

OPERADS OF NATURAL OPERATIONS I: LATTICE PATHS, BRACES AND HOCHSCHILD COCHAINS

by

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Abstract. — In this first paper of a series we study various operads of natural operations on Hochschild cochains and relationships between them.

Résumé (Opérades des opérations naturelles I: chemins brisés, opérations braces et cochaînes de Hochschild)

Dans ce premier article d'une série nous étudions et comparons plusieurs opérades munies d'une action naturelle sur les cochaînes de Hochschild d'une algèbre associative.

1. Introduction

This paper continues the efforts of [14, 3, 2] in which we studied operads naturally acting on Hochschild cochains of an associative or symmetric Frobenius algebra. A general approach to the operads of natural operations in algebraic categories was set up in [14] and the first breakthrough in computing the homotopy type of such an operad has been achieved in [3]. In [2], the same problem was approached from a combinatorial point of view, and a machinery which produces operads acting on the Hochschild cochain complex in a general categorical setting was introduced.

The constructions of [2] have some specific features in different categories which are important in applications. In this first paper of a series entitled 'Operads of Natural Operations' we begin a detailed study of these special cases.

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It is very natural to start with the classical Hochschild cochain complex of an associative algebra. This is, by far, the most studied case. It seems to us, however, that a systematic treatment is missing despite its long history and a vast amount of literature available. One of the motivations of this paper was our wish to relate various approaches in literature and to provide a uniform combinatorial language for this purpose.

Here is a short **summary** of the paper.

In section 2 we describe our main combinatorial tool: the lattice path operad \mathcal{L} and its condensation in the differential graded setting. This description leads to a careful treatment of (higher) brace operations and their relationship with lattice paths in section 3.

The lattice path operad comes equipped with a filtration by complexity [2]. The second filtration stage $\mathcal{L}_{(2)}$ is the most important for understanding natural operations on the Hochschild cochains. In section 4 we give an alternative description of $\mathcal{L}_{(2)}$ in terms of trees, closely related to the operad of natural operations from [14]. Finally, in section 5 we study various suboperads generated by brace operations. The main result is that **all these operads have the homotopy type of a chain model of the little disks operad**. For sake of completeness we add a brief appendix containing an overview of some categorical constructions used in this paper.

Convention. If not stated otherwise, by an *operad* we mean a classical symmetric (i.e. with the symmetric groups acting on its components) operad in an appropriate symmetric monoidal category which will be obvious from the context. The same convention is applied to coloured operads, substitutes, multitensors and functor-operads recalled in the appendix.

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2. The lattice path operad

As usual, for a non-negative integer m , $[m]$ denotes the ordinal $0 < \dots < m$. We will use the same symbol also for the category with objects $0, \dots, m$ and the unique morphism $i \rightarrow j$ if and only if $i \leq j$. The *tensor product* $[m] \otimes [n]$ is the category freely generated by the (m, n) -grid which is, by definition, the oriented graph with vertices (i, j) , $0 \leq i \leq m$, $0 \leq j \leq n$, and one oriented edge $(i', j') \rightarrow (i'', j'')$ if and only if $(i'', j'') = (i' + 1, j')$ or $(i'', j'') = (i', j' + 1)$.

Let us recall, closely following [2], the *lattice path operad* and its basic properties. For non-negative integers k_1, \dots, k_n, l and $n \in \mathbb{N}$ put

$$\mathcal{L}(k_1, \dots, k_n; l) := \text{Cat}_{*,*}([l+1], [k_1+1] \otimes \dots \otimes [k_n+1])$$

where \otimes is the tensor product recalled above and $\text{Cat}_{*,*}([l+1], [k_1+1] \otimes \dots \otimes [k_n+1])$ the set of functors φ that preserve the extremal points, by which we mean that

$$(1) \quad \varphi(0) = (0, \dots, 0) \quad \text{and} \quad \varphi(l+1) = (k_1+1, \dots, k_n+1).$$

A functor $\varphi \in \mathcal{L}(k_1, \dots, k_n; l)$ is given by a chain of $l + 1$ morphisms $\varphi(0) \rightarrow \varphi(1) \rightarrow \dots \rightarrow \varphi(l + 1)$ in $[k_1 + 1] \otimes \dots \otimes [k_n + 1]$ with $\varphi(0)$ and $\varphi(l + 1)$ fulfilling (1). Each morphism $\varphi(i) \rightarrow \varphi(i + 1)$ is determined by a finite oriented edge-path in the $(k_1 + 1, \dots, k_n + 1)$ -grid. For $n = 0$, $\mathcal{L}(\cdot; l)$ consists of the unique functor from $[l + 1]$ to the terminal category with one object.

2.1. Marked lattice paths. — We will use a slight modification of the terminology of [2]. For non-negative integers $k_1, \dots, k_n \in \mathbb{N}$ denote by $\mathcal{Q}(k_1, \dots, k_n)$ the integral hypercube

$$\mathcal{Q}(k_1, \dots, k_n) := [k_1 + 1] \times \dots \times [k_n + 1] \subset \mathbb{Z}^{\times n}.$$

A *lattice path* is a sequence $p = (x_1, \dots, x_N)$ of $N := k_1 + \dots + k_n + n + 1$ points of $\mathcal{Q}(k_1, \dots, k_n)$ such that x_{a+1} is, for each $0 \leq a < N$, given by increasing exactly one coordinate of x_a by 1. A *marking* of p is a function $\mu : p \rightarrow \mathbb{N}$ that assigns to each point x_a of p a non-negative number $\mu_a := \mu(x_a)$ such that $\sum_{a=1}^N \mu_a = l$.

We can describe functors in $\mathcal{L}(k_1, \dots, k_n; l)$ as marked lattice paths (p, μ) in the hypercube $\mathcal{Q}(k_1, \dots, k_n)$. The marking $\mu_a = \mu(x_a)$ represents the number of elements of the interior $\{1, \dots, l\}$ of $[l + 1]$ that are mapped by φ to the a th lattice point x_a of p . We call lattice points marked by 0 *unmarked* points so the set of marked points equals $\varphi(\{1, \dots, l\})$. For example, the marked lattice path



represents a functor $\varphi \in \mathcal{L}(3, 2; 8)$ with $\varphi(0) = (0, 0)$, $\varphi(1) = \varphi(2) = \varphi(3) = (1, 0)$, $\varphi(4) = (2, 0)$, $\varphi(5) = \varphi(6) = (3, 1)$ and $\varphi(7) = \varphi(8) = \varphi(9) = (4, 3)$. The lattice is trivial for $n = 0$, so the unique element of $\mathcal{L}(\cdot; l)$ is represented by the point marked l , i.e. by \bullet^l .

2.2. Definition. — Let $p \in \mathcal{L}(k_1, \dots, k_n; l)$ be a lattice path. A point of p at which p changes its direction is an *angle* of p . An *internal point* of p is a point that is not an angle nor an extremal point of p . We denote by $Angl(p)$ (resp. $Int(p)$) the set of all angles (resp. internal points) of p .

For instance, the path in (2) has 4 angles, 2 internal points, 4 unmarked points and 1 unmarked internal point.

Following again [2] closely, we denote, for $1 \leq i < j \leq n$, by p_{ij} the projection of the path $p \in \mathcal{L}(k_1, \dots, k_n; l)$ to the face $[k_i + 1] \times [k_j + 1]$ of $\mathcal{Q}(k_1, \dots, k_n)$; let $c_{ij} := \#Angl(p_{ij})$ be the number of its angles. The maximum $c(p) := \max\{c_{ij}\}$ is called the *complexity* of p . Let us finally denote by $\mathcal{L}_{(c)}(k_1, \dots, k_n; l) \subset \mathcal{L}(k_1, \dots, k_n; l)$ the subset of marked lattice paths of complexity $\leq c$. The case $c = 2$ is particularly interesting, because $\mathcal{L}_{(2)}(k_1, \dots, k_n; l)$ is, by [2, Proposition 2.14], isomorphic to the

space of unlabeled $(l; k_1, \dots, k_n)$ -trees recalled on page 19. For convenience of the reader we recall this isomorphism on page 20.

As shown in [2], the sets $\mathcal{L}(k_1, \dots, k_n; l)$ and their subsets $\mathcal{L}_{(c)}(k_1, \dots, k_n; l)$, $c \geq 0$, form an \mathbb{N} -coloured operad \mathcal{L} and its sub-operads $\mathcal{L}_{(c)}$. To simplify formulations, we will allow $c = \infty$, putting $\mathcal{L}_{(\infty)} := \mathcal{L}$.

2.3. Convention. — Since we aim to work in the category of abelian groups, we will make no notational difference between the sets $\mathcal{L}_{(c)}(k_1, \dots, k_n; l)$ and their linear spans.

The underlying category of the coloured operad \mathcal{L} (which coincides with the underlying category of $\mathcal{L}_{(c)}$ for any $c \geq 0$) is, by definition, the category whose objects are non-negative integers and morphism $n \rightarrow m$ are elements of $\mathcal{L}(n, m)$, i.e. non-decreasing maps $\varphi : [m+1] \rightarrow [n+1]$ preserving the endpoints.

By Joyal's duality [11], this category is isomorphic to the (skeletal) category Δ of finite ordered sets, i.e. $\mathcal{L}(n, m) = \Delta(n, m)$. The operadic composition makes the collection $\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet)$ (with $c = \infty$ allowed) a functor $(\Delta^{\text{op}})^{\times n} \times \Delta \rightarrow \text{Abel}$, i.e. n -times simplicial 1-time cosimplicial Abelian group.

Morphisms in the category Δ are generated by the cofaces $d_i : [m-1] \rightarrow [m]$ given by the non-decreasing map that misses i , and the codegeneracies $s_i : [m+1] \rightarrow [m]$ given by the non-decreasing map that hits i twice. In both cases, $0 \leq i \leq m$. Let us inspect how these generating maps act on the pieces of the operad $\mathcal{L}_{(c)}$.

2.4. Simplicial structures. — We describe the induced r th ($1 \leq r \leq n$) simplicial maps

$$\partial_i^r : \mathcal{L}_{(c)}(k_1, \dots, k_{r-1}, m, k_{r+1}, \dots, k_n; l) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_{r-1}, m-1, k_{r+1}, \dots, k_n; l),$$

where $m \geq 1$, $0 \leq i \leq m$, and

$$\sigma_i^r : \mathcal{L}_{(c)}(k_1, \dots, k_{r-1}, m, k_{r+1}, \dots, k_n; l) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_{r-1}, m+1, k_{r+1}, \dots, k_n; l),$$

where $0 \leq i \leq m$. To this end, we define, for each $m \geq 1$ and $0 \leq i \leq m$, the epimorphism of the hypercubes

$$D_i^r : \mathcal{Q}(k_1, \dots, k_{r-1}, m, k_{r+1}, \dots, k_n) \twoheadrightarrow \mathcal{Q}(k_1, \dots, k_{r-1}, m-1, k_{r+1}, \dots, k_n)$$

by

$$D_i^r(a_1, \dots, a_r, \dots, a_n) := \begin{cases} (a_1, \dots, a_r, \dots, a_n), & \text{if } a_r \leq i, \text{ and} \\ (a_1, \dots, a_r - 1, \dots, a_n), & \text{if } a_r > i, \end{cases}$$

where $(a_1, \dots, a_r, \dots, a_n) \in \mathcal{Q}(k_1, \dots, k_{r-1}, m, k_{r+1}, \dots, k_n)$ is an arbitrary point. In a similar fashion, the monomorphism

$$S_i^r : \mathcal{Q}(k_1, \dots, k_{r-1}, m, k_{r+1}, \dots, k_n) \hookrightarrow \mathcal{Q}(k_1, \dots, k_{r-1}, m+1, k_{r+1}, \dots, k_n)$$

is, for $0 \leq i \leq m$, given by

$$S_i^r(a_1, \dots, a_r, \dots, a_n) := \begin{cases} (a_1, \dots, a_r, \dots, a_n), & \text{if } a_r \leq i, \text{ and} \\ (a_1, \dots, a_r + 1, \dots, a_n), & \text{if } a_r > i. \end{cases}$$

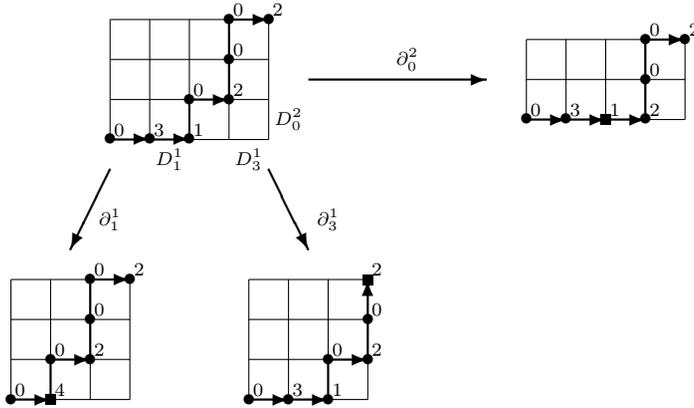


FIGURE 1. The simplicial boundaries acting on the element of (2).

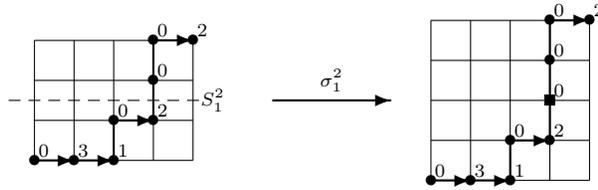


FIGURE 2. The operator σ_1^2 acting on the element of (2).

Let (p, μ) be a marked lattice path in $\mathcal{Q}(k_1, \dots, k_{r-1}, m, k_{r+1}, \dots, k_n)$ representing a functor $\varphi \in \mathcal{L}_{(c)}(k_1, \dots, k_{r-1}, m, k_{r+1}, \dots, k_n; l)$. Then $\partial_i^r(\varphi)$ is represented by the marked path $(\partial_i^r(p), \partial_i^r(\mu))$, where $\partial_i^r(p)$ is the image $D_i^r(p)$ of p in $\mathcal{Q}(k_1, \dots, k_{r-1}, m-1, k_{r+1}, \dots, k_n, l)$. The marking $\partial_i^r(\mu)$ is given by $\partial_i^r(\mu)(D_i^r(x)) := \sum_{\tilde{x} \in p} \mu(\tilde{x})$, with the sum taken over all $\tilde{x} \in p$ such that $D_i^r(\tilde{x}) = D_i^r(x)$. A less formal description of this marking is the following.

There are precisely two different points of p , say x' and x'' , such that $D_i^r(x') = D_i^r(x'')$; let us call the remaining points of p regular. The marking of $D_i^r(x)$ is the same as the marking of x if x is regular. If x' and x'' are the two non-regular points, then the marking of the common value $D_i^r(x') = D_i^r(x'')$ is $\mu(x') + \mu(x'')$. See Figure 1 in which the operator ∂_1^1 contracts the column denoted D_1^1 and decorates the point \blacksquare obtained by identifying the point $(1, 0)$ marked 3 with the point $(2, 0)$ marked 1 by $3 + 1 = 4$. The remaining operators act in the similar fashion.

To define the marked lattice path $(\sigma_i^r(p), \sigma_i^r(\mu))$ representing the degeneracy $\sigma_i^r(\varphi)$, we need to observe that the image $S_i^r(p)$ is not a lattice path in $\mathcal{Q}(k_1, \dots, k_{r-1}, m+1, k_{r+1}, \dots, k_n)$, but that it can be made one by adding a unique ‘missing’ lattice point \hat{x} . The resulting lattice path is $\sigma_i^r(p)$. The marking $\sigma_i^r(\mu)$ is given by $\sigma_i^r(\mu)(S_i^r(x)) := \mu(x)$ for $x \in p$ while $\sigma_i^r(\mu)(\hat{x}) := 0$, i.e. the newly added point \hat{x} is unmarked. See Figure 2 in which the new point \hat{x} is denoted \blacksquare . Observe that \hat{x} is always an internal point.

2.5. The cosimplicial structure. — We describe, for $l \geq 1$ and $0 \leq i \leq l$, the boundaries

$$\delta^i : \mathcal{L}_{(c)}(k_1, \dots, k_n; l-1) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_n; l)$$

and, for $0 \leq i \leq l$, the degeneracies

$$s^i : \mathcal{L}_{(c)}(k_1, \dots, k_n, l+1) \rightarrow \mathcal{L}_{(c)}(k_1, \dots, k_n; l),$$

of the induced cosimplicial structure. Let (p, μ) be a marked path in $\mathcal{Q}(k_1, \dots, k_n; l \mp 1)$ representing a functor $\varphi \in \mathcal{L}_{(c)}(k_1, \dots, k_n; l \mp 1)$. Neither δ^i nor s^i changes the underlying path, so $\delta^i(\varphi)$ is represented by $(p, \delta^i(\mu))$ and $s^i(\varphi)$ by $(p, s^i(\mu))$.

Let $\hat{x} := \varphi(i)$. Then the markings $\delta^i(\mu)$ and $s^i(\mu)$ are defined by $\delta^i(\mu)(x) = s^i(\mu)(x) = \mu(x)$ for $x \neq \hat{x}$, while $\delta^i(\mu)(\hat{x}) := \mu(\hat{x}) + 1$ and $s^i(\mu)(\hat{x}) := \mu(\hat{x}) - 1$.

3. Weak equivalences

3.1. Un-normalized totalizations. — Given an n -simplicial cosimplicial abelian group, i.e. a functor $X : \Delta^{\text{op} \times n} \times \Delta \rightarrow \text{Abel}$, denote by $X_*^\bullet = \overline{\text{Tot}}(X(\bullet_1, \dots, \bullet_n; \bullet))$ the simplicial totalization. It is a cosimplicial dg-abelian group with components

$$(3) \quad X_*^\bullet := \bigoplus_{* = -(k_1 + \dots + k_n)} X(k_1, \dots, k_n; \bullet)$$

bearing the degree +1 differential $\partial = \partial^1 + \dots + \partial^n$, where each ∂^r is induced from the boundaries of the r th simplicial structure in the standard manner. We also denote by $|X|^* = \overline{\text{Tot}}(\overline{\text{Tot}}(X(\bullet_1, \dots, \bullet_n; \bullet)))$ the cosimplicial totalization of the cosimplicial dg-abelian group X_*^\bullet . It is a dg-abelian group with components

$$|X|^* = \prod_{* = l - (k_1 + \dots + k_n)} X(k_1, \dots, k_n; l) = \prod_{l \geq 0} \bigoplus_{l - * = k_1 + \dots + k_n} X(k_1, \dots, k_n; l)$$

and the degree +1 differential $d = \delta + \partial$, where ∂ is as above and δ is the standard alternating sum of the cosimplicial boundary operators.

According to Appendix A, the dg-abelian groups $|\mathcal{L}_{(c)}|(n) := |\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet)|$ are the result of condensation and, therefore, assemble, for each $c \geq 0$, into a dg-operad $|\mathcal{L}_{(c)}| = \{|\mathcal{L}_{(c)}|(n)\}_{n \geq 0}$. Observe that $|\mathcal{L}_{(2)}|$ is isomorphic to the Tamarkin-Tsygan operad \mathcal{T} recalled on page 19.⁽¹⁾

Let us denote, for each $n, c \geq 0$, by $\mathcal{B}r_{(c)}(n)$ the simplicial totalization of the n -times simplicial abelian group $\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; 0)$, that is,

$$\mathcal{B}r_{(c)}^*(n) := \bigoplus_{* = -(k_1 + \dots + k_n)} \mathcal{L}_{(c)}(k_1, \dots, k_n; 0),$$

with the induced differential $\partial = \partial^1 + \dots + \partial^n$. Elements of $\mathcal{B}r_{(c)}(n)$ are represented by marked lattice paths $(p, 0)$ with the trivial marking $\mu = 0$ (all points of p are

⁽¹⁾ Whenever we refer to sections 4 or 5, we shall keep in mind that Convention 4.2 is used in these sections.

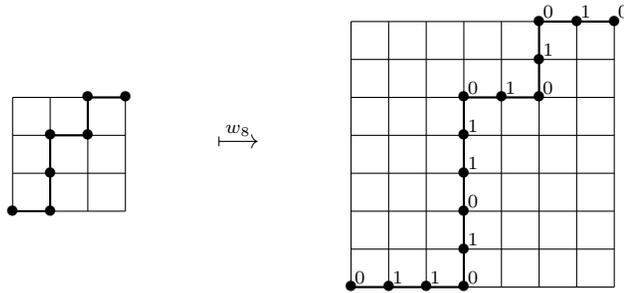


FIGURE 3. An element $p \in \mathcal{B}r_{(c)}(2)$ (left) and one of the terms in $w_s(p) \in |\mathcal{L}_{(c)}|(2)$ (right). The newly added internal points are marked by 1.

unmarked). Since the trivial marking bears no information, we will discard it from the notation. The whiskering $w : \mathcal{B}r_{(c)}(n) \rightarrow |\mathcal{L}_{(c)}|(n)$ is defined as

$$(4) \quad w(p) := \prod_{s \geq 0} w_s(p),$$

where $w_s(p) \in |\mathcal{L}_{(c)}|(n)$ is the sum of all marked paths, taken with appropriate signs, obtained from p by inserting precisely s new distinct internal lattice points marked 1. The origin of the signs is explained in Proposition 3.2 below. The action of the whiskering is illustrated in Figure 3.

For $p' \in \mathcal{L}_{(c)}(a_1, \dots, a_n; 0)$, $p'' \in \mathcal{L}_{(c)}(b_1, \dots, b_m; 0)$ and $1 \leq i \leq n$ define

$$(5) \quad p' \circ_i p'' := p' \circ_i w_{a_i}(p'') \in \bigoplus_{b'_1 + \dots + b'_m = b_1 + \dots + b_m + a_i} \mathcal{L}_{(c)}(a_1, \dots, a_{i-1}, b'_1, \dots, b'_m, a_{i+1}, \dots, a_n; 0)$$

where $w_{a_i}(p'')$ is the whiskering of the lattice path p'' by a_i points and \circ_i in the right hand side is the operadic composition in the coloured operad $\mathcal{L}_{(c)}$. By linearity, (5) extends to the operation $\circ_i : \mathcal{B}r_{(c)}(n) \otimes \mathcal{B}r_{(c)}(m) \rightarrow \mathcal{B}r_{(c)}(m + n - 1)$.

3.2. Proposition. — Operations \circ_i above make the collection $\mathcal{B}r_{(c)} = \{\mathcal{B}r_{(c)}(n)\}_{n \geq 1}$ a dg-operad. The signs in (4) can be chosen such that the map $w : \mathcal{B}r_{(c)} \hookrightarrow |\mathcal{L}_{(c)}|$ is an inclusion of dg-operads.

Proof. — The first part of the proposition can be verified directly. There is an inductive procedure to fix the signs in (4), but we decided not to include this clumsy and lengthy calculation here. A conceptual way to get the signs is to embed the dg-operad $|\mathcal{L}_{(c)}|$ into the coendomorphism operad of chains on the standard simplex of a sufficiently large dimension, cf. [2, Remark 2.20], and to require that the whiskering $w : \mathcal{B}r_{(c)} \rightarrow |\mathcal{L}_{(c)}|$ induces, via the isomorphism of Proposition 3.4 below, the action of the surjection operad, with the sign convention of [4, Section 2.2]. \square

Remark. One of the main advantages of the ‘operadic’ sign convention (see 4.1) which we use in sections 4 or 5 is that in the corresponding whiskering formula (17) all terms, quite miraculously, appear with the +1-signs.

So the operad structure of $\mathcal{B}r_{(c)}$ is induced by the operad structure of $|\mathcal{L}_{(c)}|$ and the whiskering map. Notice that $\mathcal{B}r_{(2)}$ is the brace operad $\mathcal{B}r$ recalled on page 21 and the map $w : \mathcal{B}r_{(2)} \rightarrow |\mathcal{L}_{(2)}|$ the whiskering defined in (17). Proposition 3.2 therefore generalizes Proposition 5.7.

3.3. Normalized totalizations. — Let $X(\bullet_1, \dots, \bullet_n; \bullet)$ be an n -simplicial cosimplicial abelian group as in 3.1. We will need also the traditional n -simplicial *normalized* totalization, or simplicial normalization for short, denoted $\overline{X}_* = \underline{\text{Nor}}(X(\bullet_1, \dots, \bullet_n; \bullet))$, obtained from the un-normalized totalization (3) by modding out the images of simplicial degeneracies. We then denote by $|\overline{X}|^* = \overline{\text{Nor}}(\underline{\text{Nor}}(X(\bullet_1, \dots, \bullet_n; \bullet)))$ the normalized cosimplicial totalization of the cosimplicial dg-abelian group \overline{X}_* . It is the intersection of the kernels of cosimplicial degeneracies in the un-normalized cosimplicial totalization of \overline{X}_* . As argued in [2], the n -simplicial cosimplicial normalization $|\overline{\mathcal{L}}_{(c)}|$ of the lattice path operad $\mathcal{L}_{(c)}$ is a dg-operad.

Let us denote, for each $n, c \geq 0$, by $\text{Nor}(\mathcal{B}r_{(c)})(n) = \underline{\text{Nor}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; 0))$ the simplicial normalization of the n -simplicial abelian group $\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; 0)$, with the induced differential. The explicit description of the simplicial structure in 2.4 makes it obvious that elements of $\text{Nor}(\mathcal{B}r_{(c)})(n)$ are represented by (unmarked) lattice paths with no internal points.

One defines the operadic composition on $\text{Nor}(\mathcal{B}r_{(c)}) = \{\text{Nor}(\mathcal{B}r_{(c)})(n)\}_{n \geq 0}$ and the whiskering $w : \text{Nor}(\mathcal{B}r_{(c)}) \hookrightarrow |\overline{\mathcal{L}}_{(c)}|$ by the same formulas as in the un-normalized case. The operad $\text{Nor}(\mathcal{B}r_{(2)})$ is the normalized brace operad $\text{Nor}(\mathcal{B}r)$ recalled on page 22. We leave as an exercise to verify that $\text{Nor}(\mathcal{B}r_{(1)})$ is the operad for unital associative algebras and $\text{Nor}(\mathcal{B}r_{(0)})$ the operad whose ‘algebras’ are abelian groups with a distinguished point.

3.4. Proposition. — *The operads $\text{Nor}(\mathcal{B}r_{(c)})$ are isomorphic to the suboperads $F_c\mathcal{X}$ of the surjection operad \mathcal{X} introduced in [4, 1.6.2], resp. the suboperads \mathcal{S}_c of the sequence operad \mathcal{S} introduced in [17, Definition 3.2].*

Proof. — We rely on the terminology of [4, 1.6.2]. A non-degenerate surjection $u : \{1, \dots, m\} \rightarrow \{1, \dots, n\}$, $m \geq n$, in $F_c\mathcal{X}(n)$ induces a lattice path φ_u representing an element of $\text{Nor}(\mathcal{B}r_{(c)})(n)$ as follows. For $1 \leq i \leq n$ denote by $d_i \in \mathbb{Z}^{\times n}$ the vector $(0, \dots, 1, \dots, 0)$ with 1 at the i th position, and $k_i := \#u^{-1}(i) - 1$. Then φ_u is the path in the grid $[k_1 + 1] \otimes \dots \otimes [k_n + 1]$ that starts at the ‘lower left corner’ $(0, \dots, 0)$, advances by $d_{u(1)}$, then by $d_{u(2)}$, etc., and finally by $d_{u(m)}$. It is obvious that the correspondence $u \mapsto \varphi_u$ is one-to-one. \square

The following statement follows from [2, Examples 3.10(c)] and [4, Section 1.2].

3.5. Proposition. — *The whiskering $w : \text{Nor}(\mathcal{B}r_{(c)}) \hookrightarrow |\overline{\mathcal{L}}_{(c)}|$ is an inclusion of dg-operads.*

We will need also the following statement.

3.6. Proposition. — *The natural projection $\pi : \mathcal{B}r_{(c)} \rightarrow \text{Nor}(\mathcal{B}r_{(c)})$ to the normalization is an epimorphism of dg-operads for each $c \geq 0$.*

Proof. — It is almost obvious that the operadic composition in $\mathcal{B}r_{(c)}$ preserves the number of internal points, that is, if p' (resp. p'') is a lattice path with a' (resp. a'') internal points, then $p' \circ_i p''$ is, for each i for which this expression makes sense, a linear combination of lattice paths with $a' + a''$ internal points. This implies that the degenerate subspace $Dgn(\mathcal{B}r_{(c)})$ of $\mathcal{B}r_{(c)}$ which is the subcollection spanned by lattice paths with at least one internal point, form a dg-operadic ideal in $\mathcal{B}r_{(c)}$, so the projection $\pi : \mathcal{B}r_{(c)} \rightarrow \mathcal{B}r_{(c)}/Dgn(\mathcal{B}r_{(c)}) = \text{Nor}(\mathcal{B}r_{(c)})$ is an operad map. The fact that π commutes with the differentials follows from the standard properties of the simplicial normalizations. \square

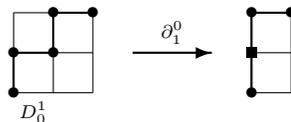
Let $\widehat{\mathcal{B}r}_{(c)} = \{\widehat{\mathcal{B}r}_{(c)}(n)\}_{n \geq 0}$ be the subcollection of $\mathcal{B}r_{(c)}$ such that $\widehat{\mathcal{B}r}_{(c)}(n) \subset \mathcal{B}r_{(c)}(n)$ is spanned by paths with no internal points, for $n \geq 1$, and $\widehat{\mathcal{B}r}_{(c)}(0) := 0$.

3.7. Proposition. — *The collection $\widehat{\mathcal{B}r}_{(c)}$ is a (non-dg) suboperad of $\mathcal{B}r_{(c)}$ for any $c \geq 0$. It is dg-closed if and only if $c \leq 2$.*

Proof. — It follows from the property stated in the proof of Proposition 3.6 that the subcollection $\widehat{\mathcal{B}r}_{(c)}$ is closed under the operad structure of $\mathcal{B}r_{(c)}$ for an arbitrary $c \geq 0$. It remains to prove that $\widehat{\mathcal{B}r}_{(c)}$ is closed under the action of the differential if and only if $c \leq 2$. Let us prove first that it is dg-closed for $c \leq 2$.

For $c = 2$ this follows from the fact that $\widehat{\mathcal{B}r}_{(2)} = \widehat{\mathcal{B}r}$ is a dg-suboperad of $\mathcal{B}r_{(2)} = \mathcal{B}r$, see Proposition 5.2 and the description of the dg-operad structures of $\mathcal{B}r$ and $\widehat{\mathcal{B}r}$ in terms of trees following that proposition, or [16]. For $c = 0, 1$, the proposition is obvious.

If $c \geq 3$, the differential may create internal points, as shown in the following picture where the piece ∂_1^0 of the differential creates the internal point \blacksquare :



So $\widehat{\mathcal{B}r}_{(c)}$ is not dg-closed if $c \geq 3$. \square

3.8. Semi-normalizations. — For each $n, c \geq 0$, one may also consider the collection $|\dot{\mathcal{L}}_{(c)}| := \{|\dot{\mathcal{L}}_{(c)}|(n)\}_{n \geq 0}$ defined by

$$|\dot{\mathcal{L}}_{(c)}|(n) := \overline{\text{Tot}}(\underline{\text{Nor}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet)))$$

i.e. as the n -simplicial normalization followed by the un-normalized cosimplicial totalization.

Observe that there is a natural projection $\pi : |\mathcal{L}_{(c)}| \rightarrow |\dot{\mathcal{L}}_{(c)}|$ of collections. We emphasize that, for $c \geq 3$, the collection $|\dot{\mathcal{L}}_{(c)}|$ has no natural dg-operad structure although it will still play an important auxiliary role in this section. We, however, have

$$\begin{array}{ccccccc}
& & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_0^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_0^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_0^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-1}^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-1}^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-1}^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-2}^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-2}^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-2}^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-3}^0 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-3}^1 & \longrightarrow & \mathcal{L}_{(c)}(n)_{-3}^2 & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}$$

FIGURE 4. The structure of the dg-operad $|\mathcal{L}_{(c)}|$.

3.9. Proposition. — For $c \leq 2$, the collection $|\dot{\mathcal{L}}_{(c)}|$ has a natural operad structure such that the projection $\pi : |\mathcal{L}_{(c)}| \rightarrow |\dot{\mathcal{L}}_{(c)}|$ is a map of dg-operads.

Proof. — The proof uses the fact that $|\dot{\mathcal{L}}_{(2)}|$ is the normalized Tamarkin-Tsygan operad $\text{Nor}(\mathcal{T})$ which is a quotient of $\mathcal{T} = |\mathcal{L}_{(2)}|$, see 4.9. This proves the proposition for $c = 2$. For $c = 0, 1$ the claim is obvious. \square

The main theorem of this section reads:

3.10. Theorem. — For each $c \geq 0$, there is the following chain of weak equivalences of dg-operads:

$$|\mathcal{L}_{(c)}| \xleftarrow{w} \mathcal{B}r_{(c)} \xrightarrow{\pi} \text{Nor}(\mathcal{B}r_{(c)}) \xrightarrow{w} |\overline{\mathcal{L}}_{(c)}|,$$

in which the maps w are the whiskerings of Propositions 3.2 and 3.5, and π is the normalization projection of Proposition 3.6.

Proof. — The map $\pi : \mathcal{B}r_{(c)}(n) \rightarrow \text{Nor}(\mathcal{B}r_{(c)})(n)$ is a homology isomorphism for each $n, c \geq 0$ because it is the normalization map of an n -simplicial abelian group, so π is a weak equivalence of dg operads.

Let us analyze the un-normalized whiskering $w : \mathcal{B}r_{(c)}(n) \hookrightarrow |\mathcal{L}_{(c)}|(n)$. The arity n piece of the dg-operad $|\mathcal{L}_{(c)}|$ can be organized into the bicomplex of Figure 4 in which the l th column $\mathcal{L}_{(c)}(n)_*^l$, $l \geq 0$, is the simplicial totalization $\underline{\text{Tot}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; l))$ and the horizontal differentials are induced from the cosimplicial structure. The dg-abelian group $|\mathcal{L}_{(c)}|(n)$ is then the corresponding $\text{Tot}\overline{\Pi}$ -total complex (see [19, Section 5.6] for the terminology).

The dg-abelian group $\mathcal{B}r_{(c)}(n)$ appears as the leftmost column of Figure 4, so one has the projection $\text{proj} : |\mathcal{L}_{(c)}|(n) \rightarrow \mathcal{B}r_{(c)}(n)$ of dg-abelian groups which is the

identity on the leftmost column and sends the remaining columns to 0. Since clearly $proj \circ w = \text{id}$, it is enough to prove that $proj$ is a homology isomorphism.

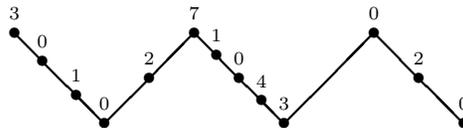
We interpret $proj : |\mathcal{L}_{(c)}|(n) \rightarrow \mathcal{B}r_{(c)}(n)$ as a map of bicomplexes, with $\mathcal{B}r_{(c)}(n)$ consisting of one column, and we prove that $proj$ induces an isomorphism of the E^2 -terms of the spectral sequences induced by the column filtrations. These filtrations are complete and exhaustive, thus the Eilenberg-Moore comparison theorem [19, Theorem 5.5.1] implies that $proj$ is a homology isomorphism.

Let (E_{**}^0, d^0) be the 0th term of column spectral sequence for $|\mathcal{L}_{(c)}|(n)$. This means that $(E_{l,*}^0, d^0) = (\text{Tot}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; l))_*, \partial)$, the l th column of the bicomplex in Figure 4 with the simplicial differential.

To calculate $E_{l,*}^1 := H_*(E_{l,*}^0, d^0)$, we recall the explicit description of the simplicial structures given in 2.4 and observe that the vertical differential $d^0 = \partial$ does not increase the number of angles of lattice paths. We therefore have, for each fixed $l \geq 0$, another spectral sequence $(\overline{E}_{**}^l, \overline{d}^l)$ induced by the filtration of $\mathcal{L}_{(c)}(n)_*$ by the number of angles. The piece \overline{E}_{uv}^0 of the initial sheet of this spectral sequence is spanned by marked paths $(p, \mu) \in \mathcal{L}_{(c)}(k_1, \dots, k_n; l)$ with $-u$ angles and $v = -u - (k_1 + \dots + k_n)$. With this degree convention, the total degree of an element of \overline{E}_{**}^l is the same as the degree of the corresponding element in $E_{l,*}^1$. By simple combinatorics, $(\overline{E}_{**}^l, \overline{d}^l)$ is a spectral sequence concentrated at the region $\{(u, v); u \leq 1 - n, u - v \geq 2 - 2n\}$ of the (u, v) -plane, thus no convergence problems occur. One easily sees that, as dg-abelian groups,

$$(6) \quad (\overline{E}_{u,*}^0, \overline{d}^0) \cong \left(\bigoplus_{\substack{p \in \text{Nor}(\mathcal{B}r_{(c)}(n)) \\ \# \text{ Angl}(p) = -u}} \bigoplus_{i_1 + \dots + i_{u+1} = n - 1 - * } \underbrace{\{\mathbb{B}_{i_1} \otimes_{\mathbb{Z}[x]} \dots \otimes_{\mathbb{Z}[x]} \mathbb{B}_{i_{u+1}}\}}_{-u + 1 \text{ factors}} t, d_{\mathbb{B}} \right),$$

where $\mathbb{B}_* = \mathbb{B}_*(\mathbb{Z}[x], \mathbb{Z}[x], \mathbb{Z}[x])$ is the un-normalized two-sided bar construction of the polynomial algebra $\mathbb{Z}[x]$ and the differential $d_{\mathbb{B}}$ is induced in the standard manner from the bar differential. The subscript l in (6) denotes the l -homogeneous part with respect to the grading induced by the number of instances of x . The factors of the direct sum are indexed by unmarked paths with no internal points representing a basis of $\text{Nor}(\mathcal{B}r_{(c)}(n))$. The isomorphism (6) is best explained by looking at the marked path



with 4 angles which is an element of $\overline{E}_{-4,-6}^0$ represented, via the isomorphism (6), by the element

$$x^3 \otimes [x^0|x^1] \otimes x^0 \otimes [x^2] \otimes x^7 \otimes [x^1|x^0|x^4] \otimes x^3 \otimes [] \otimes x^0 \otimes [x^2] \otimes x^0$$

in $\mathbb{B}_2 \otimes_{\mathbb{Z}[x]} \mathbb{B}_1 \otimes_{\mathbb{Z}[x]} \mathbb{B}_3 \otimes_{\mathbb{Z}[x]} \mathbb{B}_0 \otimes_{\mathbb{Z}[x]} \mathbb{B}_1$. It is a standard result of homological algebra that $(\mathbb{B}_* \otimes_{\mathbb{Z}[x]} \dots \otimes_{\mathbb{Z}[x]} \mathbb{B}_*, d_{\mathbb{B}})$ is acyclic in positive dimensions, thus the cohomology

of the right hand side of (6) is spanned by cycles of the form

$$(7) \quad x^l \otimes [] \otimes \cdots \otimes [] \in \overline{E}_{-\# \text{Angl}(p), n-1}^0.$$

At this point we need to observe that the differential ∂ decreases the number of angles of lattice paths p with no internal points representing elements of $\text{Nor}(\mathcal{B}r_{(c)})(n)$ by one. Indeed, it is easy to see that a simplicial boundary operator described in 2.4 may either decrease the number of angles of p by 1 or by 2. When it decreases it by 2 it creates an internal point, so the contributions of all simplicial boundaries that decrease the number of angles by 2 sum up to 0, by the standard property of the simplicial normalization. We conclude that $(\bigoplus_{*=u+v} \overline{E}_{uv}^1, \overline{d}^1) \cong (\text{Nor}(\mathcal{B}r_{(c)})_*(n), \partial)$ as dg-abelian groups and that $(\overline{E}_{**}^r, \overline{d}^r)$ collapses at this level.

Let us return to the column spectral sequence (E_{**}^r, d^r) for the bicomplex in Figure 4. It follows from the above calculation that the l th column E_{l*}^1 of the first term (E_{**}^1, d^1) equals $H_*(\text{Nor}(\mathcal{B}r_{(c)})(n))$ for each $l \geq 0$. It remains to describe the differential $d^1 : E_{l*}^1 \rightarrow E_{(l+1)*}^1$. To this end, one needs to observe that the expressions (7) representing elements of $E_{l*}^1 = H_*(\text{Nor}(\mathcal{B}r_{(c)})(n))$ correspond to marked lattice paths without internal points, whose only marked point is the initial one, marked by l . From the description of the cosimplicial structure given in 2.5 one easily obtains that

$$d^1 : E_{l*}^1 \rightarrow E_{(l+1)*}^1 = \begin{cases} 0, & \text{if } l \text{ is even and} \\ \text{id}, & \text{if } l \text{ is odd.} \end{cases}$$

We conclude that $E_{**}^2 := H_*(E_{**}^1, d^1)$ is concentrated at the leftmost column which equals $H_*(\text{Nor}(\mathcal{B}r_{(c)})(n))$ and that, from the obvious degree reasons, the column spectral sequence collapses at this stage. Since we already know that the projection $\mathcal{B}r_{(c)} \xrightarrow{\pi} \text{Nor}(\mathcal{B}r_{(c)})$ is a weak equivalence i.e., in particular, that $H_*(\mathcal{B}r_{(c)})(n) \cong H_*(\text{Nor}(\mathcal{B}r_{(c)})(n))$, the above facts imply that $\text{proj} : |\mathcal{L}_{(c)}|(n) \rightarrow \mathcal{B}r_{(c)}(n)$ induces an isomorphism of the E^2 -terms of the column spectral sequences, so it is a homology isomorphism and w is a homology isomorphism, too.

Let us finally prove that the normalized whiskering $w : \text{Nor}(\mathcal{B}r_{(c)})(n) \hookrightarrow |\overline{\mathcal{L}}_{(c)}|(n)$ is a weak equivalence. We have the composition

$$(8) \quad \text{Nor}(\mathcal{B}r_{(c)})(n) \xrightarrow{w} |\overline{\mathcal{L}}_{(c)}|(n) \xrightarrow{i} |\dot{\mathcal{L}}_{(c)}|(n)$$

in which the obvious inclusion i is a homology isomorphism by a simple lemma formulated below. As in the un-normalized case, the dg-abelian group $\text{Nor}(\mathcal{B}r_{(c)})(n)$ is the first column of the semi-normalized version of the bicomplex in Figure 4, so there is a natural projection $\text{proj} : |\dot{\mathcal{L}}_{(c)}|(n) \rightarrow \text{Nor}(\mathcal{B}r_{(c)})(n)$. This proj is a homology isomorphism by the same arguments as in the un-normalized case, only using in (6) the normalized bar construction instead. The proof is finished by observing that proj is the left inverse of the composition (8). \square

In the proof of Theorem 3.10 we used the following

3.11. Lemma. — *The inclusion $i : |\overline{\mathcal{L}}_{(c)}|(n) \hookrightarrow |\dot{\mathcal{L}}_{(c)}|(n)$ is a homology isomorphism for each $n, c \geq 0$.*

Proof. — The lemma follows from the fact that $|\overline{\mathcal{L}}_{(c)}|(n)$ is the cosimplicial normalization of the dg-cosimplicial group $|\dot{\mathcal{L}}_{(c)}|(n)$. □

In the following two sections we consider several operads. To simplify the navigation, we give a glossary of notation.

\mathcal{B} ,	big operad of all natural operations,	page 14
$\text{Nor}(\mathcal{B})$,	normalized big operad,	page 19
$\widehat{\mathcal{B}}$,	non-unital big operad,	page 18
\mathcal{T} ,	Tamarkin-Tsygan suboperad of \mathcal{B} ,	page 19
$\text{Nor}(\mathcal{T})$,	normalized Tamarkin-Tsygan operad,	page 20
$\widehat{\mathcal{T}}$,	non-unital Tamarkin-Tsygan operad,	page 20
$\mathcal{B}r$,	brace operad,	page 21
$\text{Nor}(\mathcal{B}r)$,	normalized brace operad,	page 22
$\widehat{\mathcal{B}r}$,	non-unital brace operad,	page 22

The operads mentioned in the list and their maps are organized in Figure 10 on page 28.

4. Operads of natural operations

In the previous sections we studied versions of the lattice path operad and its suboperads. We only briefly mentioned that some of these operads act on the Hochschild cochain complex of an associative algebra. The present and the following sections will be devoted to this action. It turns out that, in order to retain some nice features of the constructions in the previous section, namely the ‘whiskering’ formula (4) without signs, on one hand, and to have simple rules for the signs in formulas for natural operations on the other hand, one needs to use the ‘operadic’ degree convention, recalled in the next subsection.

4.1. Classical vs. operadic. — There are two conventions in defining the Hochschild cohomology of an associative algebra A . The *classical one* used for instance in [9] is based on the chain complex $C_{cl}^*(A; A) = \bigoplus_{n \geq 0} C_{cl}^n(A; A)$, where $C_{cl}^n(A; A) := \text{Lin}(A^{\otimes n}, A)$ (the subscript *cl* refers to “classical”). Another appropriate name would be the *(co)simplicial* convention, because $C_{cl}^*(A; A)$ is a natural cosimplicial abelian group. With this convention, the cup product \cup is a degree 0 operation and the Gerstenhaber bracket $[-, -]$ has degree -1 , see [9, Section 7] for the ‘classical’ definitions of these operations.

On the other hand, it is typical for this part of mathematics that signs are difficult to handle. A systematic way to control them is the *Koszul sign rule* requiring that whenever we interchange two “things” of odd degrees, we multiply the sign by -1 . This, however, needs the definition of the Hochschild cohomology as the operadic cohomology of associative algebras [13]. Now the underlying chain complex is

$$(9) \quad C^*(A; A) := \text{Lin}(\mathbb{T}(\downarrow A), \downarrow A)^*,$$

where \downarrow denotes the desuspension of a (graded) vector space and $\mathbb{T}(\downarrow A)$ the tensor algebra generated by A placed in degree -1 . Explicitly, $C^*(A; A) = \bigoplus_{n \geq -1} C^n(A; A)$, where $C^n(A; A) := \text{Lin}(A^{\otimes n+1}, A)$, so $C^n(A; A) = C_{cl}^{n+1}(A; A)$ for $n \geq -1$. With this convention, the cup product has degree $+1$ and the Gerstenhaber bracket degree 0 .

Depending on the choice of the convention, there are two definitions of the ‘big’ operad of natural operations, see 4.3 below. The *classical* one introduces \mathcal{B}_{cl} as a certain suboperad of the endomorphism operad $\text{End}_{C_{cl}^*(A;A)}$ of the graded vector space $C_{cl}^*(A; A)$, and the *operadic* one introduces \mathcal{B} as a suboperad of the endomorphism operad $\text{End}_{C^*(A;A)}$. Here A is a *generic*, in the sense of Definition 4.6, unital associative algebra. The difference between \mathcal{B}_{cl} and \mathcal{B} is merely conventional; the operad \mathcal{B}_{cl} is the operadic suspension $\mathfrak{s}\mathcal{B}$ of the operad \mathcal{B} [15, Definition II.3.15] while, of course, $\text{End}_{C_{cl}^*(A;A)} \cong \mathfrak{s}\text{End}_{C^*(A;A)}$.

4.2. Convention. — In sections 4 and 5 we accept the *operadic* convention because we want to rely on the Koszul sign rule. As explained above, the operads \mathcal{B} and \mathcal{B}_{cl} differ from each other only by the regrading and sign factors.

4.3. The operad of natural operations. — Recall the dg-operad $\mathcal{B} = \{\mathcal{B}(n)\}_{n \geq 0}$ of all *natural multilinear operations* on the (operadic) Hochschild cochain complex (9) of a generic associative algebra A (see Definition 4.6) with coefficients in itself introduced in [14] (but notice that we are using here the operadic degree convention, see 4.2, while [14] uses the classical one).

Let A be a unital associative algebra. A *natural operation* in the sense of [14] is a linear combination of compositions of the following ‘elementary’ operations:

(a) The insertion $\circ_i : C^k(A; A) \otimes C^l(A; A) \rightarrow C^{k+l}(A; A)$ given, for $k, l \geq -1$ and $0 \leq i \leq k$, by the formula

$$\circ_i(f, g)(a_0, \dots, a_{k+l}) := (-1)^{il} f(a_0, \dots, a_{i-1}, g(a_i, \dots, a_{i+l}), a_{i+l+1}, \dots, a_{k+l}),$$

for $a_1, \dots, a_{k+l-1} \in A$ – the sign is determined by the Koszul rule!

(b) Let $\mu : A \otimes A \rightarrow A$ be the associative product, $\text{id} : A \rightarrow A$ the identity map and $1 \in A$ the unit. Then elementary operations are also the ‘constants’ $\mu \in C^1(A; A)$, $\text{id} \in C^0(A; A)$ and $1 \in C^{-1}(A; A)$.

(c) The assignment $f \mapsto \text{sgn}(\sigma) \cdot f\sigma$ permuting the inputs of a cochain $f \in C^k(A; A)$ according to a permutation $\sigma \in \Sigma_{k+1}$ and multiplying by the signature of σ is an elementary operation.

Let $B(A)_{k_1, \dots, k_n}^l$ denote, for $l, k_1, \dots, k_n \geq 0$, the abelian group of all natural, in the above sense, operations

$$(10) \quad O : C^{k_1-1}(A; A) \otimes \dots \otimes C^{k_n-1}(A; A) \rightarrow C^{l-1}(A; A).$$

The regrading in the above equation guarantees that the super- and subscripts of $B(A)_{k_1, \dots, k_n}^l$ will all be non-negative integers. Moreover, with this definition the spaces $B(A)_{k_1, \dots, k_n}^l$ agree with the ones introduced in [3]. The system $B(A)_{k_1, \dots, k_n}^l$ clearly forms an \mathbb{N} -coloured suboperad $B(A)$ of the endomorphism operad of the \mathbb{N} -coloured collection $\{C^{n-1}(A; A)\}_{n \geq 0}$.

Recall that the Hochschild differential $d_H : C^{n-1}(A; A) \rightarrow C^n(A; A)$ is, for $n \geq 0$, given by the formula

$$\begin{aligned} d_H f(a_0 \otimes \dots \otimes a_n) &:= (-1)^{n+1} a_0 f(a_1 \otimes \dots \otimes a_n) + f(a_0 \otimes \dots \otimes a_{n-1}) a_n \\ &\quad + \sum_{i=0}^{n-1} (-1)^{i+n} f(a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n), \end{aligned}$$

for $a_i \in A$. Apparently, d_H is a natural operation belonging to $B(A)_n^{n+1}$. Therefore, if $O \in B(A)_{k_1, \dots, k_n}^l$ is as in (10), one may define $\delta O \in B(A)_{k_1, \dots, k_n}^{l+1}$ and, for $1 \leq i \leq k$, also $\partial_i O \in B(A)_{k_1, \dots, k_{i-1}, k_i-1, k_{i+1}, \dots, k_n}^l$ by

$$(11) \quad \begin{aligned} \delta O(f_1, \dots, f_n) &:= d_H O(f_1, \dots, f_n) \text{ and} \\ \partial_i O(f_1, \dots, f_n) &:= (-1)^{k_i + \dots + k_n + l + n + i} \cdot O(f_1, \dots, f_{i-1}, d_H f_i, f_{i+1}, \dots, f_n). \end{aligned}$$

The sign in the second line of the above display equals $(-1)^{\deg(f_1) + \dots + \deg(f_{i-1})} \cdot (-1)^{\deg(O)}$ as dictated by the Koszul rule.

It follows from definition that elements of $B(A)_{k_1, \dots, k_n}^l$ can be represented by linear combinations of $(l; k_1, \dots, k_n)$ -trees in the sense of the following definition in which, as usual, the *arity* of a vertex of a rooted tree is the number of its input edges and the *legs* are the input edges of a tree, see [15, II.1.5] for the terminology.

4.4. Definition. — Let l, k_1, \dots, k_n be non-negative integers. An $(l; k_1, \dots, k_n)$ -tree is a planar rooted tree with legs labeled by $1, \dots, l$ and three types of vertices:

- (a) ‘white’ vertices of arities k_1, \dots, k_n labeled by $1, \dots, n$,
- (b) ‘black’ vertices of arities ≥ 2 and
- (c) ‘special’ black vertices of arity 0 (no input edges).

We moreover require that there are no edges connecting two black vertices or a black vertex with a special vertex. For $n = 0$ we allow also the exceptional trees \mid and \blacklozenge with no vertices.

We call an internal edge whose initial vertex is special a *stub* (also called, in [12], a *tail*). It follows from definition that the terminal vertex of a stub is white; the exceptional tree \blacklozenge is not a stub. An example of an $(l; k_1, \dots, k_n)$ -tree is given in Figure 5.

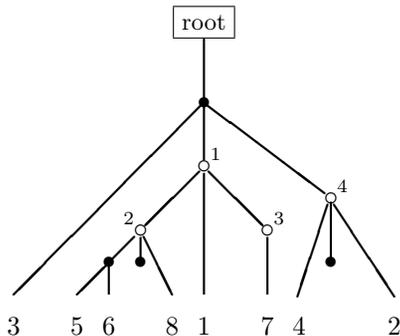


FIGURE 5. An $(8; 3, 3, 1, 3)$ -tree representing an operation in $\mathcal{B}_{3,3,1,3}^8$. It has 4 white vertices, 2 black vertices and 2 stubs. We use the convention that directed edges point upwards so the root is always on the top.

Each $(l; k_1, \dots, k_n)$ -tree T as in Definition 4.4 has its *signature* $\sigma(T) = \pm 1$ defined as follows. Since T is planar, its white vertices are naturally linearly ordered by walking around the tree counterclockwise, starting at the root. The first white vertex which one meets is the first one in this linear order, the next white vertex different from the first one is the second in this linear order, etc. For instance, the labels of the tree in Figure 5 agree with the ones given by the natural order, which of course need not always be the case.

One is therefore given a function $w \mapsto p(w)$ that assigns to each white vertex w of the tree T its position $p(w) \in \{1, \dots, n\}$ in the linear order described above. This defines a permutation $\sigma \in \Sigma_n$ by $\sigma(i) := p(w_i)$, where w_i is the white vertex labelled by i , $1 \leq i \leq n$. Let, finally, $\sigma(T)$ be the Koszul sign of σ permuting n variables v_1, \dots, v_n of degrees $k_1 - 1, \dots, k_n - 1$, respectively. In other words, $\sigma(T)$ is determined by

$$(12) \quad \sigma(T) \cdot v_1 \wedge \dots \wedge v_n = v_{\sigma(1)} \wedge \dots \wedge v_{\sigma(n)},$$

satisfied in the free graded commutative associative algebra generated by v_1, \dots, v_n .

An $(l; k_1, \dots, k_n)$ -tree T determines the natural operation $O_T \in B(A)_{k_1, \dots, k_n}^l$ given by decorating, for each $1 \leq i \leq n$, the i th white vertex by $f_i \in C^{k_i-1}(A; A)$, the black vertices by the iterated multiplication, the special vertices by the unit $1 \in A$, and performing the composition along the tree. The result is then multiplied by the signature $\sigma(T)$ defined above.

When evaluating on concrete elements, we apply the Koszul sign rule and use the ‘desuspended’ degrees, that is $f : A^{\otimes n} \rightarrow A$ is assigned degree $n - 1$ and $a \in A$ degree -1 , see 4.2. For instance, the tree in Figure 5 represents the operation

$$O(f_1, f_2, f_3, f_4)(a_1, \dots, a_8) := -a_3 f_1(f_2(a_5 a_6, 1, a_8), a_1, f_3(a_7)) f_4(a_4, 1, a_2),$$

$a_1, \dots, a_8 \in A$, where, as usual, we omit the symbol for the iteration of the associative multiplication μ . The minus sign in the right hand side follows from the Koszul rule explained above. The exceptional $(1;)$ -tree \mid represents the identity $\text{id} \in C^0(A; A)$.

Notation. — For each $l, k_1, \dots, k_n \geq 0$ denote by B_{k_1, \dots, k_n}^l the free abelian group spanned by all (l, k_1, \dots, k_n) -trees. The correspondence $T \mapsto O_T$ defines, for each associative algebra A , a linear epimorphism $\omega_A : B_{k_1, \dots, k_n}^l \rightarrow B(A)_{k_1, \dots, k_n}^l$.

Let T' be an $(l'; k'_1, \dots, k'_n)$ -tree, T'' an $(l''; k''_1, \dots, k''_m)$ -tree and assume that $l'' = k'_i$ for some $1 \leq i \leq n$. The i th vertex insertion assigns to T' and T'' the tree $T' \circ_i T''$ obtained by replacing the white vertex of T' labelled i by T'' . It may happen that this replacement creates edges connecting black vertices. In that case it is followed by collapsing these edges. The above construction extends into a linear operation

$$\circ_i : B_{k'_1, \dots, k'_n}^{l'} \otimes B_{k''_1, \dots, k''_m}^{l''} \rightarrow B_{k'_1, \dots, k'_{i-1}, k'_i, \dots, k''_m, k'_{i+1}, \dots, k'_n}^{l'}, \quad 1 \leq i \leq n, \quad l'' = k'_i.$$

Recall the following:

4.5. Proposition ([3]). — The spaces B_{k_1, \dots, k_n}^l assemble into an \mathbb{N} -coloured operad B with the operadic composition given by the vertex insertion and the symmetric group relabelling the white vertices. With this structure, the maps $\omega_A : B_{k_1, \dots, k_n}^l \rightarrow B(A)_{k_1, \dots, k_n}^l$ form an epimorphism $\omega_A : B \rightarrow B(A)$ of \mathbb{N} -coloured operads.

In [3] we formulated the following important:

4.6. Definition. — A unital associative algebra A is *generic* if the map $\omega_A : B \rightarrow B(A)$ is an isomorphism.

In [3] we also proved that generic algebras exist; the free associative unital algebra $U := \mathbb{T}(x_1, x_2, x_3, \dots)$ generated by countably many generators x_1, x_2, x_3, \dots is an example. We may therefore *define* the operad B alternatively as the operad of natural operations on the Hochschild cochain complex of a *generic algebra*.

The differentials (11) clearly translate, for a generic A , to the tree language of the operad B as follows. The component $\partial_i, 1 \leq i \leq n$, of the differential $\partial = \partial_1 + \dots + \partial_n$ replaces the white vertex of an $(l; k_1, \dots, k_n)$ -tree T labelled i with $k_i \geq 1$ inputs by the linear combination

$$(13) \quad \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} + \dots + (-1)^{k_i+1} \sum_{1 \leq s \leq k_i-1} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array}$$

in which the white vertex has $k_i - 1$ inputs and retains the label i . The result is then multiplied by the overall sign in the second line of (11). In the summation of (13), the black binary vertex is inserted into the s th input of the white vertex. If the i th white vertex of T has no inputs then $\partial_i(T) = 0$.

The differential δ replaces an $(l; k_1, \dots, k_n)$ -tree symbolized by the triangle  with l inputs by the linear combination

$$\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} + \dots + (-1)^l \sum_{1 \leq s \leq l} \begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array}$$

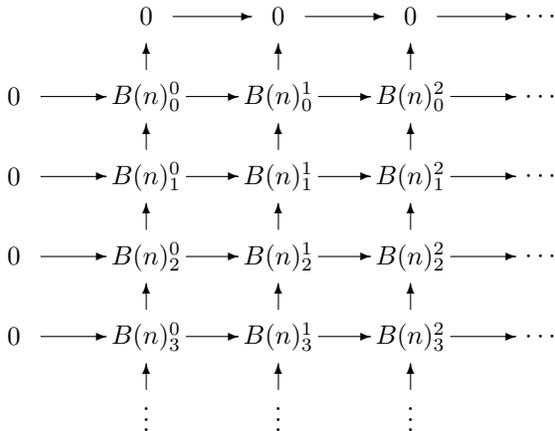


FIGURE 6. The structure of the operad \mathcal{B} . In the above diagram, $B(n)_k^m := \prod_{k_1+\dots+k_n=k} B_{k_1,\dots,k_n}^m$. The vertical arrows are the simplicial differentials ∂ and the horizontal arrows are the cosimplicial differentials δ .

If a replacement above creates an edge connecting black vertices, it is followed by collapsing these edges.

We finally define the arity n piece of the operad of natural operations as

$$\mathcal{B}^*(n) := \prod_{l-(k_1+\dots+k_n)+n-1=*} B_{k_1,\dots,k_n}^l,$$

with the degree $+1$ differential $d : \mathcal{B}^* \rightarrow \mathcal{B}^{*+1}$ defined by $d := (\partial_1 + \dots + \partial_n) - \delta$. It is evident that the collection $\mathcal{B} = \{\mathcal{B}^*(n)\}_{n \geq 0}$, with the operadic composition inherited from the inclusion $\mathcal{B} \subset \text{End}_{C^*(A;A)}$ for A generic, is a dg-operad.

The structure of the operad \mathcal{B} is visualized in Figure 6. We emphasize that the degree m -piece of $\mathcal{B}(n)$ is the direct *product*, not the direct sum, of elements on the diagonal $p + q = m - n + 1$ in the (p, q) -plane. It follows from our definitions that the Hochschild complex $C^*(A; A)$ of an arbitrary unital associative A is a natural \mathcal{B} -algebra.

4.7. Convention. — From now on, we will assume that A is a *generic algebra* in the sense of Definition 4.6 and make no distinction between natural operations on the Hochschild complex of A and the corresponding linear combinations of trees.

4.8. Variant. — An important suboperad of \mathcal{B} is the suboperad $\widehat{\mathcal{B}}$ generated by trees *without* stubs and without \blacklozenge . The operad $\widehat{\mathcal{B}}$ is the operad of all natural multilinear operations on the Hochschild complex of a *non-unital* generic associative algebra. It is generated by natural operations (a)–(c) above but without the unit $1 \in C^{-1}(A; A)$ in (b). Let us denote by $\widehat{B}_{k_1,\dots,k_n}^l$ the space of all operations (10) of this restricted type. An important feature of the operad $\widehat{\mathcal{B}}$ is that it is, in a certain sense, bounded. Indeed, one may easily prove that $\widehat{B}_{k_1,\dots,k_n}^l = 0$ if $k_1 + \dots + k_n - l \geq n$, see Figure 7.

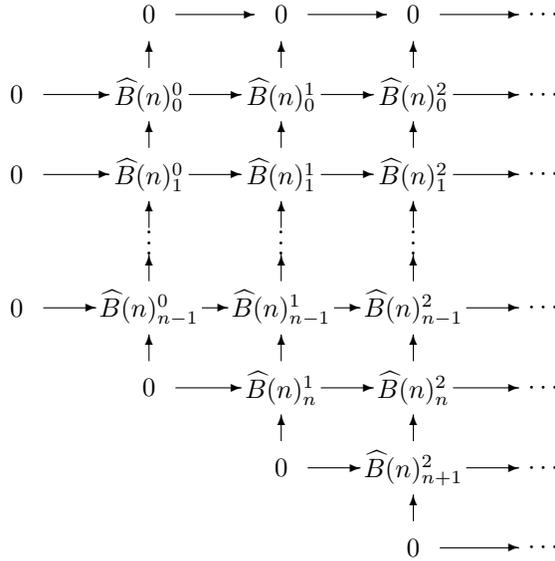


FIGURE 7. The structure of the non-unital operad $\widehat{\mathcal{B}}$. In the diagram, $\widehat{B}(n)_k^m := \prod_{k_1+\dots+k_n=k} \widehat{B}_{k_1,\dots,k_n}^m$.

One also has the quotient $\text{Nor}(\mathcal{B})$ of the collection \mathcal{B} modulo the trees with stubs. As explained in [3], $\text{Nor}(\mathcal{B})$ forms an operad which is in fact the componentwise simplicial normalization of \mathcal{B} . The operad $\text{Nor}(\mathcal{B})$ acts on the *normalized* Hochschild complex of a *unital* algebra. One has the diagram of operad maps

$$(14) \quad \widehat{\mathcal{B}} \xrightarrow{\iota} \mathcal{B} \xrightarrow{\pi} \text{Nor}(\mathcal{B}),$$

in which the projection π is a weak equivalence and the components $\pi\iota(n)$ of the composition $\pi\iota$ are isomorphisms for each $n \geq 1$. If \mathfrak{U} denotes the functor that replaces the arity zero component of a dg-operad by the trivial abelian group, then $\mathfrak{U}(\pi\iota)$ is a dg-operad isomorphism $\mathfrak{U}(\widehat{\mathcal{B}}) \cong \mathfrak{U}(\text{Nor}(\mathcal{B}))$.

4.9. Tamarkin-Tsygan operad. — There is also a suboperad \mathcal{T} of \mathcal{B} generated by elementary operations of types (a) and (b) only, without the use of permutations in (c). Its arity- n piece equals

$$\mathcal{T}^*(n) := \prod_{l-(k_1+\dots+k_n)+n-1=*} T_{k_1,\dots,k_n}^l,$$

where operations in T_{k_1,\dots,k_n}^l are represented by linear combinations of *unlabeled* $(l; k_1, \dots, k_n)$ -trees, that is, planar trees as in Definition 4.4 but without the labels of the legs. The inclusion $T_{k_1,\dots,k_n}^l \hookrightarrow B_{k_1,\dots,k_n}^l$ is realized by labeling the legs of an unlabeled tree from the left to the right in the orientation given by the planar embedding. The groups T_{k_1,\dots,k_n}^l form a coloured operad T and the inclusion above is the inclusion of operads $T \hookrightarrow B$.

The operad \mathcal{T} is the condensation of T and it is a chain version of the operad considered in [18, Section 3]. There is also the operad $\widehat{\mathcal{T}} := \widehat{\mathcal{B}} \cap \mathcal{T}$ generated by unlabeled trees without stubs and without \downarrow . It is clear that $\widehat{\mathcal{T}}$ is bounded in the same way as $\widehat{\mathcal{B}}$. We finally have the normalized Tamarkin-Tsygan operad $\text{Nor}(\mathcal{T})$ defined as the image of \mathcal{T} under the canonical projection $\pi : \mathcal{B} \rightarrow \text{Nor}(\mathcal{B})$. One has the diagram $\widehat{\mathcal{T}} \xrightarrow{\iota} \mathcal{T} \xrightarrow{\pi} \text{Nor}(\mathcal{T})$ with the properties analogous to that of (14).

Summing up, we have the following \mathbb{N} -coloured operads:

- the operad B whose piece B_{k_1, \dots, k_n}^l equals the span of the set of all $(l; k_1, \dots, k_n)$ -trees,
- the operad \widehat{B} whose piece $\widehat{B}_{k_1, \dots, k_n}^l$ is the span of the set of all $(l; k_1, \dots, k_n)$ -trees without stubs and without \downarrow if $n = l = 0$,
- the operad T whose piece T_{k_1, \dots, k_n}^l equals the span of the set of all unlabeled $(l; k_1, \dots, k_n)$ -trees, and
- the operad $\widehat{T} = T \cap \widehat{B}$ whose piece $\widehat{T}_{k_1, \dots, k_n}^l$ is the span of the set of all unlabeled $(l; k_1, \dots, k_n)$ -trees without stubs and without \downarrow if $n = l = 0$.

We close this section by recalling the isomorphism between the set of unlabeled $(l; k_1, \dots, k_n)$ -trees and $\mathcal{L}_{(2)}(k_1, \dots, k_n; l)$ constructed in the proof of [2, Proposition 2.14]. Let T be an unlabeled $(l; k_1, \dots, k_n)$ -tree. We run around T counterclockwise via the unique edge-path that begins and ends at the root and goes through each edge of T exactly twice (in opposite directions). The lattice path $\varphi_T : [l + 1] \rightarrow [k_1 + 1] \otimes \dots \otimes [k_n + 1]$ corresponding to T starts at the ‘lower left’ corner with coordinates $(0, \dots, 0)$ and advances according the following rules:

- when the edge-path hits the white vertex labeled i , $1 \leq i \leq n$, we advance φ_T in the direction of the vector $d_i := (0, \dots, 1, \dots, 0)$ (1 at the i th place),
- when the edge-path hits the leg, we do not move but increase the marking of our position by one.

The correspondence $T \mapsto \varphi_T$ is illustrated in Figure 8.

4.10. Proposition ([2], Proposition 2.14). — *The above correspondence induces an isomorphism of coloured operads T and $\mathcal{L}_{(2)}$, and hence, the isomorphism between $\mathfrak{s}\mathcal{T}$ and $|\mathcal{L}_{(2)}|$.⁽²⁾*

More conceptually, the difference between the \mathbb{N} -coloured operads T and B and the corresponding operads \mathcal{T} and \mathcal{B} can be explained as follows. Let \mathcal{O} and \mathcal{O}_1 be the categories of operads and of nonsymmetric operads in the category of chain complexes \mathcal{Chain} correspondingly. There is the forgetful functor $\text{Des}_1 : \mathcal{O} \rightarrow \mathcal{O}_1$ which forget the symmetric group actions. Let \mathcal{M} be the *nonsymmetric* operad for unital monoids.

4.11. Definition. — The category of *multiplicative nonsymmetric operads* is the comma-category $\mathcal{M}/\mathcal{O}_1$, see [10]. The category of *multiplicative operads* is the comma-category \mathcal{M}/Des_1 .

⁽²⁾ The operadic suspension \mathfrak{s} applied to \mathcal{T} is a consequence of Convention 4.2.

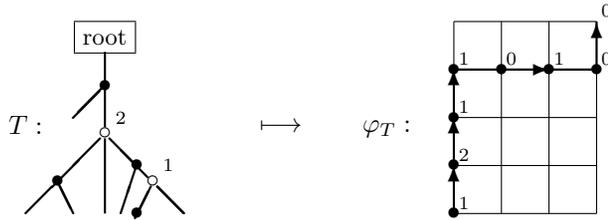


FIGURE 8. An unlabeled $(6; 2, 2)$ -tree T and the corresponding lattice path $\varphi_T \in \mathcal{L}_{(c)}(2, 2; 6)$.

So, a multiplicative operad is an operad A equipped with a structure morphism $p : \mathcal{M} \rightarrow Des_1(A)$. Equivalently, by adjunction, a structure morphism can be replaced by a morphism $UAss \rightarrow A$, where $UAss$ is the operad for unital associative algebras.

The description in [5, 1.5.6] of the coloured operad whose algebras are symmetric operads, readily implies the following proposition which illuminates the main result of [3].

4.12. Proposition. — *The category of algebras over the coloured operad T is isomorphic to the category of multiplicative nonsymmetric operads. The category of algebras of the coloured operad B is isomorphic to the category of multiplicative operads. Under this identification, the inclusion $T \hookrightarrow B$ induces the forgetful functor from multiplicative operads to nonsymmetric multiplicative operads.*

5. Operads of braces

Throughout this section we use Convention 4.7. There is another very important suboperad $\mathcal{B}r$ of \mathcal{B} generated by braces, cup-products and the unit whose normalized version was introduced in [16, Section 1] under the notation \mathcal{H} . Let us recall its definition. The operad $\mathcal{B}r$ is the suboperad of the operad \mathcal{B} generated by the following operations.

- (a) The *cup product* $-\cup - : C^*(A; A) \otimes C^*(A; A) \rightarrow C^*(A; A)$ defined by $f \cup g := \mu(f, g)$.
- (b) The constant $1 \in C^{-1}(A; A)$.
- (c) The *braces* $-\{-, \dots, -\} : C^*(A; A)^{\otimes n} \rightarrow C^*(A; A)$, $n \geq 2$, given by

$$(15) \quad f\{g_2, \dots, g_n\} := \sum f(\text{id}, \dots, \text{id}, g_2, \text{id}, \dots, \text{id}, g_n, \text{id}, \dots, \text{id}),$$

where id is the identity map of A and the summation runs over all possible substitutions of g_2, \dots, g_n (in that order) into f .

Notice that, for $f \in C^k(A; A)$ and $g \in C^l(A; A)$, the cup product $f \cup g \in C^{k+l+1}(A; A)$ evaluated at $a_0, \dots, a_{k+l+1} \in A$ equals

$$(16) \quad (f \cup g)(a_0, \dots, a_{k+l+1}) = (-1)^{(k+1)l} f(a_0, \dots, a_k)g(a_{k+1}, \dots, a_{k+l+1}),$$

with the sign dictated by the Koszul rule. This formula differs from the original one [9, Section 7] due to a different degree convention used here, see 4.2. We leave as an exercise to write a similar explicit formula for the brace.

The brace operad has also its non-unital version $\widehat{\mathcal{B}r} := \widehat{\mathcal{B}} \cap \mathcal{B}r$ generated by elementary operations (a) and (c). One can verify that both $\mathcal{B}r$ and $\widehat{\mathcal{B}r}$ are indeed dg-suboperads of \mathcal{B} , see [16]. We also denote by $\text{Nor}(\mathcal{B}r) \subset \text{Nor}(\mathcal{B})$ the image of $\mathcal{B}r$ under the projection $\mathcal{B} \rightarrow \text{Nor}(\mathcal{B})$. One has again an analog $\widehat{\mathcal{B}r} \xrightarrow{\iota} \mathcal{B}r \xrightarrow{\pi} \text{Nor}(\mathcal{B}r)$ of (14).

Let us describe the operad $\mathcal{B}r$, its suboperad $\widehat{\mathcal{B}r}$ and its quotient $\text{Nor}(\mathcal{B})$ in terms of trees.

5.1. Definition. — Let k_1, \dots, k_n be integers. An *amputated* (k_1, \dots, k_n) -tree is an $(0; k_1, \dots, k_n)$ -tree in the sense of Definition 4.4. We denote by A_{k_1, \dots, k_n} the (finite) set of all amputated (k_1, \dots, k_n) -trees, by $\text{Nor}(A)_{k_1, \dots, k_n}$ its subset consisting of amputated (k_1, \dots, k_n) -trees without stubs and $\widehat{A}_{k_1, \dots, k_n}$ the set that equals $\text{Nor}(A)_{k_1, \dots, k_n}$ for $n \geq 1$ and is \emptyset for $n = 0$.

5.2. Proposition. — For each $n \geq 0$ and $d \leq n - 1$, there is a natural isomorphism

$$w : \text{Span}(\{A_{k_1, \dots, k_n}; n - 1 - (k_1 + \dots + k_n) = d\}) \cong \mathcal{B}r^d(n)$$

which restricts to the isomorphism (denoted by the same symbol)

$$w : \text{Span}(\{\widehat{A}_{k_1, \dots, k_n}; n - 1 - (k_1 + \dots + k_n) = d\}) \cong \widehat{\mathcal{B}r}^d(n).$$

and projects into the isomorphism (denoted again by the same symbol)

$$w : \text{Span}(\{\text{Nor}(A)_{k_1, \dots, k_n}; n - 1 - (k_1 + \dots + k_n) = d\}) \cong \text{Nor}(\mathcal{B}r)^d(n).$$

The map w is defined in formula (17) below. From the reasons apparent later we call it the *whiskering*. The proof of the proposition is postponed to page 27. Before we give the definition of w , we illustrate the notion of amputated trees in the following:

5.3. Example. — The space $\mathcal{B}r^*(0)$ is concentrated in degree -1 ,

$$\mathcal{B}r^*(0) = \mathcal{B}r^{-1}(0) = \text{Span}(\downarrow),$$

while $\widehat{\mathcal{B}r}^*(0) = 0 = \text{Span}(\emptyset)$. The space $\mathcal{B}r^{-d}(1)$ is, for $d \geq 0$, the span of the single element



while $\widehat{\mathcal{B}r}^*(1) = \widehat{\mathcal{B}r}^0(1) = \text{Span}(\downarrow)$. Similarly

$$\mathcal{B}r^1(2) = \widehat{\mathcal{B}r}^1(2) = \text{Span} \left(\begin{array}{c} \downarrow \\ \circ_1 \quad \circ_2 \end{array}, \begin{array}{c} \downarrow \\ \circ_2 \quad \circ_1 \end{array} \right), \quad \widehat{\mathcal{B}r}^0(2) = \text{Span} \left(\begin{array}{c} \downarrow \\ \circ_1 \\ \circ_2 \end{array}, \begin{array}{c} \downarrow \\ \circ_2 \\ \circ_1 \end{array} \right)$$

and

$$\mathcal{B}r^0(2) = \widehat{\mathcal{B}r^0}(2) \oplus \text{Span} \left(\begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \text{---} \\ | \\ \circ_2 \text{---} \circ_1 \\ | \quad | \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array}, \begin{array}{c} \text{---} \\ | \\ \circ_2 \text{---} \circ_1 \\ | \quad | \\ \bullet \quad \bullet \end{array} \right).$$

5.4. Definition. — We call an unlabeled $(l; k_1, \dots, k_n)$ -tree *amputable* if all terminal vertices of its legs are white. For such a tree T we denote by $\text{amp}(T)$ the amputated (k_1, \dots, k_n) -tree obtained from T by removing all its legs.

5.5. Example. — The $(1; 1, 1)$ -tree  is amputable, and

$$\text{amp} \left(\begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array} \right) = \begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array}.$$

The $(2; 1)$ -tree  is not amputable.

For each amputated (k_1, \dots, k_n) -tree S we define the whiskering to be the product

$$(17) \quad w(S) := \coprod (T; T \text{ is an amputable tree such that } \text{amp}(T) = S).$$

Recall that, by Convention 4.7, we interpret the unlabeled trees in the right hand side as operations in $\mathcal{T}^d(n) \subset \mathcal{B}^d(n)$, $d = n - 1 - (k_1 + \dots + k_n)$, via the correspondence $T \leftrightarrow O_T$ introduced on page 16. An equivalent definition in terms of the whiskered insertion into a corolla is given in (20).

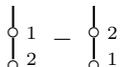
5.6. Example. — Of course, $w(\downarrow) = \downarrow$ represents the unit $1 \in C^{-1}(A; A)$. The element given by the whiskering of ,

$$w(\downarrow) = \coprod_{d \geq 0} \underbrace{\begin{array}{c} \text{---} \\ | \\ \circ_1 \\ / \quad \backslash \\ \dots \end{array}}_{d\text{-times}} \in \widehat{\mathcal{T}} \subset \widehat{\mathcal{B}}(1),$$

is the identity $f \mapsto f$, i.e., the unit of the operad \mathcal{B} . The whiskering of ,

$w(\begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array}) = \begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array} \sqcup \begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array} \sqcup \begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array} \sqcup \begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array} \sqcup \begin{array}{c} \text{---} \\ | \\ \circ_1 \text{---} \circ_2 \\ | \quad | \\ \bullet \quad \bullet \end{array} \sqcup \dots,$ gives the cup product (16). The whiskering of the element



gives the brace (15). In particular,  gives Gerstenhaber's \circ -product and  gives the Gerstenhaber bracket. Observe that the whiskering of the tree

$$(18) \quad \begin{array}{c} \text{---} \\ | \\ \circ_2 \text{---} \circ_1 \\ | \quad | \\ \bullet \quad \bullet \end{array}$$

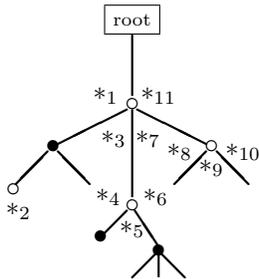


FIGURE 9. Angles of a tree symbolized by $*_1, \dots, *_{11}$. Their linear order, indicated by the subscripts, is given by walking around the tree counter-clockwise, starting at the root. Unlike [12, Section 5.2], black vertices do not have angles. The labels of white vertices are not shown.

is the operation that assigns to $f \in C^m(A; A)$ and $g \in C^n(A; A)$ the expression $(-1)^{mn}g \cup f$. The sign comes from the tree signature factor (12) in the definition of the operation O_T , because the order of the white vertices of the tree (18) and its whiskering does not agree with the natural planar one.

We are going to define operations ∂ and \circ_i acting on amputated trees that translate, via the whiskering (17), into the dg-operad structure of $\mathcal{B}r$. For an amputated (k_1, \dots, k_n) -tree S as in Definition 5.1 denote $\partial(S) := \partial_1(S) + \dots + \partial_n(S)$, where $\partial_i(S)$ is, for $k_i \geq 1$, the linear combination of amputated trees obtained by replacing the i th white vertex of S by (13) followed by the contraction of edges connecting black vertices if necessary. For $k_i = 0$ we put $\partial_i(S) = 0$.

The description of the \circ_i -operations is more delicate. Following [12, Section 5.2], define the *set of angles* of an amputated (k_1, \dots, k_n) -tree S to be the disjoint union

$$Angl(S) := \bigsqcup_{1 \leq i \leq n} \{0, \dots, k_i\}.$$

Angles come with a natural linear order whose definition is clear from Figure 9 borrowed from [12]. Now, for an amputated (k'_1, \dots, k'_n) -tree S' , an amputated (k''_1, \dots, k''_m) -tree S'' and $1 \leq i \leq n$, define $S' \circ_i S''$ to be the linear combination

$$(19) \quad S' \circ_i S'' := \sum_{\beta} (S' \circ_i S'')_{\beta},$$

where the sum runs over all (non-strictly) monotonic maps $\beta : In(w'_i) \rightarrow Angl(S'')$ from the set of incoming edges of the vertex w'_i of S' labelled i , to the set of angles of S'' . In the sum, $(S' \circ_i S'')_{\beta}$ is the tree obtained by removing the vertex w'_i from S' and replacing it by S'' , with the incoming edges of w'_i glued into the angles of S'' following β . An important particular case is $k'_i = 0$ when w'_i has no input edges. Then $S' \circ_i S''$ is defined as the tree obtained by amputating w'_i from S' and grafting the root of S'' at the place of w'_i .

We call the operation \circ_i the *whiskered insertion*. A similar operation defines in [6] the structure of the operad for pre-Lie algebras. As observed in [12], the whiskering

of Proposition 5.2 can also be expressed as the product

$$(20) \quad w(S) = \prod_{d \geq 0} \left(\begin{array}{c} | \\ \circlearrowleft \\ \dots \\ \circlearrowright \\ | \end{array} \begin{array}{c} 1 \\ \dots \\ d\text{-times} \end{array} \right) \circ_1 S.$$

The following proposition can be verified directly.

5.7. Proposition. — *With ∂ and \circ_i as defined above, the whiskering of Proposition 5.2 satisfies*

$$w(\partial S) = d(w(S)) \text{ and } w(S' \circ_i S'') = w(S') \circ_i w(S''),$$

for all amputated trees S, S', S'' and for all i for which the second equation makes sense.

5.8. Example. — We show how the classical calculations of [9] can be concisely performed in the language of amputated trees (but recall that we are using a different sign and degree convention, see 4.2). Let us start by calculating the differentials of trees representing the cap product, the circle product and the Gerstenhaber bracket. By definition, one has

$$(21) \quad \partial \left(\begin{array}{c} | \\ \bullet \\ \circlearrowleft \\ \circlearrowright \\ | \end{array} \begin{array}{c} 1 \\ \bullet \\ 2 \end{array} \right) = 0.$$

Since (13) replaces $\begin{array}{c} | \\ \circlearrowleft \\ | \end{array} 1$ by $\begin{array}{c} | \\ \bullet \\ \circlearrowleft \\ | \end{array} 1 + \begin{array}{c} | \\ \bullet \\ \circlearrowright \\ | \end{array} 1$, one gets

$$(22) \quad \partial \left(\begin{array}{c} | \\ \circlearrowleft \\ | \end{array} \begin{array}{c} 1 \\ \circlearrowleft \\ 2 \end{array} \right) = \begin{array}{c} | \\ \bullet \\ \circlearrowleft \\ | \end{array} \begin{array}{c} 1 \\ \bullet \\ 2 \end{array} + \begin{array}{c} | \\ \bullet \\ \circlearrowright \\ | \end{array} \begin{array}{c} 1 \\ \bullet \\ 2 \end{array}$$

which implies that

$$(23) \quad \partial \left(\begin{array}{c} | \\ \circlearrowleft \\ | \end{array} \begin{array}{c} 1 \\ \circlearrowleft \\ 2 \end{array} - \begin{array}{c} | \\ \circlearrowright \\ | \end{array} \begin{array}{c} 2 \\ \circlearrowright \\ 1 \end{array} \right) = \begin{array}{c} | \\ \bullet \\ \circlearrowleft \\ | \end{array} \begin{array}{c} 1 \\ \bullet \\ 2 \end{array} + \begin{array}{c} | \\ \bullet \\ \circlearrowright \\ | \end{array} \begin{array}{c} 2 \\ \bullet \\ 1 \end{array} - \begin{array}{c} | \\ \bullet \\ \circlearrowleft \\ | \end{array} \begin{array}{c} 2 \\ \bullet \\ 1 \end{array} - \begin{array}{c} | \\ \bullet \\ \circlearrowright \\ | \end{array} \begin{array}{c} 2 \\ \bullet \\ 1 \end{array} = 0.$$

We want to interpret these equations in terms of operations. To save the space, let us agree that in the rest of this example f will be an element of $C^m(A; A)$, g an element of $C^n(A; A)$ and h an element of $C^k(A; A)$, $m, n, k \geq -1$ arbitrary. By Proposition 5.7, (21) means that the differential of the cup product \cup recalled in (16) and considered as an element of $\mathcal{B}(2)$ is zero, $d(\cup) = 0$, which, by the definition (11) of the differential in \mathcal{B} means that

$$-d_H(f \cup g) = d_H f \cup g + (-1)^m f \cup d_H g.$$

We recognize [9, Eqn. (20)] saying that \cup is a chain operation. Since $\begin{array}{c} | \\ \circlearrowleft \\ | \end{array} 1$ represents the \circ -product, (22) means that

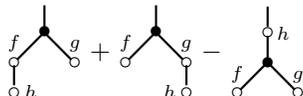
$$f \cup g + (-1)^{mn} g \cup f = d_H f \circ g + (-1)^m f \circ d_H g - d_H(f \circ g),$$

which is the graded commutativity⁽³⁾ of the cup product up to the homotopy $- \circ$ -proved in [9, Theorem 3]. The origin of the sign factor at the second term in the right hand side is explained in Example 5.6. The meaning of (23) is that

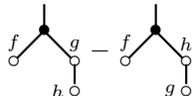
$$d_H[f, g] = [d_H f, g] + (-1)^m [f, d_H g],$$

so the bracket $[-, -]$ is a chain operation.

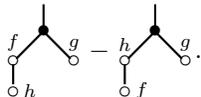
Let us investigate the compatibility between the cup product and the bracket. Since, in $\mathcal{B}(3)$, $[- \cup -, -] = [-, -] \circ_1 (- \cup -)$, the description of the \circ_i -operations in terms of amputated trees gives that $[f \cup g, h]$ is represented by



where we, for ease of reading, replaced the labels of white vertices by the corresponding cochains. Similarly, since $- \cup [-, -] = (- \cup -) \circ_2 [-, -]$ in $\mathcal{B}(3)$, $f \cup [g, h]$ is represented by



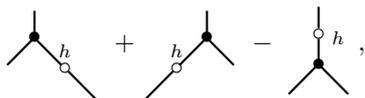
and, by the same reason, $[f, h] \cup g$ is represented by



Combining the above, one concludes that the expression $[f \cup g, h] - f \cup [g, h] - [f, h] \cup g$ is represented by

(24)

Because, by (13), ∂ replaces by



the expression in (24) equals

$$\partial \left(\begin{array}{c} | \\ \circ \\ / \quad \backslash \\ \circ \quad \circ \\ f \quad g \end{array} \right).$$

⁽³⁾ Since we use the convention in which the cup product has degree +1, its commutativity is the antisymmetry.

The meaning of the above calculations is that the bracket and the cup product are compatible up to the homotopy given by the brace $-\{-, -\}$.

Proof of Proposition 5.2. — It follows from Proposition 5.7 that the image of w contains $\mathcal{B}r$. Indeed, $Im(w)$ is a suboperad of \mathcal{B} which, by Example 5.6, contains the generators of $\mathcal{B}r$, i.e. the cup product, braces and 1. The map w is clearly a monomorphism, since each amputated (k_1, \dots, k_n) -tree S equals the amputated part (i.e. the component belonging to $\coprod B_{k_1, \dots, k_n}^0$) of its whiskering $w(S)$.

Therefore it remains to prove that $Im(w) \subset \mathcal{B}r$ or, more specifically, that $w(S) \in \mathcal{B}r(n)$ for each amputated (k_1, \dots, k_n) -tree S and $n \geq 0$. We need to show that each such S is build up, by the iterated whiskered insertions \circ_i of (19) and relabelings of white vertices, from the ‘atoms’

$$(25) \quad \downarrow, \bullet, \cup := \begin{array}{c} \bullet \\ | \\ \circ_1 \quad \circ_2 \end{array} \quad \text{and} \quad br_d := \begin{array}{c} \circ_1 \\ | \\ \circ_2 \quad \dots \quad \circ_{d+1} \end{array}, \quad d \geq 1,$$

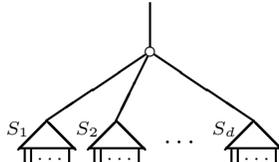
representing the generators of $\mathcal{B}r$. Since the whiskering w is an operad homomorphism and the atoms are mapped to $\mathcal{B}r$, this would indeed imply that $Im(w) \subset \mathcal{B}r$.

The first step is to get rid of the stubs. If S has $s \geq 1$ stubs, we denote by \bar{S} the tree S with each stub replaced by \downarrow . Let us label these new white vertices of \bar{S} by $n + 1, \dots, n + s$. Then clearly

$$S = \pm(\dots((\bar{S} \circ_{n+1} \bullet) \circ_{n+2} \bullet) \dots) \circ_{n+s} \bullet.$$

The sign in the above expression, not important for our purposes, is a consequence of the Koszul sign rule, since \bullet represents $1 \in A$ placed in degree -1 . So we may suppose that S has no stubs and proceed by induction on the number of internal edges. Assume that S has e internal edges. If $e \leq 1$ then S is either \downarrow or br_1 , so we may assume that $e \geq 2$. We distinguish two cases.

Case 1. The root vertex (i.e. the vertex adjacent to the root edge) is white; assume it has $d \geq 1$ input edges. The tree S looks as:



where S_1, \dots, S_d are suitable amputated trees. It is then clear that S can be obtained from

$$(\dots((br_d \circ_1 S_1) \circ_2 S_2) \dots) \circ_d S_d,$$

where br_d is the tree in (25), by relabeling the white vertices and changing the sign if necessary. Clearly, each S_1, \dots, S_d has less than e internal edges, and the induction goes on.

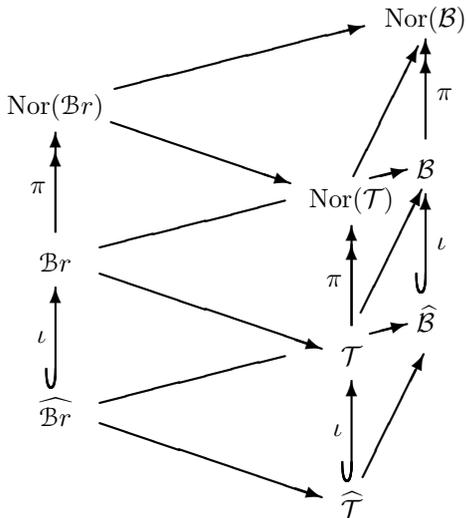
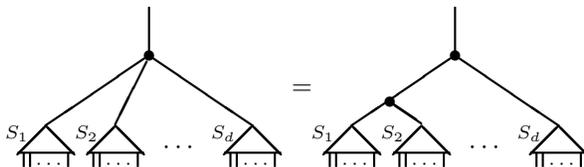


FIGURE 10. Operads of natural operations and their maps; see also the glossary on page 13. The horizontal maps are inclusions.

Case 2. The root vertex is black, with $d \geq 2$ inputs. If $d = 2$, we argue as in Case 1, only using \cup instead of br_2 . If $d \geq 3$, we use the equality



and argue as if $d = 2$. This finishes the proof. □

We finish this section by completing the proof of the following theorem of [3].

5.9. Theorem. — *The operads introduced above can be organized into the diagram in Figure 10. In this diagram:*

1. *Operads in the two upper triangles have the chain homotopy type of the operad $C_{-*}(\mathcal{D})$ of singular chains on the little disks operad \mathcal{D} with the inverted grading. In particular, the big operad \mathcal{B} of all natural operations has the homotopy type of $C_{-*}(\mathcal{D})$,*
2. *all morphisms between vertices of the two upper triangles are weak equivalences,*
3. *operads in the bottom triangle of Figure 10 have the chain homotopy type of the operad $C_{-*}(\mathcal{D})$ with the component of arity 0 replaced by the trivial abelian group, and*
4. *all morphisms in Figure 10 become weak equivalences after the application of the functor \mathfrak{A} that replaces the component of arity 0 of a dg-operad by the trivial abelian group.*

Proof. — The only piece of information that was missing in [3] and for which we had to refer to this paper was that the whiskering $w : \mathcal{B}r \rightarrow \mathcal{T}$ is a weak equivalence. This fact follows from Theorem 3.10, the identification $s\mathcal{T} \cong |\mathcal{L}_{(2)}|$ established in Proposition 4.10, and the induced identification $s\mathcal{B}r \cong \mathcal{B}r_{(2)}$ of suboperads. \square

5.10. Remark. — Theorem 5.9 shows that, up to homotopy, there is no difference between actions on the Hochschild cochains of the operads \mathcal{B} , \mathcal{T} and $\mathcal{B}r$, resp. $\widehat{\mathcal{B}}$, $\widehat{\mathcal{T}}$ and $\widehat{\mathcal{B}r}$ in the nonunital case, resp. $\text{Nor}(\mathcal{B})$, $\text{Nor}(\mathcal{T})$ and $\text{Nor}(\mathcal{B}r)$ in the normalized case.

Appendix A

Substitudes, convolution and condensation

In this appendix we briefly remind the reader of some categorical definitions and constructions we use in the paper. Most of the material is contained in [8],[7],[17] and [2].

Let V be a symmetric monoidal closed category. Let A be a small V -category and let $[A, V]$ be the V -category of V -functors from A to V . The enriched *Hom*-functor $\text{Nat}_A(F, G)$ is given by the end:

$$\text{Nat}_A(F, G) := \int_{X \in A} V(F(X), G(X)).$$

We also define the tensor product of the V -functors $F : A^{op} \rightarrow V$ and $G : A \rightarrow V$ by the coend

$$F \otimes_A G := \int^{X \in A} F(X) \otimes G(X).$$

A.1. Definition. — A *V*-substitute (P, A) is a small V -category A together with a sequence of V -functors:

$$P_n : \underbrace{A^{op} \otimes \cdots \otimes A^{op}}_{n\text{-times}} \otimes A \rightarrow V, \quad n \geq 0,$$

$$P_n(X_1, \dots, X_n; X) = P_{X_1, \dots, X_n}^X$$

equipped with

- a V -natural family of substitution operations

$$\mu : P_{X_1, \dots, X_n}^X \otimes P_{X_{11}, \dots, X_{1m_1}}^{X_1} \otimes \cdots \otimes P_{X_{n1}, \dots, X_{nm_n}}^{X_n} \rightarrow P_{X_{11}, \dots, X_{nm_n}}^X$$

- a V -natural family of morphisms (unit of substitute)

$$\eta : A(X, Y) \rightarrow P_1(X; Y) = P_X^Y$$

- for each permutation $\sigma \in S_n$ a V -natural family of isomorphisms

$$\gamma_\sigma : P_{X_1, \dots, X_n}^X \rightarrow P_{X_{\sigma(1)}, \dots, X_{\sigma(n)}}^X,$$

satisfying some associativity, unitality and equivariancy conditions [7].

Notice that P_1 is a V -monad on A in the bicategory of V -bimodules (V -profunctors or V -distributors). The Kleisli category of this monad is called *the underlying category of P* .

The concept of substitute generalizes operads and symmetric lax-monoidal categories. Indeed, any coloured operad P in V with the set of colours S is naturally a substitute $(P, U(P))$ with $U(P)$ equal the V -category with the set of objects S and the object of morphisms $U(P)(X, Y) = P(X; Y) \in V$. The substitution operation in the coloured operad P makes the assignment $P_n(X_1, \dots, X_n; X) = P_{X_1, \dots, X_n}^X$ a functor

$$P_n : \underbrace{U(P)^{op} \otimes \dots \otimes U(P)^{op}}_{n\text{-times}} \otimes U(P) \rightarrow V, \quad n \geq 0,$$

and the sequence of these functors form a substitute. The category $U(P)$ is the underlying category of this substitute also called the underlying category of the coloured operad P . In fact, a substitute is a coloured operad P together with a small V -category A and a V -functor $\eta : A \rightarrow U(P)$ [8, Prop. 6.3].

A.2. Definition. — [1, 7] A *symmetric lax-monoidal structure* or a *multitensor* on a V -category C is a sequence of V -functors

$$E_n : \underbrace{C \otimes \dots \otimes C}_{n\text{-times}} \rightarrow C$$

equipped with

- a family of V -natural transformations:

$$\mu : E_n(E_{m_1}, \dots, E_{m_k}) \rightarrow E_{m_1 + \dots + m_k};$$

- A V -natural transformation (unit)

$$Id \rightarrow E_1;$$

- an action of symmetric group

$$\gamma_\sigma : E_n(X_1, \dots, X_n) \rightarrow E_n(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n)}),$$

satisfying some natural associativity, unitarity and equivariance conditions.

A.3. Definition. — [17] A multitensor is called a *functor-operad* if its unit is an isomorphism.

McClure and Smith observed in [17] that functor-operads can be used to define operads. Their observation works also for multitensors. Let $\delta \in C$ be an object of C then the coendomorphism operad of δ with respect to a multitensor E is given by a collection of objects in V

$$Coend^E(\delta)(n) = C(\delta, E_n(\delta, \dots, \delta)).$$

Substitudes and multitensors are related by the following convolution operation [8, 7].

A.4. Definition. — Let (P, A) be a substitute. We define a multitensor E^P on $C = [A, V]$ as follows:

$$(26) \quad E_n^P(\phi_1, \dots, \phi_n)(X) = P_{-, \dots, -}^X \otimes_A \phi_1(-) \otimes_A \dots \otimes_A \phi_n(-).$$

A special case of this construction is when A is equal to the underlying category of P . In this case the convolution operation produces a functor-operad.

Let (P, A) be a substitute and let $\delta : A \rightarrow V$ be a V -functor.

A.5. Definition. — By a δ -condensation of the substitute (P, A) we mean the operad $C^{(P,A)}(\delta) = Coend^{E^P}(\delta)$. So, as a collection it is given by

$$C^{(P,A)}(\delta)(n) = Nat_A(\delta, E_n^P(\delta, \dots, \delta)).$$

The operad $C^{(P,A)}(\delta)$ naturally acts on the objects of the form

$$Tot_\delta(\phi) = Nat_A(\delta, \phi)$$

for an arbitrary V -functor $\phi : A \rightarrow V$ (δ -totalization of ϕ) [17, 2].

Let $i : B \rightarrow A$ and $\delta : B \rightarrow V$ be two V -functors. Let $Lan_i(\delta)$ be a (V -enriched) left Kan extension of δ along i . Then

$$Tot_{Lan_i(\delta)}(\phi) = Nat_A(Lan_i(\delta), \phi) = Nat_B(\delta, i^*(\phi)) = Tot_\delta(i^*(\phi)),$$

where i^* is the restriction functor induced by i .

There is a similar formula which expresses the condensation with respect to $Lan_i(\delta)$. Let (P, A) be a substitute and let $i_{*, \dots, *}^A(P)$ be a sequence of functors

$$\begin{aligned} i_{*, \dots, *}^A(P)_n &: B^{op} \otimes \dots \otimes B^{op} \otimes A \rightarrow V, \\ i_{*, \dots, *}^A(P)_{B_1, \dots, B_n}^A &= P_{i(B_1), \dots, i(B_n)}^A \quad . \end{aligned}$$

We define a sequence of functors

$$E_n^{i_{*, \dots, *}^A(P)} : [B, V] \otimes \dots \otimes [B, V] \rightarrow [A, V]$$

by the formula similar to formula (26). We also define i^*P as the substitute (i^*P, B) obtained from P by restricting P_n along i .

A.6. Proposition. — For the functors $\phi_1, \dots, \phi_n \in [B, V]$ the following V -natural isomorphisms hold:

$$E_n^P(Lan_i(\phi_1), \dots, Lan_i(\phi_n)) = E_n^{i_{*, \dots, *}^A(P)}(\phi_1, \dots, \phi_n).$$

In particular,

$$\begin{aligned} C^{(P,A)}(Lan_i(\delta))(n) &= Tot_\delta(i^*E_n^{i_{*, \dots, *}^A(P)}(\delta, \dots, \delta)) \\ &= Tot_\delta(E_n^{i^*(P)}(\delta, \dots, \delta)) = C^{(i^*(P), B)}(\delta)(n). \end{aligned}$$

This result allows to see many of the operads in this paper as the result of δ -condensation of some substitutes. For us V will be the category of chain complexes Ch . Our category A will be the category of nonempty ordinals Δ (linearized) or the crossed interval category $(IS)^{op}$ [3] (also linearized). B can be Δ or its subcategory of injective order preserving maps Δ_{in} . These categories are related by the canonical inclusions:

$$\Delta_{in} \xrightarrow{i} \Delta \xrightarrow{j} (IS)^{op}.$$

Let $\delta : \Delta \rightarrow Ch$ be the cosimplicial chain complex of normalized chains on standard simplices. It is classical that the totalization of a cosimplicial chain complex X^\bullet with respect to δ is the normalized cosimplicial totalization $\overline{\text{Nor}}(X^\bullet)$ and the tensor product $X_\bullet \otimes_{\Delta} \delta$ for a simplicial chain complex X_\bullet is the normalized simplicial realization $\underline{\text{Nor}}(X_\bullet)$. Hence, the condensation of the lattice path operad $\mathcal{L}_{(c)}$ with respect to δ is precisely the n -simplicial cosimplicial normalization

$$|\overline{\mathcal{L}_{(c)}}| = \overline{\text{Nor}}(\underline{\text{Nor}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet))) = C^{(\mathcal{L}_{(c)}, \Delta)}(\delta).$$

Proposition A.6 shows that the condensation of the lattice path operad $\mathcal{L}_{(c)}$ with respect to $\text{Lan}_i(i^*(\delta))$ is the unnormalized n -simplicial cosimplicial totalization

$$|\mathcal{L}_{(c)}| = \overline{\text{Tot}}(\underline{\text{Tot}}(\mathcal{L}_{(c)}(\bullet_1, \dots, \bullet_n; \bullet))) = C^{(i^*(\mathcal{L}_{(c)}), \Delta_{in})}(i^*(\delta)) = C^{(\mathcal{L}_{(c)}, \Delta)}(\text{Lan}_i(i^*(\delta)))$$

Analogously, for the operad of natural operations on the Hochschild cochains we use the condensation with respect to $\text{Lan}_j(\delta)$ for the normalized version and with respect to $\text{Lan}_{j_i}(i^*\delta)$ for the unnormalized version that is

$$\mathcal{B} = C^{(B, (IS)^{op})}(\text{Lan}_i(i^*(\delta))).$$

In [2] similar calculations were applied to the cyclic version of the lattice path operad.

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