C}(A, C)$*. A lax* 3*-category is* strict *iff composition factorizes through the Cartesian product* $\mathcal{C}(A, B) \times \mathcal{C}(B, C)$.

The commutativity constraint for the horizontal composite of two 2-cells in a strict 3-category is "relaxed" inside a lax 3-category $\mathcal{C}$ to an "interchange" 3-cell relating the two in general different composites :

$$\begin{array}{ccc} g_0 f_0 & \xrightarrow{1_{g_0} \star \phi} & g_0 f_1 \\ {\scriptstyle \psi \star 1_{f_0}} \Big\Downarrow & \Big\Downarrow \phi \otimes \psi & \Big\Downarrow {\scriptstyle \psi \star 1_{f_1}} \\ g_1 f_0 & \xrightarrow{1_{g_1} \star \phi} & g_1 f_1 \end{array}$$

We call this dimension raising composition of 2-cells in a lax 3-category the *Gray composition* of 2-cells.

Remark 2.4. There is a subtle point in the definition of a lax 3-category which we passed over : the *associativity* of Gray's tensor product of 2-categories which is certainly a necessary condition for consistency [19]. A natural way to establish this associativity is to define n-fold Gray tensor products of 2-categories independently of a chosen bracketing. This amounts essentially to giving a bracketing independent description of all possible horizontal compositions of n consecutive 2-cells in a lax 3-category.

So, let $\phi_i : f_i \Rightarrow g_i$ be 2-cells with $f_i, g_i : A_{i-1} \to A_i$ for $i = 1, \ldots, n$. Then there are 2^n different compositions of the 1-cells to define an arrow from A_0 to A_n. Regarding the 2-cells ϕ_i as directions of a n-cube, each of the 2^n compositions corresponds to a well defined vertex in the n-cube and conversely, each edge of the cube to a well defined 2-cell in $\mathcal{C}$. Hence, there are exactly $n!$ 2-cells from $f_n f_{n-1} \cdots f_1$ to $g_n g_{n-1} \ldots g_1$ obtained by composing the different

ϕ_i in an arbitrary order. The crucial point now is that the interchange 3-cells of the lax 3-category $\mathcal{C}$ *partially order* the set of 2-cells between $f_n f_{n-1} \cdots f_1$ to $g_n g_{n-1} \ldots g_1$ according to the weak Bruhat order, and that conversely the weak Bruhat order gives all of them independently of any chosen bracketing. To see this, regard the interchange 3-cells as relators; the 1-functoriality of Gray's tensor product then states precisely that the so defined (reflexive) relation is transitive and hence a partial order.

Notation 2.5. For a braided monoidal category $(\mathcal{C}, \square, U)$, let $\Omega^{-2}\mathcal{C}$ denote the 1-reduced lax 3-category defined by shifting up dimension twice :

$$(\Omega^{-2}\mathcal{C})_0 = \{*\}, (\Omega^{-2}\mathcal{C})_1 = \{1_*\}, (\Omega^{-2}\mathcal{C})_2 = \mathcal{C}_0, (\Omega^{-2}\mathcal{C})_3 = \mathcal{C}_1,$$

where vertical (and thus horizontal) composition of 2-cells in $\Omega^{-2}\mathcal{C}$ is given by the monoidal structure on $\mathcal{C}$ and the interchange 3-cell between the two different composites of 2-cells is given by the braiding. Naturality, transitivity and unitarity of the braiding transform into 2-, 1- and 0-functoriality of Gray's tensor product under the assignment $\mathcal{C} \mapsto \Omega^{-2}\mathcal{C}$. Braided monoidal functors transform into 3-functors, so that this *categorical double delooping* Ω^{-2} defines a fully faithful embedding of the category of braided monoidal categories into the category of lax 3-categories. Indeed, the former category is equivalent to the full subcategory of the latter formed by the 1-reduced lax 3-categories.

We saw in the first section that the geometric realization of a braided monoidal category $\mathcal{C}$ admits a double delooping. We shall show below that under favourable circumstances $\Omega^{-2}\mathcal{C}$ can be regarded as a categorical model of such a double delooping. But, what's the space or simplicial set represented by a lax 3-category ? For strict 3-categories (resp. ω-categories), R. Street [40] gave a very convincing answer, extending each ordinal $[n] = \{0 \to 1 \to 2 \to \ldots \to n\}$ to a 3-categorical object $\mathcal{O}_n$ (called *n-th oriental*) such that n-simplices in a 3-category $\mathcal{D}$ are just 3-functors $\mathcal{O}_n \to \mathcal{D}$. In particular, the family $(\mathcal{O}_n)_{n\geq 0}$ defines a cosimplicial object in the category of 3-categories (resp. ω-categories).

In our lax setting, we have to replace Street's 3-categorical orientals by lax ones, which take into account the non-commutative horizontal composition of 2-cells in a lax 3-category. The geometric idea behind the following definition of a cosimplicial object in 3-$\mathrm{Cat}_\otimes$ is taken from O. Leroy's paper [29] :

Definition 2.6. *For $0 \leq i < j \leq n$, let $\|n\|(i,j)$ be the following 2-categorical $(j-i-1)$-cube :*

$$\|n\|(i,j)_0 = \{\textit{factorizations of } i \to j \textit{ in } [n]\}$$
$$\|n\|(i,j)_1 = \{\textit{refinement sequences}\}$$
$$\|n\|(i,j)_2 \leftrightarrow \textit{weak Bruhat order}$$

Composition $\|n\|(i,j) \otimes \|n\|(j,k) \to \|n\|(i,k)$ is defined by concatenation and endows $\|n\|$ with the structure of a lax 3-category.

The simplicial operators $[m] \to [n]$ canonically extend to 3-functors so as to define a cosimplicial object $(\|n\|)_{n\geq 0}$ in 3-$Cat_\otimes$.

Some comments are in order :
Factorizations $i \to i_1 \to \cdots \to i_k \to j$ of $i \to j$ are in bijection with subsets $\{i_1, \dots, i_k\} \subseteq \{i+1, i+2, \dots, j-1\}$ and thus naturally form the vertex-set of a $(j-i-1)$-cube whose edges are oriented by subset-inclusion. A *refinement sequence* $I_1 \to I_r$ is then a sequence of subsets

$$\emptyset \subseteq I_1 \subset I_2 \subset \cdots \subset I_r \subseteq \{i+1, i+2, \dots, j-1\}$$

such that $|I_k| = |I_{k-1}|+1$ for $k = 2, \dots, r$. Refinement sequences are in bijection with (oriented) edge-paths in the $(j-i-1)$-cube.
The *weak Bruhat order* on refinement sequences is finally given by the following relators :

$$(I_1 \subset I_2 \subset \cdots \subset I_r) \Rightarrow (J_1 \subset J_2 \cdots \subset J_r)$$

if and only if $I_1 = J_1$ and $I_r = J_r$ and

$$[I_2 \backslash I_1, I_3 \backslash I_2, \dots, I_r \backslash I_{r-1}] \leq [J_2 \backslash J_1, J_3 \backslash J_2, \dots, J_r \backslash J_{r-1}] \text{ in } (\mathfrak{S}_{I_r \backslash I_1}, \leq).$$

The Gray composition $I \otimes J$ of 0-cells $I \in \|n\|(i,j)_0, J \in \|n\|(j,k)_0$ is simply concatenation of the corresponding factorizations (in fact the disjoint union of subsets); Gray composition of 1-cells is given by the following formula :

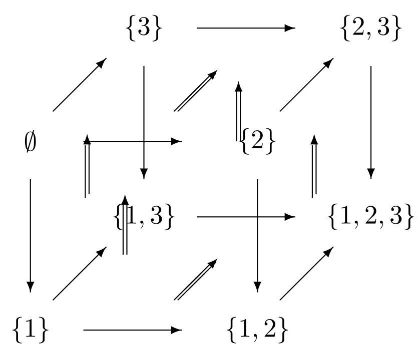
$$\begin{array}{ccc} \|n\|(i,j)_1 \otimes \|n\|(j,k)_1 & \longrightarrow & \|n\|(i,k)_2 \\ (I_1 \subset \cdots \subset I_r) \otimes (J_1 \subset \cdots \subset J_r) & \mapsto & (I_1 \otimes J_1 \subset \cdots I_r \otimes J_1 \subset \cdots I_r \otimes J_s) \\ & \Rightarrow & (I_1 \otimes J_1 \subset \cdots I_1 \otimes J_s \subset \cdots I_r \otimes J_s) \end{array}$$

Under the bijection between 2-cells and relators of the weak Bruhat order, Gray composition corresponds to the multiplication of the permutation operad :

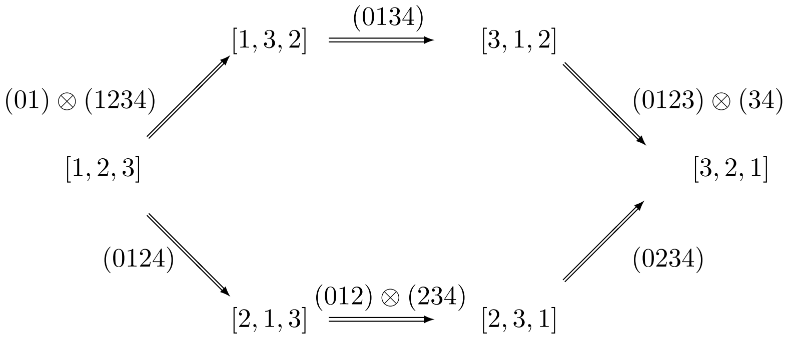
$$\begin{array}{rcl} m^{\mathfrak{S}}_{|I_r \backslash I_1|, |J_s \backslash J_1|} : \mathfrak{S}_2 \times \mathfrak{S}_{|I_r \backslash I_1|} \times \mathfrak{S}_{|J_s \backslash J_1|} & \longrightarrow & \mathfrak{S}_{I_r \otimes J_s \backslash I_1 \otimes J_1} \\ ([1,2] \to [2,1]; \sigma_1, \sigma_2) & \mapsto & (\sigma_1 \oplus \sigma_2 \to \sigma_2 \oplus \sigma_1) \end{array}$$

In particular, associativity (resp. monotony) of $m^{\mathfrak{S}}$ implies the 1- (resp. 2-) functoriality of Gray composition.

Example 2.7. *The* 2*-categorical* 3*-cube* $\|4\|(0,4)$.



Among the six faces of the cube, five are images of $\|3\|(0,3)$ by simplicial operators, and one (the right hand face) is the image of $\|4\|(0,2) \otimes \|4\|(2,4)$ by Gray composition. We get the following commutative Yang-Baxter hexagon of 2-cells of $\|4\|(0,4)$ which illustrates well the intimate relationship between braiding, Gray tensor product and weak Bruhat order (cf. 1.1, 2.4 and especially Baues' *cellular string functor* [6]) :



Remark 2.8. Stasheff's *associativity pentagon* is obtained by contracting the decomposable "interchange cell" $(012) \otimes (234)$ into an identity (cf. [38]). This contraction is part of a natural *strictifying* 3-functor $\|n\| \to \mathcal{O}_n$ which contracts all interchange cells of $\|n\|$ into identities ending up with Street's 3-categorical n-th oriental $\mathcal{O}_n$. Extending the preceeding example, it follows from the definition that the category of 2-cells $\|n+1\|(0,n+1)(\emptyset,\{1,2,\ldots,n\})$ is canonically isomorphic to the weak Bruhat order $(\mathfrak{S}_n,\leq)$, which is also the 1-skeleton of Milgram's *permutohedron* P_n (cf. [35], [32]). The strictifying 3-functor defines then a surjective map of partially ordered sets

$$\|n+1\|(0,n+1)(\emptyset,\{1,2,\ldots,n\}) \longrightarrow \mathcal{O}_{n+1}(0,n+1)(\emptyset,\{1,2,\ldots,n\})$$

whose image is exactly the 1-skeleton of Stasheff's associahedron $\mathcal{A}_{n+1}$. This 1-skeletal map extends to a convex map $P_n \to \mathcal{A}_{n+1}$, as shown by A. Tonks [41]. It seems likely that *higher order categorical laxness* is related to the geometry of this convex projection. In particular, Kapranov-Voevodsky's idea [25] of *deriving pasting schemes* which is a categorical version of Baues' *cellular string functor* [6], may shed some light on this "relaxing" process from $\mathcal{O}_n$ to $\|n\|$.

Lemma 2.9. *The* 3*-nerve* N_3 : *3-Cat*$_\otimes \to$ SimpSets : $\mathcal{D} \mapsto$ *3-Cat*$_\otimes(\|-\|,\mathcal{D})$ *has a left adjoint* t_3 : SimpSets $\to$ *3-Cat*$_\otimes$ *given by the coend formula :*

$$t_3(X) = \int^{n\in\gneqq} X_n \otimes \|n\| = \varinjlim_{\Delta[-]\downarrow X} \|-\|$$

so that we have a natural bijection of morphism-sets

$$\textit{3-Cat}_\otimes(t_3(X),\mathcal{D}) \overset{\sim}{\leftrightarrow} \mathrm{SimpSets}(X,\mathcal{N}_3(\mathcal{D})).$$

Proof. – The adjunction formula follows immediately from the universal property of the coend (cf. [31]). It remains to show that colimits *exist* in 3-$\mathrm{Cat}_\otimes$. A powerful method to establish such existence results, consists in realizing (up to equivalence) the category under consideration, as a category of T-algebras over a cocomplete base category with respect to a *filtered colimits preserving* monad T. We then dispose of an "algebraic" construction of colimits like for example in the case of groups over sets, see M. Kelly [27].

In our case we have two possible choices : neglecting *all* compositional laws defines a forgetful functor U into the category of (3-dimensional) globular sets which is (as a presheaf category over sets) cocomplete. Forthcoming work of M. Batanin [5] implies that U is *monadic*, i.e. lax 3-categories are T-algebras for some monad $T = UF$ on the category of globular sets, where F is a "free functor" left adjoint to U. The monad T preserves filtered colimits since T can be constructed as a coproduct of some finitely iterated pullback constructions.

Alternatively, one can neglect just the compositional laws of 3-cells which defines a forgetful functor into the category of graphs enriched over $2-\mathrm{Cat}_\otimes$, the category of 2-categories with the closed monoidal structure induced by Gray's tensor product. It is well known that categories are T-algebras over graphs for some filtered colimits preserving monad T on the category of graphs, and this remains true in our enriched context, cf. H. Wolff [44]. □

Recall that a simplicial set X is called a *simplicial n-type* if the homotopy groups $\pi_k(|X|)$ are trivial for $k > n$. A small lax 3-category $\mathcal{D}$ whose cells of positive dimension are (strictly) invertible is called a *lax 3-groupoid.* Since the interchange cells of a lax 3-groupoid are invertible, lax 3-groupoids are precisely semi-strict 3-groupoids in the terminology of Baez-Neuchl [2] and Gray groupoids in the terminology of Gordon-Power-Street [17].

Proposition 2.10. *The 3-nerve of a lax 3-groupoid is a fibrant simplicial 3-type.*

Proof. – Two n-simplices $x, y : \|n\| \to \mathcal{G}$ of a lax 3-groupoid $\mathcal{G}$ which coincide on their 3-skeleta are equal, since there exist no cells of dimension greater than 3 in $\mathcal{G}$. A *fibrant* simplicial set with this property is a simplicial 3-type. It remains thus to show that $\mathcal{N}_3\mathcal{G}$ is fibrant, i.e. for all couples (k, n) with $0 \le k \le n$, all (k, n)-horns have fillers in $\mathcal{N}_3(\mathcal{G})$ (cf. [16]).

If $n \le 3$, this is an easy consequence of the invertiblity of 1- and 2-cells in $\mathcal{G}$. If $n \ge 5$, a (k, n)-horn in $\mathcal{N}_3(\mathcal{G})$ defines by adjunction a partial map from $\|n\|$ to $\mathcal{G}$. The domain of this partial map (i.e. the lax 3-categorical (k, n)-horn) contains the 3-skeleton of $\|n\|$, so that the partial map extends uniquely to $\|n\|$ defining the required filler. If $n = 4$, a (k, n)-horn in $\mathcal{N}_3(\mathcal{G})$ defines by adjunction, a 2-functor on all but one face of the 3-cube $\|4\|(0, 4)$ (2.7). The commutativity of this cube gives an equation for the missing face, which can be solved thanks to the invertibility of the 3-cells in $\mathcal{G}$. □

Proposition 2.11. *Let $\mathcal{C}$ be a braided monoidal category whose categorical double delooping $\Omega^{-2}\mathcal{C}$ is a lax 3-groupoid. Then the nerve of $\mathcal{C}$ is homotopy equivalent to the double loop space of the 3-nerve of $\Omega^{-2}\mathcal{C}$.*

Proof. – Since $\mathcal{N}_3(\Omega^{-2}\mathcal{C})$ is fibrant, the simplicial functional object

$$\underline{\mathrm{SimpSets}_*}(S^2, \mathcal{N}_3(\Omega^{-2}\mathcal{C}))$$

has the homotopy type of $\Omega^2|\mathcal{N}_3(\Omega^{-2}\mathcal{C})|$, where S^2 denotes the reduced simplicial 2-sphere $\Delta[2]/\dot{\Delta}[2]$ (cf. [16]). By adjunction, the same functional object can be defined over $\underline{3\text{-}\mathrm{Cat}_{\otimes}}_*(t_3(S^2), \Omega^{-2}\mathcal{C})$ where the simplicial structure is induced by the cosimplicial object $t_3(S^2 \rightrightarrows S^2 \times \Delta[1] \cdots)$ in 3-$\mathrm{Cat}_{\otimes}$. Since the lax 3-groupoid $\Omega^{-2}\mathcal{C}$ is 1-reduced, we can identify 3-functors $t_3(S^2 \times \Delta[n]) \to \Omega^{-2}\mathcal{C}$ with simplicial collections of 3-cells in $\Omega^{-2}\mathcal{C}$ or equivalently (by shifting down dimension twice) with simplicial collections of 1-cells in $\mathcal{C}$. More precisely, the simplicial functional object $\underline{\mathrm{SimpSets}_*}(S^2, \mathcal{N}_3(\Omega^{-2}\mathcal{C}))$ is isomorphic to the *simplicial subdivision* of the nerve $\mathcal{NC}$ induced by the cosimplicial object $S^{-2}(S^2 \rightrightarrows S^2 \times \Delta[1]/\{*\} \times \Delta[1] \cdots)$ in SimpSets. Since simplicial subdivision preserves the homotopy type we get the required homotopy equivalence. □

Remark 2.12. A braided monoidal category $\mathcal{C}$ whose double delooping $\Omega^{-2}\mathcal{C}$ is a lax 3-groupoid is precisely what Joyal-Street [22] call a strict braided categorical group. We shall call it here (for shortness) a *braided groupoid*. A braided groupoid is thus a *group in the category of groupoids endowed with a natural family of braidings* in the sense of definition 1.1.

It is shown in [22] that the category of braided groupoids is equivalent to the category of Whitehead's crossed modules endowed with a bracket operation in Conduché's sense [13]. This category is equivalent to the category of stable crossed modules, which in turn models $(n-2)$-connected homotopy n-types for $n \geq 3$. Our functor $\mathcal{N}_3\Omega^{-2}$ defines a realization functor from braided groupoids into 1-reduced simplicial 3-types and the preceeding proof implies that the algebraically defined π_0 and π_1 of the braided groupoid are canonically isomorphic to the geometrically defined π_2 and π_3 of the associated 1-reduced 3-type.

3 Algebraic 3-type of space

There is a vast literature on algebraic models of homotopy n-types, beginning with the pioneering work of H. Poincaré and J.H.C. Whitehead [43]. As our concern is merely to point out some relationship between the combinatorics of 2-fold iterated loop spaces and of homotopy 3-types, this chapter will be rather sketchy and primarily intends to give some complementary details to O. Leroy's paper [29]. We believe that lax 3-groupoids are "good" models for homotopy 3-types insofar as they furnish "fibrant" models out of some natural "cellular bases". R. Brown's and also A. Grothendieck's emphasis on the wellsuitedness of groupoids for homotopy types gets full justification here : we reduce the required equivalence of homotopy categories to a structural statement about the fundamental groupoid of the double loop space of a simply connected 3-type : it is braided in the sense of remark 2.12.

Conduché's *crossed modules of length* 2 [13] and Baues' *quadratic modules* [7] are closely related to lax 3-groupoids as suggested by the work of Brown-Gilbert

[11] on automorphism structures of crossed modules [43]. In fact, to each vertex of a lax 3-groupoid $\mathcal{G}$ is associated a 3-stage differential group

$$\pi_3(\mathcal{G}_3, \mathcal{G}_2) \to \pi_2(\mathcal{G}_2, \mathcal{G}_1) \to \pi_1(\mathcal{G}_1),$$

which turns out to be a "2-nilpotent" crossed module of length 2, i.e. it carries a natural structure of quadratic module.

Let 3-$\mathrm{Grp}_\otimes$ denote the category of lax 3-groupoids and $\hat{t}_3 : \mathrm{SimpSets} \to$ 3-$\mathrm{Grp}_\otimes$ denote the composition of $t_3 : \mathrm{SimpSets} \to$ 3-$\mathrm{Cat}_\otimes$ with the reflection into 3-$\mathrm{Grp}_\otimes$ (defined by formally inverting all cells). Recall then the following corollary of a theorem of Sjoerd Crans [14] about the *transfer of closed model structures* along an adjunction :

Proposition 3.1. *The category 3-Grp$_\otimes$ of lax 3-groupoids carries a Quillen closed model structure [42] induced by the closed structure of* SimpSets *through the adjunction $\hat{t}_3 \dashv \mathcal{N}_3$.*
Fibrations *(resp.* weak equivalences*) are those 3-functors ϕ for which $\mathcal{N}_3(\phi)$ is a Kan fibration (resp. weak equivalence) in* SimpSets*.* Cofibrations *are 3-functors which have the right lifting property with respect to trivial fibrations.*

Proof. – All we need to check is that the composite functor $\mathcal{N}_3\hat{t}_3 : \mathrm{SimpSets} \to \mathrm{SimpSets}$ transforms trivial cofibrations of into weak equivalences. This follows from the fact that trivial cofibrations in SimpSets are "generated" by the set of inclusions of (k, n)-horns into $\Delta[n]$, and that for the latter, the statement is true. □

Proposition 3.2. *The unit $\eta_X : X \to \mathcal{N}_3\hat{t}_3(X)$ is a weak equivalence if and only if X is a simplicial 3-type.*

Proof. – The necessity for X to be a 3-type follows from proposition 2.10. The sufficiency is at the heart of the comparison between our geometric and algebraic homotopy categories. We shall sketch a proof.

The first step is a reduction to simply connected 3-types. Indeed, there is a natural fibration from each lax 3-groupoid $\mathcal{G}$ to its underlying fundamental groupoid $\mathcal{G}[1]$ defined by "contracting" all 2- and 3-cells of $\mathcal{G}$ into identities. For a simplicial set X, this defines precisely the fundamental groupoid $\Pi(X) = \hat{t}_3(X)[1]$ of Gabriel-Zisman [16]. The unit η_X is a weak equivalence iff the homotopy fiber X' of the composite $X \stackrel{\eta_x}{\to} \mathcal{N}_3\hat{t}_3 \stackrel{p[1]}{\to} \mathcal{N}_3\hat{t}_3(X)[1]$ is weakly equivalent (via η_X) to the fiber of $p[1]$, but this amounts to proving that $\eta_{X'}$ is a weak equivalence for the simply connected 3-type X'. So assume $X = X'$.

The second step is a reduction to *minimal fibrant* models. This is possible because each simplicial set is contained in a fibrant model which in turn contains a minimal fibrant one as a simplicial deformation retract [16]. Minimality means that two simplices in X coincide as soon as they are homotopic relative to the boundary. Minimal fibrant models X of simply connected spaces are in particular 1-reduced, i.e. $X_0 = X_1 = \{*\}$.

The third step is a double looping process. For a 1-reduced simplicial set X, the simplicial functional object $\underline{\mathrm{SimpSets}}_*(S^2, X)$ is (as in proposition 2.11) a subdivision of a simplicial set $\Omega^2 X$ which has the homotopy type of the double loops on X for fibrant X. The lax 3-groupoid $\hat{t}_3(X)$ is also 1-reduced. Double looping of $\eta_X : X \to \mathcal{N}_3\hat{t}_3(X)$ yields thus a canonical inclusion of $\Omega^2 X$ into the nerve of $\Omega^2\hat{t}^3(X)$. It remains to show that this inclusion is a weak equivalence. The universal property of $\hat{t}_3$ together with the full embedding of braided groupoids (cf. 2.12) into 3-$\mathrm{Grp}_\otimes$ (via Ω^{-2}) implies that $\Omega^2\hat{t}^3(X)$ is actually the *free braided groupoid* generated by $\Omega^2 X$. Since braided groupoids form an algebraic enrichment of groupoids, the inclusion into the free braided groupoid factors through the free groupoid on $\Omega^2 X$ which is simply the *fundamental groupoid* of $\Omega^2 X$. We are thus finally reduced to showing that for $\Omega^2 X$, the inclusion of the fundamental groupoid into the free braided groupoid is a weak equivalence. This will follow from a natural family of braidings on the fundamental groupoid of $\Omega^2 X$ for *minimal* fibrant 1-reduced X.

Indeed, the 2-skeleton of X is a bunch of 1-reduced 2-spheres being in one-to-one correspondence with the vertices of $\Omega^2 X$. By minimality of X, each arrow of the fundamental groupoid on $\Omega^2 X$ corresponds to a unique 3-simplex of X obtained by "composition of homotopies". The group structure on $\pi_2(X)$ induces (again by minimality) a group structure on the 2-spheres in X and hence on the vertices of $\Omega^2 X$, natural with respect to arrows of the fundamental groupoid. The braiding $c_{A,B} : A\Box B \to B\Box A$ is defined by the following geometric construction: represent the corresponding 2-spheres S^2_A, S^2_B of X as loops γ_A, γ_B on ΩX. Then there is a commuting homotopy from $\gamma_A \star \gamma_B$ to $\gamma_B \star \gamma_A$ which yields a simplicial homotopy from $S^2_A + S^2_B$ to $S^2_B + S^2_A$ and hence an arrow $c_{A,B}$ in the fundamental groupoid. This arrow defines an invertible braiding, which in general is *not* symmetric, i.e. $c^{-1}_{A,B} \neq c_{B,A}$.

The fundamental groupoid is thus a braided subgroupoid of the free braided groupoid on $\Omega^2 X$. In particular, the inclusion admits a canonical retraction, which is easily identified with the counit of the adjunction between groupoids and braided groupoids. This *co*unit is a weak equivalence by formal reasons similar to those used in the proof of the following theorem. □

Theorem 3.3. (Leroy [29], Joyal-Tierney [23])
The adjunction $\hat{t}_3 \dashv \mathcal{N}_3$ induces an equivalence between the homotopy category $\mathbf{Ho}$(*3-Grp*$_\otimes$) *of lax* 3*-groupoids and the homotopy category* $\mathbf{Ho}$(SimpSets[3]) *of simplicial* 3*-types.*

Proof. According to Quillen [42], we have to check that
– the functor $\hat{t}_3$ preserves cofibrations and weak equivalences between cofibrant objects (this follows by adjunction from the definition of the closed structure of 3-$\mathrm{Grp}_\otimes$ together with proposition 3.2),
– the functor $\mathcal{N}_3$ preserves fibrations and weak equivalences between fibrant objects (immediate),
– the unit maps $\eta_X : X \to \mathcal{N}_3\hat{t}_3(X)$ and counit maps $\epsilon_\mathcal{G} : \hat{t}_3\mathcal{N}_3(\mathcal{G}) \to \mathcal{G}$ are weak equivalences (again proposition 3.2 joined to the fact that it is enough to show that $\mathcal{N}_3\epsilon_\mathcal{G}$ is a weak equivalence, which follows from the adjunction formula $\mathcal{N}_3\epsilon_\mathcal{G} \circ \eta_{\mathcal{N}_3\mathcal{G}} = 1_\mathcal{G}$). □

Corollary 3.4. *A connected homotopy* 3*-type* X *is represented by a strict* 3*-groupoid if and only if the third Postnikov invariant* $k_X^3 \in H^4(X[2], \pi_3(X))$ *vanishes, i.e. iff* X *is homotopy equivalent to the Cartesian product of the underlying* 2*-type* $X[2]$ *with the Eilenberg-MacLane space* $K(\pi_3(X), 3)$.

Proof. – The double loop space of the universal cover of X has a braided fundamental groupoid weakly equivalent to the braided groupoid derived from the representing lax 3-groupoid $\mathcal{G}_X$ by shifting down dimension twice after passage to the universal cover (cf. 3.2). Hence, $\mathcal{G}_X$ is strictifiable iff all braidings are symmetric iff the derived braided groupoid is symmetric iff the double loop space of the universal cover of X is a product of (abelian) Eilenberg-MacLane spaces iff $k_X^3 = 0$. □

Remark 3.5. For a lax 3-groupoid $\mathcal{G}$ the homotopy groups with respect to a base point $* \in \mathcal{G}_0$ can be defined algebraically by

$$\begin{aligned}\pi_1(\mathcal{G}) &= Aut_{\mathcal{G}}(*)/\sim,\\ \pi_2(\mathcal{G}) &= Aut_{\mathcal{G}}(1_*)/\sim,\\ \pi_3(\mathcal{G}) &= Aut_{\mathcal{G}}(1_{1_*}),\end{aligned}$$

where the equivalence relation is induced by the cells of the next higher dimension. The compatibility of this definition with the 3-nerve functor follows from proposition 2.10 and Kan's definition [24] of the homotopy groups of a fibrant simplicial set. The homotopy groups can also be defined as homology groups of the 3-stage differential group

$$\pi_3(\mathcal{G}_3, \mathcal{G}_2) \stackrel{src_2}{\to} \pi_2(\mathcal{G}_2, \mathcal{G}_1) \stackrel{src_1}{\to} \pi_1(\mathcal{G}_1), \text{ where}$$

$$\begin{aligned}\pi_3(\mathcal{G}_3, \mathcal{G}_2) &= \{x \in \mathcal{G}_3 \,|\, trg_2(x) = 1_{1_*}\},\\ \pi_2(\mathcal{G}_2, \mathcal{G}_1) &= \{x \in \mathcal{G}_2 \,|\, trg_1(x) = 1_*\},\\ \pi_1(\mathcal{G}_1) &= \{x \in \mathcal{G}_1 \,|\, trg_0(x) = * = src_0(x)\}.\end{aligned}$$

We denote by $src_n(x)$ (resp. $trg_n(x)$) the n-dimensional source (resp. target) of the $(n+1)$-cell x. The *group* structures of $\pi_3(\mathcal{G}_3, \mathcal{G}_2)$ and $\pi_2(\mathcal{G}_2, \mathcal{G}_1)$ are induced by *horizontal composition* in $\mathcal{G}$ and thus governed by Gray's tensor product of 2-categories. For 2-cells $x, y \in \pi_2(\mathcal{G}_2, \mathcal{G}_1)$, the interchange 3-cell $x \otimes y$ induces actually two different group structures on $\pi_2(\mathcal{G}_2, \mathcal{G}_1)$ which we denote by

$$\begin{aligned}y \dashv x &= src_2(x \otimes y) = (y \star 1_{trg_1(x)}) \circ (1_{src_1(y)} \star x) \text{ and}\\ y \vdash x &= trg_2(x \otimes y) = (1_{trg_1(y)} \star x) \circ (y \star 1_{src_1(x)}).\end{aligned}$$

Both group structures share the same unit 1_{1_*} and are related by the commutation formula $y \dashv x = {}^{src_1(y)}x \vdash y$, where *horizontal conjugation* is defined to be ${}^{src_1(y)}x = 1_{src_1(y)} \star x \star 1_{src_1(y)^{-1}}$. In particular, horizontal conjugation induces group actions of $\pi_1(\mathcal{G}_1)$ on $\pi_2(\mathcal{G}_2, \mathcal{G}_1)$ and on $\pi_3(\mathcal{G}_3, \mathcal{G}_2)$ as well as a group action of $\pi_2(\mathcal{G}_2, \mathcal{G}_1)$ on $\pi_3(\mathcal{G}_3, \mathcal{G}_2)$. These group actions endow the differentials

src_1 and src_2 with the structure of *precrossed modules* (see [13], [43]). The second differential src_2 is even a crossed module endowed with a generalized bracket operation (cf. [13], [22] and remark 2.12) :

$$\begin{array}{rcl} \{-,-\}: \pi_2(\mathcal{G}_2,\mathcal{G}_1)\times\pi_2(\mathcal{G}_2,\mathcal{G}_1) & \to & \pi_3(\mathcal{G}_3,\mathcal{G}_2) \\ (x,y) & \mapsto & (x\otimes y)\vdash x^{-1}\vdash y^{-1}. \end{array}$$

The interchange 3-cell $x\otimes y$ from $y\dashv x$ to $y\vdash x$ simultanously induces a bracket operation on src_2 and measures the failure of src_1 to be crossed; indeed, the bracket $\{x,y\}$ yields a 3-cell from ${}^{src_1(y)}x\vdash y\vdash x^{-1}\vdash y^{-1}$ to 1_{1_*}, i.e. a lifting of a categorical *Peiffer commutator*. Its being an identity for all $x,y\in\pi_2(\mathcal{G}_2,\mathcal{G}_1)$ is equivalent to src_1 being crossed, which occurs precisely in the case of a strict 3-groupoid. We leave it to the reader to check that the relations between the different source-target functions as well as the 0- and 1-functoriality of Gray's tensor product lead exactly to what Conduché [13] calls a crossed module of length 2. The 2-functoriality of Gray's tensor product moreover implies the 2-nilpotency of $\pi_2(\mathcal{G}_2,\mathcal{G}_1)$ with respect to Peiffer commutators $src_2\{-,-\}$ so that we actually have defined a quadratic module in the sense of Baues [7].

The results of Conduché and Baues imply that the 3-type defined by a connected lax 3-groupoid $\mathcal{G}$ is already contained in the associated quadratic module $(\pi_3(\mathcal{G}_3,\mathcal{G}_2)\to\pi_2(\mathcal{G}_2,\mathcal{G}_1)\to\pi_1(\mathcal{G}_1),\{-,-\}:\pi_2\times\pi_2\to\pi_3)$. We conjecture that a groupoid object in the category of quadratic modules naturally gives rise to a lax 3-groupoid, inducing thus an explicit equivalence of the corresponding homotopy categories.

Example 3.6. – *The homotopy* 3*-type of the* 2*-sphere.*
The adjunction $\hat{t}_3\dashv\mathcal{N}_3$ implies that for each simplicial set X, the lax 3-groupoid $\hat{t}_3(X)$ models the homotopy 3-type of the geometric realization of X. We shall explicitly describe the lax 3-groupoid defined by the reduced simplicial 2-sphere $S^2=\Delta[2]/\dot{\Delta}[2]$.

The colimit presentation of $t_3(S^2)$ shows that the generating cells of $\hat{t}_3(S^2)$ are those which correspond either to non-degenerate simplices of S^2 or to Gray-composites of the latter, so that we get :

$$\hat{t}_3(S^2)_0=\{*\},\quad \hat{t}_3(S^2)_1=\{1_*\},\quad \hat{t}_3(S^2)_2=\{\sigma^k,k\in\mathbb{Z}\},$$
$$\hat{t}_3(S^2)_3(1_{1_*},1_{1_*})=\{\{\sigma,\sigma\}^k,k\in\mathbb{Z}\},$$

where the interchange 3-cell $\sigma^k\otimes\sigma^m$ is a horizontal translate of the bracket-power $\{\sigma,\sigma\}^{km}$, due to the bilinearity of the bracket $\{-,-\}$, (cf. remark 3.5 and [13]). We see in particular that this model of the 3-type of S^2 incorporates in a comparatively simple way the basic facts that $\pi_3(S^2)\cong\mathbb{Z}$ and that the generator $[\eta]$ of the latter group (represented by the Hopf fibration $\eta:S^3\to S^2$) induces for each space X a quadratic map $\eta^*:\pi_2(X)\to\pi_3(X)$.

Remark 3.7. – We conclude with a few speculations about possible generalizations to higher dimension. What follows is much influenced by conversations with M. Batanin and S. Crans. A lax 3-category is probably the most restrictive extension of the concept of 3-category, for which the associated groupoid models homotopy 3-types. We recall that the basic idea is to replace the commutativity constraint for the horizontal composite of two 2-cells by an interchange 3-cell, whose source and target cells form thus a sort of *duplication* of horizontal composition. Forthcoming work of M. Batanin [5] shows that there is indeed some operadic *tree-calculus* which allows one to deduce all axioms of a lax 3-category from this idea of duplication together with a suitable *contractibility* condition.

In the same spirit, there should be an essentially unique notion of lax n-category in which all associativities are strict and in which horizontal composition is governed by some higher-dimensional interchange cells. The involved combinatorics however become more and more complicated with increasing n, since there is a whole hierarchy of horizontal compositions depending on the degree of internality (i.e. the dimension of the intersection cell). M. Batanin's tree-calculus and S. Crans' higher-order laxness conditions suggest that the horizontal composite of an n-cell and an m-cell along a k-cell should be governed by an $(n+m-k-1)$-dimensional interchange cell. In particular, this would imply that the "correct" definition of the horizontal composite of two 3-cells only takes place inside a lax 5-category, so that the 2-categorical braid analogue (we looked for in the introduction) is probably given by shifting down three times a 2-reduced lax 5-category.

We conjecture that in general the action of the n-th filtration-term $E\mathfrak{S}^{(n)}$ of the universal simplicial E_∞-operad $E\mathfrak{S}$ detects nerves of n-times shifted down $(n-1)$-reduced lax $(2n-1)$-categories.

For the moment, we only have some indication on the possible definition of a lax 4-category. Indeed, braided monoidal 2-categories have been studied by Kapranov-Voevodsky [26] and Baez-Neuchl [2] and should correspond to twice shifted down 1-reduced lax 4-categories with invertible interchange cells. Since in our setting the interchange cells are not assumed invertible, some of the defining 2-cells of a braided monoidal 2-category are forced to be identities in order to obtain a coherent definiton; actually the *transitivity* of the braiding has to be strict in our setting, whereas the *naturality* of the braiding is replaced by a 2-cell like in Kapranov-Voevodsky's approach. The Yang-Baxter hexagon is then commutative only up to a well defined 2-cell which, according to the "two proofs of the Yang-Baxter-hexagon", is given by the equality of 2-cells $c_{A,c_{B,C}} = c_{c_{A,B},C}$. Here, some principle of orientation has to be found, implicitly contained in the 2-categorical structure of the permutohedra (cf. [32], [26]). Finally, this should define an *associative* horizontal composition of 2-cells in a lax 4-category. It still remains to define coherently the horizontal composition of 3- and 4-cells in a lax 4-category. We hope to come back to this topic in a subsequent paper.

References

[1] E. Artin – *Theory of braid*, Annals of Math. 48 (1947), pp. 101-126.

[2] J. Baez and M. Neuchl – *Higher- Dimensional Algebra I : Braided monoidal 2-categories*, Advances in Math. 121 (1996), pp. 196-244.

[3] C. Balteanu, Z. Fiedorowicz, R. Schwänzl and R. M. Vogt – *Iterated monoidal categories*, preprint (1997).

[4] M. G. Barratt and P. J. Eccles – Γ^+*-structures I, II, II*, Topology 13 (1974), pp. 25-45, 113-126, 199-207.

[5] M. A. Batanin – *Monoidal globular categories as natural environment for the theory of weak n-category*, Macquarie University report (1997).

[6] H.-J. Baues – *Geometry of loop spaces and the cobar-construction*, Memoirs AMS 230 (1980).

[7] H.-J. Baues – *Combinatorial homotopy and 4-dimensional complexes*, De Gruyter Expositions in Math. 2 (1991).

[8] C. Berger – *Opérades cellulaires et espaces de lacets itérés*, Annales de l'Inst. Fourier 46 (1996), pp. 1125-1157.

[9] C. Berger – *Combinatorial models of real configuration spaces and E_n-operads*, Contemp. Math. 202 (1997), pp. 37-52.

[10] J. M. Boardman and R. M. Vogt – *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math. 347 (1973), Springer Verlag.

[11] R. Brown and N. D. Gilbert – *Algebraic models of 3-types and automorphism structures for crossed modules*, Proc. London Math. Soc. (3) 59 (1989), pp. 51-73.

[12] F. R. Cohen – *The homology of C_{n+1}-spaces*, Lecture Notes in Math. 533 (1976), pp. 207-351.

[13] D. Conduché – *Modules croisés généralisés de longueur* 2, J. Pure Appl. Algebra 34 (1984), pp. 155-178.

[14] S. E. Crans – *Quillen closed model structures for sheaves*, J. Pure Appl. Algebra 101 (1995), pp. 35-57.

[15] Z. Fiedorowicz – *The symmetric bar-construction*, to appear.

[16] P. Gabriel and M. Zisman – *Calculus of fractions and homotopy theory*, Ergebnisse der Mathematik, vol. 35, Springer Verlag (1967).

[17] R. Gordon, A. J. Power and R. H. Street – *Coherence for tricategories*, Memoirs AMS 558 (1995).

[18] J. W. Gray – *Formal Category Theory : Adjointness for* 2*-categories*, Lecture Notes in Math. 391 (1974).

[19] J. W. Gray – *Coherence for the tensor product of* 2*-categories, and braid groups*, Algebra, Topology and Category theory (a collection in honour of Samuel Eilenberg), Academic Press (1976), pp. 63-76.

[20] A. Grothendieck – *Pursuing stacks*, manuscript.

[21] A. Joyal and R. Street – *The geometry of tensor calculus I*, Advances in Math. 88 (1991), pp. 55-112.

[22] A. Joyal and R. Street – *Braided tensor categories*, Advances in Math. 102 (1993), pp. 20-78.

[23] A. Joyal and M. M. Tierney – *Algebraic homotopy types*, in preparation.

[24] D. M. Kan – *A combinatorial definition of homotopy groups*, Annals of Math. 67 (1958), pp. 282-312.

[25] M. M. Kapranov and V. A. Voevodsky – *Combinatorial-geometric aspects of polycategory theory : pasting schemes and higher Bruhat orders*, Cahiers topologie et géometrie diff. catégoriques 32 (1991), pp. 11-27.

[26] M. M. Kapranov and V. A. Voevodsky – *Braided monoidal* 2*-categories and Manin-Schechtman higher braid groups*, J. Pure Appl. Math. 92 (1994), pp. 241-267.

[27] G. M. Kelly – *A unified treatment of transfinite construtions for free algebras, free monoids, colimits, associated sheaves and so on*, Bull. Australian Math. Soc. 22 (1980), pp. 1-84.

[28] G. M. Kelly and R. Street – *Review of the elements of* 2*-categories*, Lecture Notes in Math. 420 (1974).

[29] O. Leroy – *Sur une notion de* 3*-catégorie adaptée a l'homotopie*, prepub. AGATA, Univ. Montpellier II (1994).

[30] S. MacLane – *Natural associativity and commutativity*, Rice Univ. Stud. 49 (1963), pp. 28-46.

[31] S. MacLane – *Categories for the Working Mathematician*, Graduate Texts in Math., vol. 5, Springer Verlag (1971).

[32] Y. I. Manin and V. V. Schechtman – *Arrangements of hyperplans, higher braid groups and higher Bruhat orders*, Adv. Stud. Pure Math. 17 (1989), pp. 289-308.

[33] J. P. May – *The Geometry of iterated loop spaces*, Lecture Notes in Math. 271 (1972).

[34] J. P. May – E_∞*-spaces, group completions and permutative categories*, London Math. Soc. Lecture Notes 11 (1974), pp. 61-94.

[35] R. J. Milgram – *Unstable homotopy from the stable point of view*, Lecture Notes in Math. 368 (1974).

[36] G. Segal – *Configuration spaces and iterated loop spaces*, Inventiones Math 21 (1973), pp. 213-221.

[37] J. H. Smith – *Simplicial Group Models for* $\Omega^n S^n X$, Israel J. Math. 66 (1989), pp. 330-350.

[38] J. D. Stasheff – *Homotopy associativity of* H*-spaces*, Trans. AMS 108 (1963), pp. 275-312.

[39] J. D. Stasheff – *Differential graded Lie algebras, quasi-Hopf algebras and higher homotopy algebras*, Euler Int. Math. Inst. Leningrad 1510 (1992), pp. 120-137.

[40] R. Street – *The algebra of oriented simplices*, J. Pure Appl. Math. 49 (1987), pp. 283-335.

[41] A. Tonks – *Relating the associahedron and the permutohedron*, Contemp. Math. 202 (1997), pp. 33-36.

[42] D. Quillen – *Homotopical Algebra*, Lecture Notes in Math. 43 (1967).

[43] J. H. C. Whitehead – *Combinatorial homotopy I,II*, Bull. AMS 55 (1949), pp. 213-245, 453-496.

[44] H. Wolff – V*-cat and* V*-graph*, J. Pure Appl. Algebra 4 (1974), pp. 123-135.