Combinatorial models for E_n-operads and iterated loop spaces

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Let $\mathrm{S}^n \subset \mathbb{R}^{n+1}$ be the *unit n-sphere* based at $(1, 0, \ldots, 0)$.
Let $(X, *)$ be a based topological space.

The *n-th loop space* of X is $\Omega^n X = \mathrm{map}_*(S^n, X)$.
The *n-th homotopy group* of X is $\pi_n(X) = \pi_0(\Omega^n X)$.

Let $\mathrm{S}^{n-1} \subset \mathrm{S}^n$ be the "equator" $\mathrm{S}^{n-1} = \mathrm{S}^n \cap \{x_{n+1} = 0\}$.

Lemma

The quotient map $p : \mathrm{S}^n \to \mathrm{S}^n/\mathrm{S}^{n-1} \cong \mathrm{S}^n \vee \mathrm{S}^n$ defines

$$\Omega^n X \times \Omega^n X = \mathrm{map}_*(\mathrm{S}^n \vee \mathrm{S}^n, X) \xrightarrow{p^*} \mathrm{map}_*(\mathrm{S}^n, X) = \Omega^n X$$

inducing a *group structure* on $\pi_n(X)$ for $n \geq 1$.

Proposition (Eckmann-Hilton)

$\pi_n(X)$ is abelian for $n \geq 2$.

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For (based) spaces X, Y, Z one has a trinatural bijection

$$\mathrm{Top}(X \times Y, Z) \cong \mathrm{Top}(X, \mathrm{map}(Y, Z))$$
$$\text{resp. } \mathrm{Top}_*(X \wedge Y, Z) \cong \mathrm{Top}_*(X, \mathrm{map}_*(Y, Z))$$

where $X \wedge Y = (X \times Y)/(X \times \{*_Y\}) \cup (\{*_X\} \times Y)$.

The *n-th suspension of* X is $\Sigma^n X = X \wedge \mathrm{S}^n$.

Corollary

$\mathrm{Top}_*(\Sigma^n X, Z) \cong \mathrm{Top}_*(X, \Omega^n Z)$ whence a map $X \longrightarrow \Omega^n \Sigma^n X$

Theorem (Freudenthal)

$\pi_k(X) \to \pi_k(\Omega\Sigma X)$ isomorphism if $k \leq 2 \cdot \mathit{connectivity}(X)$.

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Definition (stable homotopy groups)

- $\Omega^\infty\Sigma^\infty X = \mathrm{colim}(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \cdots)$
- $\pi_k^{st}(X) = \pi_k(\Omega^\infty\Sigma^\infty X)$

The *stable homotopy groups* share some of the good properties of the homology groups (abelianess, exact cofibration sequences).

Corollary

$\pi_k^{st}(X) = \pi_k(\Omega^n\Sigma^n X)$ for $n \geq k+2$.

Stable homotopy groups remain difficult to compute; calculating $\pi_k^{st}(\mathrm{S}^0)$ is one of the major problems in algebraic topology.
The groups are known only for $k \leq 64$:

k	0	1	2	3	4	5	6	7
$\pi_k^{st}(\mathrm{S}^0)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$	0	0	$\mathbb{Z}/240\mathbb{Z}$	$\mathbb{Z}/4\mathbb{Z}$

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Since $\mathrm{S}^n = \overbrace{\mathrm{S}^1 \wedge \cdots \wedge \mathrm{S}^1}^{n}$, any n-fold loop space $\Omega^n X$ carries n different, yet compatible multiplications induced by

$$\mathrm{S}^1 \wedge \cdots \wedge \mathrm{S}^1 \wedge \cdots \wedge \mathrm{S}^1 \to \mathrm{S}^1 \wedge \cdots \wedge (\mathrm{S}^1 \vee \mathrm{S}^1) \wedge \cdots \wedge \mathrm{S}^1.$$

The two pinch maps $\mathrm{S}^2 \to \mathrm{S}^2 \vee \mathrm{S}^2$ are given by:

A space of pinch maps $\mathcal{C}_2(2) \subset \mathrm{map}_*(\mathrm{S}^2, \mathrm{S}^2 \vee \mathrm{S}^2)$ is given by:

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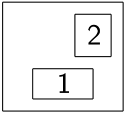

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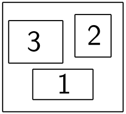

Definition

A topological operad $\mathcal{O}$ is a family of $\mathfrak{S}_k$-spaces $\mathcal{O}(k)$, $k \geq 0$, equipped with a unit $1 \in \mathcal{O}(1)$ and with substitution maps

$$\mathcal{O}(k) \times \mathcal{O}(n_1) \times \cdots \times \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$

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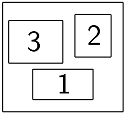

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Example (Boardman-Vogt '68)

The family $\mathcal{C}_2(k)$, $k \geq 0$, defines an operad, the *little squares operad* $\mathcal{C}_2$. Similarly, one defines the *little n-cubes operad* $\mathcal{C}_n$.

$$\mathcal{C}_2(2) \quad \times \quad \mathcal{C}_2(2) \quad \times \quad \mathcal{C}_2(1) \quad \longrightarrow \quad \mathcal{C}_2(2+1)$$

Remark

The little n-cubes operad $\mathcal{C}_n$ is a *suboperad* of

$$\mathrm{Coend}(\mathrm{S}^n)(k) = \mathrm{map}_*(\mathrm{S}^n, \overbrace{\mathrm{S}^n \vee \cdots \vee \mathrm{S}^n}^{k}), \; k \geq 0.$$

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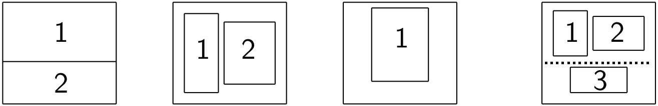

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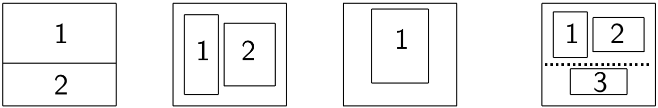

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Definition

An $\mathcal{O}$-action on a space X consists of maps

$$\mathcal{O}(k) \times X^k \to X, \quad k \geq 0,$$

satisfying natural equivariance, associativity and unit constraints.

Example

Any n-fold loop space $\Omega^n X$ carries a canonical $\mathcal{C}_n$-action.

$$\begin{array}{ccc} \mathcal{C}_n(k) \times (\Omega^n X)^k & \longrightarrow & \mathrm{map}_*(\mathrm{S}^n, (\mathrm{S}^n)^{\vee k}) \times \mathrm{map}_*((\mathrm{S}^n)^{\vee k}, X) \\ \downarrow & & \downarrow \\ \Omega^n X & = & \mathrm{map}_*(S^n, X) \end{array}$$

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Definition

A space X is an *E_n-space* if X comes equipped with an action by an *E_n-operad* (i.e. a $\mathfrak{S}$-cofibrant operad weakly equivalent to $\mathcal{C}_n$).

Theorem (Boardman-Vogt '73, May '72, Segal '74)

Any connected E_n-space is weakly homotopy equivalent to an n-fold loop space.

Theorem (May '72)

For any connected space $(X, *)$, the *free* $\mathcal{C}_n$-space generated by X

$$\mathcal{C}_n(X) = \left(\coprod_{k \geq 0} \mathcal{C}_n(k) \times X^k \right) / \sim$$

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Lemma (Künneth)

For a field K, the functor $H_*(-;K) : (\text{spaces}) \to (K\text{-vector spaces})$ is *strong monoidal*, i.e. $H_*(X \times Y; K) \cong H_*(X;K) \otimes_K H_*(Y;K)$.

Corollary

The functor $H_*(-;K)$ takes (co)algebraic structures in spaces to corresponding (co)algebraic structures in K-vector spaces.

Example

If X is a topological group then $H_*(X;K)$ is a Hopf algebra over K.

Theorem (F. Cohen '76)

If X is an E_2-space then $H_*(X;K)$ is a Gerstenhaber K-algebra.

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If X is an E_2-space then $H_*(X;K)$ is a Gerstenhaber K-algebra.

Lemma (Künneth)

For a field K, the functor $H_*(-;K) : (\text{spaces}) \to (K\text{-vector spaces})$ is *strong monoidal*, i.e. $H_*(X \times Y; K) \cong H_*(X;K) \otimes_K H_*(Y;K)$.

Corollary

The functor $H_*(-;K)$ takes (co)algebraic structures in spaces to corresponding (co)algebraic structures in K-vector spaces.

Example

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A *Gerstenhaber K-algebra* $(H, \cup, \{-,-\})$ is a graded-commutative K-algebra with Lie bracket of degree -1 such that

$$\{f, g \cup h\} = \{f, g\} \cup h + (-1)^{|f|(|g|-1)} g \cup \{f, h\}.$$

Remark

Cup product resp. Lie bracket are induced by the generators of $H_0(\mathcal{C}_2(2); K)$ resp. $H_1(\mathcal{C}_2(2); K)$ using that $\mathcal{C}_2(2) \simeq S^1$.

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For any associative K-algebra A, the Hochschild cohomology $HH^*(A; A)$ is a Gerstenhaber K-algebra.

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$$C^n(A;M) = \mathrm{Hom}_K(A^{\otimes n}, M), \quad n \geq 0,$$

where for $f \in C^n(A;M)$,

$$(\partial_i f)(a_1,\ldots,a_{n+1}) = \begin{cases} a_1 f(a_2,\ldots,a_{n+1}) & i=0; \\ f(a_1,\ldots,a_i a_{i+1},\ldots,a_{n+1}) & i=1,\ldots,n; \\ f(a_1,\ldots,a_n)a_{n+1} & i=n+1. \end{cases}$$

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The Hochschild cohomology $HH^*(A;M)$ is the cohomology of the cochain complex of the cosimplicial K-module $C^*(A;M)$.

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There is a *cup product*

$$- \cup - : C^m(A;A) \otimes_K C^n(A;A) \to C^{m+n}(A;A)$$

$$(f \cup g)(a_1, \ldots, a_{m+n}) = f(a_1, \ldots, a_m) g(a_{m+1}, \ldots, a_{m+n})$$

and a *brace operation*

$$-\{-\} : C^m(A;A) \otimes_K C^n(A;A) \to C^{m+n-1}(A;A)$$

where $f\{g\}(a_1, \ldots, a_{m+n-1})$ is defined by

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The bracket $\{f, g\} = f\{g\} - (-1)^{(|f|-1)(|g|-1)} g\{f\}$ induces a Lie bracket of degree -1 on $HH^*(A;A)$.

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Problem

What is the origin of the Gerstenhaber structure on $HH^*(A;A)$?

Theorem (conjectured by Deligne '93)

The Gerstenhaber structure on $HH^*(A;A)$ derives from an E_2*-operad action* on the Hochschild cochain complex $C^*(A;A)$.

Proofs have been given by Voronov '00, Kontsevich-Soibelman '00, McClure-Smith '01, B-F '02, Kaufmann-Schwell '07, B-B '09.

Remark

$HH^0(A;A) = ZA = \{a \in A \mid ab = ba \, \forall b \in A\}$ is the *center* of A. The Hochschild cochain complex $C^*(A;A)$ is thus a kind of *homotopy center* of A and the Deligne conjecture states:

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Aim

- "Conceptual" proof of Deligne conjecture
- "Universal" construction of E_n-operads

Definition

For any object X, the *endomorphism operad* $\mathrm{End}(X)$ is defined by

$$\mathrm{End}(X)(k) = \mathrm{Hom}(X^{\otimes k}, X), \quad k \geq 0.$$

Definition

A *multiplicative operad* is a non-symmetric operad $\mathcal{O}$ equipped with a "multiplicative system" of elements $m_k \in \mathcal{O}(k)$, $k \geq 0$.

Example

For each monoid A, $\mathrm{End}(A)$ is a multiplicative operad.

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$$C^k(A;A) = \mathrm{Hom}_K(A^{\otimes k}, A) = \mathrm{End}(A)(k) \quad (k \geq 0)$$

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Any multiplicative operad $\mathcal{O}$ carries canonical cosimplicial operators $\partial_i : \mathcal{O}(k) \to \mathcal{O}(k+1)$ and $s_i : \mathcal{O}(k+1) \to \mathcal{O}(k) \quad (k \geq 0)$.

Theorem (McClure-Smith '04, Kaufmann-Schwell '07, B-B '09)

The cosimplicial totalisation of a multiplicative operad $\mathcal{O}$ in spaces or chain complexes carries a canonical action by an E_2-operad.

For a K-algebra A and $\mathcal{O} = \mathrm{End}(A)$ the cosimplicial totalisation yields $C^*(A;A)$ so that the theorem implies the Deligne conjecture. Our proof of the theorem is based on the *lattice path operad*.

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Definition

An $\mathbb{N}$-*coloured operad* $\mathcal{L}$ is given by a family of objects $\mathcal{L}(n_1, \dots, n_k; n)$, where $(n_1, \dots, n_k, n) \in \mathbb{N}^{k+1}$, together with units, $\mathfrak{S}_k$-actions and substitution maps

$$\mathcal{L}(n_1, \dots, n_k; n) \otimes \mathcal{L}(m_1, \dots, m_l; n_i) \xrightarrow{\circ_i} \mathcal{L}(n_1, \dots, n_{i-1}, m_1, \dots, m_l, n_{i+1}, \dots, n_k; n),$$

which are unital, associative and equivariant.

The *underlying category* $\mathcal{L}_u$ has as objects the natural numbers and as morphisms the "unary" operations: $\mathcal{L}_u(n, n') = \mathcal{L}(n; n')$.
An $\mathcal{L}$-*algebra* X consists of a graded object $X(n)$, $n \geq 0$, together with (equivariant, unital, associative) action maps $\mathcal{L}(n_1, \dots, n_k; n) \otimes X(n_1) \otimes \cdots \otimes X(n_k) \to X(n)$.
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The $\mathbb{N}$-coloured operad $\mathcal{L}$ induces a *multitensor* on $\mathcal{L}_u$-diagrams:

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Each $\mathcal{L}_u$-diagram δ defines a symmetric (uncoloured) operad

$$\mathrm{Coend}_{\mathcal{L}}(\delta)(k) = \mathrm{Hom}_{\mathcal{L}_u}(\delta, \overbrace{\delta \otimes_{\mathcal{L}} \cdots \otimes_{\mathcal{L}} \delta}^{k}) \quad (k \geq 0).$$

Proposition (δ-condensation)

Let X be an $\mathcal{L}$-algebra and δ be a $\mathcal{L}_u$-diagram.
Then the "δ-totalisation" $\mathrm{Hom}_{\mathcal{L}_u}(\delta, X)$ is equipped with a canonical action by the "δ-condensed" operad $\mathrm{Coend}_{\mathcal{L}}(\delta)$.

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$$\mathcal{L}(n_1, .., n_k; n) = \mathrm{Cat}_{*,*}([n+1], [n_1+1] \otimes \cdots \otimes [n_k+1]).$$

Example. Let $x \in \mathcal{L}(2,1;3)$ be the following lattice path:

$$\begin{array}{ccccccc}
(0,2) & \text{——} & (1,2) & \text{——} & (2,2) & \text{——} & x(4) \\
| & & | & & | & & \Uparrow 2 \\
(0,1) & \text{——} & \cdot & \xRightarrow{1} & x(2) & \xRightarrow{1} & x(3) \\
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x(0) & \xRightarrow{1} & x(1) & \text{——} & (2,0) & \text{——} & (3,0)
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The path is determined by the sequence of "directions" and "stops": $x = 1|21|1|2$.

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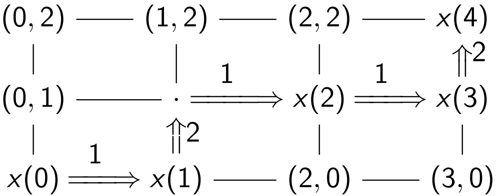

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$\mathcal{L}_u(m,n) = \mathrm{Cat}_{*,*}([n+1],[m+1]) = \Delta([m],[n])$.

Proposition

- The lattice path operad $\mathcal{L}$ is *filtered* by *complexity*, i.e. by the number of angles of the lattice paths;
- $\mathcal{L}^{(0)}$-algebras are cosimplicial objects;
- $\mathcal{L}^{(1)}$-algebras are $\square$-monoids in cosimplicial objects;
- $\mathcal{L}^{(2)}$-algebras are multiplicative operads. (Tamarkin)

Theorem

For the standard cosimplicial object δ in spaces or in chain complexes, δ-condensation of $\mathcal{L}^{(n)}$ yields an E_n-operad.

For $n = 2$ we get the previous theorem.

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