Higher complements of combinatorial sphere arrangements

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Combinatorial Structures in Algebra and Topology
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A (central) *hyperplane arrangement* $\mathcal{A}$ in euclidean space V is a finite family $(H_\alpha)_{\alpha\in\mathcal{A}}$ of hyperplanes of V containing the origin. The arrangement is *essential* if its center $\bigcap_{\alpha\in\mathcal{A}} H_\alpha$ is trivial.

The *complement* $\mathcal{M}(\mathcal{A}) = V\setminus(\bigcup_{\alpha\in\mathcal{A}} H_\alpha)$ decomposes into path components, called *chambers* (or *topes*): $\mathcal{C}_\mathcal{A} = \pi_0(\mathcal{M}(\mathcal{A}))$.

Denote by s_α the *orthogonal symmetry* with respect to H_α. If $(H_\alpha)_{\alpha\in\mathcal{A}}$ is stable under s_β for all $\beta\in\mathcal{A}$, the arrangement is called a *Coxeter arrangement*. We write $\mathcal{A} = \mathcal{A}_W$ where W is the subgroup $W = < s_\alpha, \alpha\in\mathcal{A} >$ of $O_n(\mathbb{R})$. This is justified by

Proposition (Coxeter, Tits)

There is a one-to-one correspondence between essential Coxeter arrangements $\mathcal{A}_W$ and finite Coxeter groups W. The latter are classified by their Coxeter diagrams.

The Coxeter group W acts simply transitively on $\mathcal{C}_{\mathcal{A}_W}$.

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Definition

The *k-th complement* of a hyperplane arrangement $\mathcal{A}$ is

$$\mathcal{M}_k(\mathcal{A}) = V^k \setminus \bigcup_{\alpha \in \mathcal{A}} (H_\alpha)^k.$$

Example

$V = \mathbb{R}^n$, $\mathcal{A} = (H_{ij})_{1 \leq i < j \leq n}$ where $H_{ij} = \{x \in \mathbb{R}^n | x_i = x_j\}$. This is the Coxeter arrangement $\mathcal{A}_{\mathfrak{S}_n}$ for the *symmetric group* $\mathfrak{S}_n$. The center is $\mathbb{R}.(1, \ldots, 1)$. The higher complements are configuration spaces: $\mathcal{M}_k(\mathcal{A}_{\mathfrak{S}_n}) = F(\mathbb{R}^k, n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{kn} | x_i \neq x_j\}$.

Proposition (Brieskorn '71)

$\pi_1(\mathcal{M}_2(\mathcal{A}_W)) = Ker(A_W \to W)$ (the *pure Artin group* of W).

Theorem (Deligne '72)

For any simplicial arrangement, $\mathcal{M}_2(\mathcal{A})$ is aspherical.

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Purpose of the talk

Define a *finite cell complex* $\mathcal{S}^{(k)}_{\mathcal{A}}$ of the homotopy type of $\mathcal{M}_k(\mathcal{A})$.

- Fox-Neuwirth '62 and Milgram '66 construct $\mathcal{S}^{(k)}_{\mathcal{A}_{\mathfrak{S}_n}}$ for any k;
- Salvetti '87 constructs $\mathcal{S}^{(2)}_{\mathcal{A}}$ for any arrangement $\mathcal{A}$.

Theorem (Randell '02, Dimca-Papadima '03, S-S '07)

The complement of a complex hyperplane arrangement admits a *minimal CW-structure*. The minimal CW-structure of $\mathcal{M}_2(\mathcal{A})$ derives from $\mathcal{S}^{(2)}_{\mathcal{A}}$ through combinatorial Morse theory.

Remark (Gel'fand-Rybnikov '90)

The complex $\mathcal{S}^{(2)}_{\mathcal{A}}$ only depends on the *oriented matroid* $\mathcal{F}_{\mathcal{A}}$ of $\mathcal{A}$.

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Orient a hyperplane arrangement $\mathcal{A}$ in V, by choosing for each H_α two half-spaces $H_\alpha^\pm$ such that $H_\alpha^+ \cap H_\alpha^- = H_\alpha$ and $H_\alpha^+ \cup H_\alpha^- = V$. Then each point $x \in V$ defines a sign vector $sgn_x \in \{0, \pm\}^{\mathcal{A}}$ by

$$sgn_x(\alpha) = \begin{cases} 0 & \text{if } x \in H_\alpha; \\ \pm & \text{if } x \in H_\alpha^\pm \backslash H_\alpha. \end{cases}$$

The *oriented matroid* $\mathcal{F}_\mathcal{A} \subset \{0, \pm\}^{\mathcal{A}}$ is the set of all such sign vectors sgn_x, $x \in V$, equipped with the *partial order* induced from the product order on $\{0, \pm\}^{\mathcal{A}}$ where $0 < +$ and $0 < -$.

Each $P \in \mathcal{F}_\mathcal{A}$ defines a *facet* $c_P = \{x \in V | sgn_x = P\}$. The facets are convex subsets of V, open in their closure. By definition,

$$\overline{c}_P \subseteq \overline{c}_Q \text{ in } V \text{ iff } P \leq Q \text{ in } \mathcal{F}_\mathcal{A}.$$

The unit-sphere S_V gets a *CW-structure* with cell poset $\mathcal{F}_\mathcal{A} \backslash \{0\}$.

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The subset $\mathcal{F}_\mathcal{A} \subset \{0, \pm\}^\mathcal{A}$ of sign vectors of the arrangement $\mathcal{A}$ fulfills the following defining properties of an *oriented matroid*:

1. $0 \in \mathcal{F}_\mathcal{A}$;
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3. $P, Q \in \mathcal{F}_\mathcal{A}$ implies $PQ \in \mathcal{F}_\mathcal{A}$;
4. Any $\alpha \in \mathcal{A}$ which separates $P, Q \in \mathcal{F}_\mathcal{A}$ supports an $R \in \mathcal{F}_\mathcal{A}$ sth. $R(\beta) = (PQ)(\beta) = (QP)(\beta)$ for non separating $\beta \in \mathcal{A}$.

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For $P, Q \in \mathcal{F}_\mathcal{A}$ we define a sign vector $PQ \in \{0, \pm\}^\mathcal{A}$ by

$$(PQ)(\alpha) = \begin{cases} P(\alpha) & \text{if } P(\alpha) \neq 0; \\ Q(\alpha) & \text{if } P(\alpha) = 0. \end{cases}$$

The subset $\mathcal{F}_\mathcal{A} \subset \{0, \pm\}^\mathcal{A}$ of sign vectors of the arrangement $\mathcal{A}$ fulfills the following defining properties of an *oriented matroid*:

1. $0 \in \mathcal{F}_\mathcal{A}$;
2. $P \in \mathcal{F}_\mathcal{A}$ implies $-P \in \mathcal{F}_\mathcal{A}$;
3. $P, Q \in \mathcal{F}_\mathcal{A}$ implies $PQ \in \mathcal{F}_\mathcal{A}$;
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A *sphere arrangement* in V is a collection $(S_\alpha)_{\alpha\in\mathcal{A}}$ of centrally symmetric subspheres of codimension one of S_V such that

1. The closures $S_\alpha^\pm$ of the two components of $S_V \backslash S_\alpha$ are balls;
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Theorem (Folkman-Lawrence '78, Edmonds-Mandel '78)

Any simple oriented matroid $\mathcal{F}_\mathcal{A} \subset \{0,\pm\}^\mathcal{A}$ is the oriented matroid of an essentially unique sphere arrangement in $V = \mathbb{R}^{\mathrm{rk}(\mathcal{F}_\mathcal{A})}$.

Definition

The k-th complement of a sphere arrangement $(S_\alpha)_{\alpha\in\mathcal{A}}$ in V is

$$\mathcal{M}_k(\mathcal{A}) = V^k \backslash \bigcup_{\alpha\in\mathcal{A}} (\mathbb{R}.S_\alpha)^k \simeq \underbrace{S_V * \cdots * S_V}_{k} \backslash \bigcup_{\alpha\in\mathcal{A}} \underbrace{S_\alpha * \cdots * S_\alpha}_{k}.$$

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The chamber system $\mathcal{C}_{\mathcal{A}}$ is the *discrete* subposet of $\mathcal{F}_{\mathcal{A}}$ consisting of the *maximal* facets. In particular, $|\mathcal{C}_{\mathcal{A}}| \simeq \mathcal{M}(\mathcal{A})$.

$\mathcal{F}_{\mathcal{A}} \times \mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A} \oplus \mathcal{A}}$ where $\mathcal{A} \oplus \mathcal{A} = (\mathcal{A} \times V) \cup (V \times \mathcal{A})$ in $V \times V$.

Definition (Orlik '91)

$\mathcal{C}^{(2)}_{\mathcal{A}} := \{(P, Q) \in \mathcal{F}_{\mathcal{A}} \times \mathcal{F}_{\mathcal{A}} \mid PQ \in \mathcal{C}_{\mathcal{A}}\}^{\mathrm{op}}$

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For subcomplexes K_1, K_2 of a simplicial complex L sth. $Vert(L) = Vert(K_1) \sqcup Vert(K_2)$, one has: $|L| \backslash |K_1| \simeq |K_2|$. Thus,

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Definition (Salvetti '87)

$\mathcal{S}_{\mathcal{A}}^{(2)} = \{(P, C) \in \mathcal{F}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{A}} \mid P \leq C\}$
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Theorem (Salvetti '87, Arvola '91)

$|\mathcal{S}_{\mathcal{A}}^{(2)}| \simeq \mathcal{M}_2(\mathcal{A})$.

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The map $(P, Q) \mapsto (P, PQ)$ is a hpty eq. of posets $\mathcal{C}_{\mathcal{A}}^{(2)} \xrightarrow{\sim} \mathcal{S}_{\mathcal{A}}^{(2)}$.
Indeed, by Quillen's Theorem A, it suffices to show that the hpty fibers $c_{(P,C)} = \{Q \in \mathcal{F}_{\mathcal{A}} \mid PQ \leq C\}$ are contractible.
For $\mathcal{A}_{|P|} = \{\alpha \in \mathcal{A} \mid P(\alpha) = 0\}$ we get the identification $c_{(P,C)} = \{Q \in \mathcal{F}_{\mathcal{A}} \mid Q(\alpha) \leq C(\alpha), \alpha \in \mathcal{A}_{|P|}\}$. Thus, $c_{(P,C)}$ maps to the closure of a chamber in $\mathcal{F}_{\mathcal{A}/|P|}$ via $\mathcal{F}_{\mathcal{A}} \setminus \mathcal{F}_{|P|} \simeq \mathcal{F}_{\mathcal{A}/|P|}$. □

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Let $st_{(P,C)} = \{(x_1, x_2) \in V \times V \mid x_1 \in c_P;\ x_2 \in c_C \bmod |P|\}$. These are *convex* subsets of $\mathcal{M}_2(\mathcal{A})$, open in their closure. They define a *stratification* of $\mathcal{M}_2(\mathcal{A})$ labelled by $\mathcal{S}^{(2)}_{\mathcal{A}}$ such that

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The intersection of two closed strata is a union of closed strata. Any closed stratum is contractible. Moreover, inclusions of closed strata are closed cofibrations. This implies by a homotopy colimit argument (Reedy '73) that $\mathcal{M}_2(\mathcal{A}) \simeq |\mathcal{S}_{\mathcal{A}}^{(2)}|$.

Equivalently, let $V_{(P,C)} = \bigcup_{(P,C)\geq(P',C')} st_{(P',C')}$. This defines an *open* cover of $\mathcal{M}_2(\mathcal{A})$ used by Deligne '72. The $V_{(P,C)}$ are contractible and $V_{(P,C)} \subseteq V_{(P',C')}$ iff $(P,C) \leq (P',C')$. Moreover, each $V_{(P,C)} \cap V_{(P',C')}$ is a union of $V_{(P'',C'')}$'s. A homotopy colimit argument (McCord '67) yields $\mathcal{M}_2(\mathcal{A}) \simeq |\mathcal{S}_{\mathcal{A}}^{(2)}|$.

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Definition

$$\mathcal{C}_{\mathcal{A}}^{(k)} = \{(P_1, \ldots, P_k) \in (\mathcal{F}_{\mathcal{A}})^k \mid P_1 \cdots P_k \in \mathcal{C}_{\mathcal{A}}\}^{\mathrm{op}}$$
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Theorem

$|\mathcal{C}_{\mathcal{A}}^{(k)}| \simeq \mathcal{M}_k(\mathcal{A})$ and $(P_1, \ldots, P_k) \mapsto (P_1, P_1 P_2, \ldots, P_1 P_2 \cdots P_k)$ defines a homotopy equivalence of posets $\mathcal{C}_{\mathcal{A}}^{(k)} \xrightarrow{\simeq} \mathcal{S}_{\mathcal{A}}^{(k)}$.

Proof.

The homotopy fibers $c_{(P_1, \ldots, P_{k-1}, C)}$ are homotopy colimits over $\{Q \mid P_1 Q \leq P_2\}$ of homotopy fibers $c_{(P_2, \ldots, P_{k-1}, C)}$. □

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$\mathcal{S}_{\mathcal{A}}^{(k)} \cong \{(C, \mu) \in \mathcal{C}_{\mathcal{A}} \times \{0, 1, \ldots, k-1\}^{\mathcal{A}} \,|\, \mu \text{ is } C\text{-admissible}\}$,

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Corollary

For *simplicial* arrangements, $\mathcal{S}_{\mathcal{A}}^{(k)} \cong \mathcal{C}_{\mathcal{A}} \times \{0, \ldots, k-1\}^{\mathrm{rk}(\mathcal{F}_{\mathcal{A}})}$.
For *Coxeter* arrangements, $\mathcal{S}_{\mathcal{A}_W}^{(k)} \cong W \times \{0, \ldots, k-1\}^{\mathrm{rk}(W)}$.

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For each $\mathcal{A}$, the *adjacency graph* $\mathcal{G}_{\mathcal{A}}$ has *vertex set* $\mathcal{C}_{\mathcal{A}}$ and *edge set* $\{(C, C') \in \mathcal{C}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{A}} \mid \exists P \in \mathcal{F}_{\mathcal{A}} : P \prec C \text{ and } P \prec C'\}$. Since $P(\alpha) = 0$ for a unique $\alpha \in \mathcal{A}$, the edges of $\mathcal{G}_{\mathcal{A}}$ are labelled by $\mathcal{A}$.

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Proposition (Björner-Edelman-Ziegler '90)

The face poset $\mathcal{F}_{\mathcal{A}}$ is determined by the adjacency graph $\mathcal{G}_{\mathcal{A}}$.

Definition

Let $E_{\mathcal{A}}$ be the simplicial set whose d-simplices are $(d+1)$-tupels $(C_0, C_1, \ldots, C_d)$ of chambers. $(C_0, C_1, \ldots, C_d) \in E_{\mathcal{A}}^{(k)}$ iff $(S(C_0, C_1), \ldots, S(C_{d-1}, C_d))$ contains $< k$ times each $\alpha \in \mathcal{A}$.

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- $E_{\mathcal{A}\oplus\mathcal{B}} \cong E_{\mathcal{A}} \times E_{\mathcal{B}}$ compatible with filtrations.

Theorem (Smith '89, Kashiwabara '93, B. '96)

$|E_{\mathcal{A}_{\mathfrak{S}_n}}^{(k)}| \simeq \mathcal{M}_k(\mathcal{A}_{\mathfrak{S}_n})$. For varying n, the operad on the left has the homotopy type of Boardman-Vogt's *operad of little k-cubes*.

Conjecture (Fiedorowicz)

For any finite Coxeter group W, one has $|E_{\mathcal{A}_W}^{(k)}| \simeq \mathcal{M}_k(\mathcal{A}_W)$.

This would extend the operad structure of the B/C/D-Coxeter groups to the higher complements of their Coxeter arrangement.

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This would extend the operad structure of the B/C/D-Coxeter groups to the higher complements of their Coxeter arrangement.

- $E_{\mathcal{A}}$ is contractible, filtered by simplicial subsets $E_{\mathcal{A}}^{(k)}$;
- $E_{\mathcal{A}_W} = EW$ and $E_{\mathcal{A}_W}/W = BW$;
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- $E_{\mathcal{A}\oplus\mathcal{B}} \cong E_{\mathcal{A}} \times E_{\mathcal{B}}$ compatible with filtrations.

Theorem (Smith '89, Kashiwabara '93, B. '96)

$|E_{\mathcal{A}_{\mathfrak{S}_n}}^{(k)}| \simeq \mathcal{M}_k(\mathcal{A}_{\mathfrak{S}_n})$. For varying n, the operad on the left has the homotopy type of Boardman-Vogt's *operad of little k-cubes*.

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