Eine Wanderung durch Rainer Vogt’s mathematisches Schaffen

Clemens Berger

Bonn MPIM, February 11, 2016
1942: Rainer Max Vogt is born in Stuttgart
1960-1965: Graduate studies in Frankfurt am Main
1965-1968: PhD supervised by J. M. Boardman in Warwick
1968-1974: Visiting Professor in Aarhus, Heidelberg and Saarland
1974-2015: Professor for Topology in Osnabrück

*PhD students:* K. Below, O. Blömer, M. Brinkmeier, J. Hollender, T. Hüttemann, H. Wellen, X. Yang

*50 publications with over 500 citations* in AMS MathSciNet basis
1. Convenient categories of topological spaces (1971)

2. Homotopy limits and colimits (1973)

3. Homotopy invariant algebraic structures (\& Boardman 1973)

4. $\text{THH}(R) = R \otimes S^1$ (\& McClure and Schwänzl 1997)

5. Iterated monoidal categories (BFSV 2003)

6. An additivity theorem for the interchange (\& Fiedorowicz 2015)
Problem

Find a category $\mathcal{T}$ of topological spaces which is *cartesian closed*, i.e. such that $\mathcal{T}(X \times Y, Z) = \mathcal{T}(X, Z^Y)$ for a functional space $Z^Y$.

Proposition (Vogt 1971)

Let $C$ be a class of topological spaces fulfilling

- $C$ is closed under binary product in Top;
- for any $X$ in $C$ and $Y$ in Top, evaluation (with respect to compact-open topology) $(Y^X)_{co} \times X \to Y$ is continuous.

Then the coreflective hull $\overline{C}$ in Top is cartesian closed.

$\overline{C}$ is the coreflective subcategory of Top consisting of the spaces with the final topology with respect to the class of maps out of $C$.

*Examples:* $C_1 =$ (compact Hausdorff spaces), and $C_2 =$ (locally compact spaces), and $C_3 =$ (exponentiable spaces, Day-Kelly 1970).
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Find a category $\mathcal{T}$ of topological spaces which is \textit{cartesian closed}, i.e. such that $\mathcal{T}(X \times Y, Z) = \mathcal{T}(X, Z^Y)$ for a functional space $Z^Y$.

Proposition (Vogt 1971)

Let $\mathcal{C}$ be a class of topological spaces fulfilling

- $\mathcal{C}$ is closed under binary product in $\text{Top}$;
- for any $X$ in $\mathcal{C}$ and $Y$ in $\text{Top}$, evaluation (with respect to compact-open topology) $(Y^X)_{co} \times X \to Y$ is continuous.

Then the coreflective hull $\overline{\mathcal{C}}$ in $\text{Top}$ is cartesian closed.

$\overline{\mathcal{C}}$ is the coreflective subcategory of $\text{Top}$ consisting of the spaces with the final topology with respect to the class of maps out of $\mathcal{C}$.

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**Examples:** $\mathcal{C}_1 = \text{(compact Hausdorff spaces)}$, and $\mathcal{C}_2 = \text{(locally compact spaces)}$, and $\mathcal{C}_3 = \text{(exponentiable spaces, Day-Kelly 1970)}$. 
Let $F : \mathcal{A} \rightarrow \text{Top}$ be an $\mathcal{A}$-diagram of topological spaces.


$$\text{hocolim}_\mathcal{A} F \overset{\text{def}}{=} \left( \coprod_{x_0 \in \mathcal{A}} \coprod_{n \geq 0} F(x_0) \times \mathcal{A}^{n+1}(x_0, x_{n+1}) \times [0, 1]^n \right) / \sim$$


$hocolim_\mathcal{A} : \text{Top}^\mathcal{A} \rightarrow \text{Top}$ takes pointwise homotopy equivalences to homotopy equivalences.

**Definition (W-construction of a category, Vogt 1973)**

There is a *topologically enriched* category $W \mathcal{A}$ sth.

- $\text{Ob}(W \mathcal{A}) = \text{Ob}(\mathcal{A})$;
- $(W \mathcal{A})(x, y) = (\coprod_{n \geq 0} \mathcal{A}^{n+1}(x, y) \times [0, 1]^n) / \sim$. 
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**Definition (Vogt 1973)**

A *homotopy coherent $\mathcal{A}$-diagram* is a top. functor $W\mathcal{A} \to \mathcal{T}$. Let $\mathcal{T}^{h\mathcal{A}}$ be the category of homotopy coherent $\mathcal{A}$-diagrams.

There is an enriched adjunction $\operatorname{colim}_{W\mathcal{A}} : \mathcal{T}^{h\mathcal{A}} \dashv \mathcal{T} : c_{\mathcal{A}}$. Define $\epsilon : W\mathcal{A} \to \mathcal{A}$ by $(W\mathcal{A})(x, y) \mapsto \mathcal{A}(x, y) = \pi_0(W\mathcal{A})(x, y)$.

**Proposition (Vogt 1973)**

$$\hocolim_{\mathcal{A}}(F) \cong \operatorname{colim}_{W\mathcal{A}} \epsilon^*(F)$$


There is a 2-category $\omega[n]$ with same objects as $[n]$ such that

$$(\omega[n])(k, l) = \text{Fact}([n]; k, l) \text{ if } k \leq l.$$

where $\text{Fact}([n]; k, l)$ is the factorization category of $k \to l$ in $[n]$. 
Definition (Vogt 1973)

A **homotopy coherent A-diagram** is a top. functor $WA \to T$. Let $T^hA$ be the category of homotopy coherent $A$-diagrams.

There is an enriched adjunction $\text{colim}_{WA} : T^hA \rightleftarrows T : c_A$. Define $\epsilon : WA \to A$ by $(WA)(x, y) \mapsto A(x, y) = \pi_0(WA)(x, y)$.

Proposition (Vogt 1973)

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Remark (simplicial vs cubical, cf. Baues 1983)

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A *homotopy coherent* $\mathcal{A}$-*diagram* is a top. functor $\mathcal{W}\mathcal{A} \to \mathcal{T}$. Let $\mathcal{T}^{h\mathcal{A}}$ be the category of homotopy coherent $\mathcal{A}$-diagrams.

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Lemma

- $\omega[n](k, l) \cong [1]^{l-k-1}$ if $k < l$
- $|\text{nerve}(\omega[n])| \cong W[n]$
- $\Delta \to \mathbf{sCat} : [n] \mapsto C[n] := \text{nerve}(\omega[n])$

Definition (Homotopy coherent nerve)

$$\text{Hom}_{\mathbf{sCat}}(\mathbb{C}[-], -) : \mathbf{sCat} \rightleftarrows \mathbf{sSets} : - \otimes_{\Delta} \mathbb{C}[-]$$

Theorem (Joyal 2007, Lurie 2009)

This is a Quillen equivalence between the Bergner model structure on simplicial cat’s and the Joyal model structure on simplicial sets.
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Definition (Vogt 1968, Boardman-Vogt 1973)

An operator category in normal form is a strictly associative symmetric monoidal subcategory \((\mathcal{B}, \oplus, 0)\) of \((\mathcal{T}, \times, \ast)\) such that

- The objects of \(\mathcal{B}\) are the natural numbers \(m \oplus n = m + n\)
- for all \(n = n_1 + \cdots + n_k\) there is a canonical isomorphism \(\mathcal{B}(n_1, 1) \times \cdots \times \mathcal{B}(n_k, 1) \times \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k} \mathfrak{S}_n \cong \mathcal{B}(n, k)\).

Definition (May 1972)

An operator category in normal form \((\mathcal{B}(n, k))_{n,k \in \mathbb{N}}\) determines, and is determined by, a symmetric operad \((\mathcal{O}(n) = \mathcal{B}(n, 1))_{n \in \mathbb{N}}\).

The categorical structure of \(\mathcal{B}\) amounts to a substitutional structure of \(\mathcal{O}\), i.e. a unit \(1 \in \mathcal{O}(1)\), and a multiplication

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\mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)
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satisfying associativity, unitarity and equivariance constraints.
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\[O(k) \times O(n_1) \times \cdots \times O(n_k) \to O(n_1 + \cdots + n_k)\]

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Definition

Each topological space $X$ has an endo-operad $\mathcal{E}_X(k) = \mathcal{I}(X^k, X)$. A $\mathcal{O}$-algebra structure on $X$ is an operad map $\mathcal{O} \to \mathcal{E}_X$.

Remark

An $\mathcal{O}$-algebra structure on $X \iff \mathcal{O}(k) \times X^k \to X$, $k \geq 0$.

Remark

Topological monoids are algebras over a symmetric operad; topological groups are not! Compare: Monoids can be defined in any symmetric monoidal category, while groups cannot.
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Topological monoids are algebras over a symmetric operad; topological groups are not! Compare: Monoids can be defined in any symmetric monoidal category, while groups cannot.
**Example (Iterated loop spaces and coendomorphism operads)**

A $k$-ary operation on $\Omega^n X = T_*(S^n, X)$ amounts to a map

$$T_*(S^n \vee \cdots \vee S^n, X) = T_*(S^n, X)^k \rightarrow T_*(S^n, X).$$

Such $k$-ary operations are induced by points in

$$\text{Coend}(S^n)(k) = T_*(S^n, S^n \vee \cdots \vee S^n).$$

Any suboperad of $\text{Coend}(S^n)$ acts on $n$-fold loop spaces.

**Definition (operad of little $n$-cubes)**

A little $n$-cube is an affine embedding $f : [0, 1]^n \rightarrow [0, 1]^n$ preserving the direction of the axes. $C_n(k)$ is the space of $k$-tuples $(f_1, \ldots, f_k)$ of little $n$-cubes with pairwise disjoint interiors. This defines a suboperad $C_n$ of $\text{Coend}(S^n)$ acting on $n$-fold loop spaces.
Example (Iterated loop spaces and coendomorphism operads)

A $k$-ary operation on $\Omega^n X = \mathcal{T}_*(S^n, X)$ amounts to a map

$$\mathcal{T}_*(S^n \vee \cdots \vee S^n, X) = \mathcal{T}_*(S^n, X)^k \to \mathcal{T}_*(S^n, X).$$

Such $k$-ary operations are induced by points in

$$\text{Coend}(S^n)(k) = \mathcal{T}_*(S^n, S^n \vee \cdots \vee S^n).$$

Any suboperad of $\text{Coend}(S^n)$ acts on $n$-fold loop spaces.

Definition (operad of little $n$-cubes)

A little $n$-cube is an affine embedding $f : [0, 1]^n \to [0, 1]^n$ preserving the direction of the axes. $C_n(k)$ is the space of $k$-tuples $(f_1, \ldots, f_k)$ of little $n$-cubes with pairwise disjoint interiors. This defines a suboperad $C_n$ of $\text{Coend}(S^n)$ acting on $n$-fold loop spaces.
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**Theorem (Boardman-Vogt, May, Segal)**

Any connected $C_n$-algebra is weakly equivalent to an $n$-fold loop space.

**Theorem (May, Segal)**

The free $C_n$-algebra generated by a pointed connected space $X$ is weakly equivalent to $\Omega^n \Sigma^n X$. The operad inclusions $C_n \subset C_{n+1}$ correspond to the stabilization maps $\Omega^n \Sigma^n X \to \Omega^{n+1} \Sigma^{n+1} X$.

**Definition (Boardman-Vogt, May)**

An $E_\infty$-operad is a top. operad $\mathcal{O}$ such that for each $k$, $\mathcal{O}(k)$ is a universal principal $\mathfrak{S}_k$-bundle.

**Theorem (Boardman-Vogt, May)**

The colimit $C_\infty = \colim_n C_n$ is an $E_\infty$-operad. Every algebra over an $E_\infty$-operad is (up to group completion) an infinite loop space.
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B-V construct a functorial resolution $\epsilon : W(\mathcal{O}) \overset{\sim}{\longrightarrow} \mathcal{O}$ such that $W(\mathcal{O})$-algebra structures are “homotopical” $\mathcal{O}$-algebra structures.

Theorem (Boardman-Vogt 1973)

For each top. operad $\mathcal{O}$ such that all $\mathcal{O}(k)$ are principal $\mathfrak{S}_k$-bundles, $W(\mathcal{O})$-algebra structures are homotopy-invariant.

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In any monoidal model category with suitable interval $H$, operads have a functorial $W$-resolution $W_H(\mathcal{O}) \overset{\sim}{\longrightarrow} \mathcal{O}$. If $\mathcal{O}$ is a well-pointed $\mathfrak{S}$-cofibrant operad then $W_H(\mathcal{O})$ is a cofibrant operad and B-V’s homotopy-invariance property holds.
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Remark (Brave new algebra, Waldhausen, EKMM 1997, HSS 2000)

- Infinite loop spaces are connective $\Omega$-spectra $(X_n)_{n \geq 0}$.
  (i.e. $X_n \sim \Omega X_{n+1}$ and $\pi_k(X_n) = *$ for $k < n$).
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Definition (THH, Böckstedt 1985)

$THH(E_\infty$-ring spectrum $R)$ defined like $HH$(comm. ring $A$) using the sphere spectrum $(S^n)_{n \geq 0}$ instead of the integers $\mathbb{Z}$.


$THH(R) = R \otimes S^1 \overset{def}{=} |R \wedge \Sigma^\infty(\Delta[1]/\partial\Delta[1])_+|.$
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There exists a symmetric monoidal structure on spectra prolonging smash product of pointed spaces and tensor product of abelian groups.

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Eine Wanderung durch Rainer Vogt’s mathematisches Schaffen

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Remark

A double loop space is the same as a loop space in loop spaces:
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Definition (BFSV 2003)

A 2-monoidal category is a monoid in the category of strict monoidal categories and normal lax monoidal functors.

A 2-monoidal category is a category equipped with two strictly associative tensors \( \otimes_1, \otimes_2 \) sharing the same unit, and interrelated by an *interchange morphism*

\[
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An $n$-monoidal category is a monoid in the cat. of $(n-1)$-monoidal categories and normal lax $(n-1)$-monoidal functors.

**Theorem (BFSV 2003)**

The category of $n$-monoidal categories is the category of algebras for a categorical operad $\mathcal{M}_n$.

This operad is an $E_n$-operad in posets, in particular $|\mathcal{M}_n| \simeq C_n$.

A typical morphism in $\mathcal{M}_2(4)$:

$$(((1 \otimes_1 2) \otimes_2 4) \otimes_1 3) \to (2 \otimes_2 ((1 \otimes_1 3) \otimes_2 4))$$

**Theorem (BFSV 2003 and FSV 2014, cf. Thomason for $n = 1, \infty$)**

The nerve of an $n$-monoidal category is up to group completion an $n$-fold loop space. Any $n$-fold loop space arises in this way.
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Remark (Special case $n = 2$)

$$|\mathcal{M}_2(k)| \simeq C_2(k) \simeq F(\mathbb{R}^2, k) \simeq B(P\text{Br}(k))$$

Example ($E_2$-operads and braided monoidal categories)

\[ \mathcal{M}_2 \xrightarrow{\sim} \text{CoBr} \leftrightarrow \text{PaBr} \]

- 2-monoidal
- braided
- braided monoidal
- strict monoidal
- strict unit
- strict unit


$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \subseteq \hat{GT} \overset{\text{def}}{=} \text{Aut}_0^{\text{Grpd}}(\text{PaBr})$$
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Theorem (Dunn 1980)

For the operad $\mathcal{D}_n$ of decomposable little $n$-cubes, one has an operad isomorphism $\mathcal{D}_m \otimes_{BV} \mathcal{D}_n \cong \mathcal{D}_{m+n}$.

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The $BV$-tensor product $- \otimes_{BV} -$ does not preserve weak equivalences so that $E_m \otimes_{BV} E_n \not\simeq E_{m+n}$ although $(m+n)$-fold loop spaces are $m$-fold loop spaces in $n$-fold loop spaces.
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Let $A, B$ be top. operads. There exists a top. operad $A \otimes_{BV} B$ sth. $A \otimes_{BV} B$-algebras are the same as $A$-algebras in $B$-algebras.

Theorem (Dunn 1980)

For the operad $\mathcal{D}_n$ of decomposable little n-cubes, one has an operad isomorphism $\mathcal{D}_m \otimes_{BV} \mathcal{D}_n \cong \mathcal{D}_{m+n}$.

Problem

The $BV$-tensor product $- \otimes_{BV} -$ does not preserve weak equivalences so that $E_m \otimes_{BV} E_n \not\simeq E_{m+n}$ although $(m + n)$-fold loop spaces are $m$-fold loop spaces in $n$-fold loop spaces.
**Theorem (Fiedorowicz-Vogt 2015)**

If $\mathcal{A}$ is a cofibrant $E_m$-operad and $\mathcal{B}$ is a cofibrant $E_n$-operad then $\mathcal{A} \otimes_{BV} \mathcal{B}$ is an $E_{m+n}$-operad.

**Proof.**

Analysis of the cellular structure of $W_{\text{red}}|M_m| \otimes_{BV} W_{\text{red}}|M_n|$.

**Remark (Generalized operads – Moerdijk-Weiss 2007, Lurie 2012)**

(dendroidal sets)

(top. operads)

($\infty$-operads)
Theorem (Fiedorowicz-Vogt 2015)
If $\mathcal{A}$ is a cofibrant $E_m$-operad and $\mathcal{B}$ is a cofibrant $E_n$-operad then $\mathcal{A} \otimes_{BV} \mathcal{B}$ is an $E_{m+n}$-operad.

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Proof.

Analysis of the cellular structure of $W_{red} | \mathcal{M}_m | \otimes_{BV} W_{red} | \mathcal{M}_n |$. 

Remark (Generalized operads – Moerdijk-Weiss 2007, Lurie 2012)

(dendroidal sets) → (top. operads) → (∞-operads)
Eine Wanderung durch Rainer Vogt's mathematisches Schaffen

An additivity theorem for the interchange (Fiedorowicz 2015)