Dold-Kan categories & Catalan monoids

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CATS60 – celebrating Carlos Simpson’s 60th birthday

\textsuperscript{1}joint with Christophe Cazanave and Ingo Waschkies
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Theorem (Dold 1958, Kan 1958)

\[ M : \text{Ab}^\Delta^{\text{op}} \cong \text{Ch}(\mathbb{Z}) : K \]

Remark

The functor \( K \) takes homology to homotopy. The \( K \)-image of the chain complex \((A, n) = (0 \leftarrow \cdots \leftarrow 0 \leftarrow A_n \leftarrow 0 \leftarrow \cdots)\) is a simplicial model for an Eilenberg-MacLane space of type \( K(A, n) \).

Purpose of the talk

- categorical explanation for Dold-Kan correspondence
- chain models for \( K(A, n) \)'s via Joyal's cell categories \( \Theta_n \)
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**Definition (simplex category \( \Delta \))**

\[ \text{Ob} \Delta = \{ [n] = \{0, 1, \ldots, n\}, \, n \geq 0 \}, \, \text{Mor} \Delta = \{ \text{monotone maps} \} \]

**Remark (\( \mathcal{E}-\mathcal{M} \) factorisation system)**

The category \( \Delta \) is generated by elementary:

- face operators \( \epsilon_i^n : [n - 1] \to [n], \, 0 \leq i \leq n, \) and
- degeneracy operators \( \eta_i^n : [n + 1] \to [n], \, 0 \leq i \leq n. \)

Every simplicial operator \( \phi : [m] \to [n] \) factors as

\[
\begin{array}{ccc}
[m] & \xrightarrow{\phi} & [n] \\
\downarrow \downarrow \downarrow epi & & \downarrow \downarrow \downarrow mono \\
[p] & & [n]
\end{array}
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```
  [m] --\phi--> [n]
     | epi       mono
     v
    [p]
```
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\[\phi\]
Definition (Milnor 1957 – geometric realisation)

The functor $Δ \to \text{Top} : [n] \mapsto Δ_n$ yields by left Kan extension geometric realisation $|−|_Δ : \text{Sets}^{Δ^\text{op}} \to \text{Top}$. Each $|X|$ is a $CW$-complex with one cell per non-degenerate simplex of $X$.

Definition (Eilenberg 1944 – simplicial homology)

$$\begin{aligned}
\text{Sets}^{Δ^\text{op}} &\xrightarrow{\text{C}} \text{Ab}^{Δ^\text{op}} &\xrightarrow{C} &\text{Ch}(\mathbb{Z}) &\xrightarrow{\epsilon} &\text{Ab}^\mathbb{N} \\
X_• &\xrightarrow{\epsilon} &\mathbb{Z}[X_•] &\xrightarrow{\epsilon} & (C_•(X), d_•) &\xrightarrow{\epsilon} &H_•(X)
\end{aligned}$$

There are canonical isomorphisms

$$C_n^{\text{cell}}(|X|) \cong C_n(X) = \mathbb{Z}[X_n]/\mathbb{Z}[D_n(X)] \cong \bigcap_{0 \leq k < n} \ker(\epsilon^n_k) = M_n(X)$$
Dold-Kan categories & Catalan monoids
The simplex category $\Delta$

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$$\begin{align*}
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$\begin{align*}
\text{Sets}^{\Delta^{\text{op}}} & \longrightarrow \text{Ab}^{\Delta^{\text{op}}} \xrightarrow{C} \text{Ch}(\mathbb{Z}) \longrightarrow \text{Ab}^N \\
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Definition (Dold-Kan category)

\( \mathcal{C} = (\mathcal{E}, \mathcal{M}, (\cdot)^*) \) is a DK-category whenever \((\cdot)^* : \mathcal{E}^{\text{op}} \to \mathcal{M}\) is a faithful identity-on-objects functor sth.

1. \(ee^* = 1\) (the idempotent \(e^*e\) is called an \(E\)-projector);
2. the morphisms \(f^*e\) (for \(e, f \in \mathcal{E}\)) form a subcategory of \(\mathcal{C}\);
3. Inessential \(M\)-maps form an ideal in \(\mathcal{M}\);
4. \(\text{Proj}_E(A)\) is finite. Primitive \(E\)-projectors can be enumerated in such a way that \(\phi_j\phi_i\) is an \(E\)-projector for \(i < j\).

Definition (primitive \(E\)-projectors \(e^*e\))

Whenever \(e = e_2e_1\) then either \(e_1\) or \(e_2\) is invertible.

Definition (essential and inessential \(M\)-maps)

An \(M\)-map \(m : A \to B\) is called essential if \(1_B\) is the only \(E\)-projector of \(B\) fixing \(m\). Otherwise \(m\) is called inessential.
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Dold-Kan categories & Catalan monoids
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\( \mathcal{C} = (\mathcal{E}, \mathcal{M}, (\_\_)^*) \) is a DK-category whenever \((\_\_)^* : \mathcal{E}^{op} \to \mathcal{M}\) is a faithful identity-on-objects functor such that

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**Remark (DK-category structure for $\Delta$)**

Each epi $e : [m] \to [n]$ has a *maximal* section $e^* : [n] \to [m]$. The primitive $E$-projectors of $[n]$ are the $\eta_i^* \eta_i = \epsilon_i \eta_i$, $0 \leq i < n$.

**Remark (essential $M$-maps of $\Delta$)**

are precisely the "last" face operators $\epsilon_n^* : [n - 1] \to [n]$.

**Lemma (quotienting out inessential $M$-maps)**

By axiom (3), there is a *locally pointed* category $\Xi_C = M / M_{iness}$.

**Remark (description of $\Xi_\Delta = M / M_{iness}$)**

$$
\begin{array}{cccccc}
0 & \to & 0 & \to & 0 & \to \\
[0] & \leftrightarrow & [1] & \leftrightarrow & [2] & \leftrightarrow \\
& & & \sim & \Xi^{-\infty} \Delta \to \text{Ab}_\star = \text{Ch}(\mathbb{Z})
\end{array}
$$
Remark (DK-category structure for $\Delta$)

Each epi $e : [m] \to [n]$ has a maximal section $e^* : [n] \to [m]$.
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\[ 0 \rightarrow [1] \rightarrow [2] \rightarrow [3] \rightarrow [4] \cdots \sim [\Xi_\Delta^{op}, \text{Ab}]_* = \text{Ch}(\mathbb{Z}) \]
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Remark (description of $\Xi_\Delta = \mathcal{M}/\mathcal{M}_{inest}$)

$[0] \xrightarrow{0} [1] \xrightarrow{0} [2] \xrightarrow{0} [3] \xrightarrow{0} [4] \cdots \sim [\Xi_\Delta^{\text{op}}, \text{Ab}]_* = \text{Ch}(\mathbb{Z})$
Remark (DK-category structure for \( \Delta \))

Each epi \( e : [m] \rightarrow [n] \) has a maximal section \( e^* : [n] \rightarrow [m] \).
The primitive \( \mathcal{E} \)-projectors of \([n]\) are the \( \eta_i^* \eta_i = \epsilon_i \eta_i, \ 0 \leq i < n \).

Remark (essential \( \mathcal{M} \)-maps of \( \Delta \))

are precisely the “last” face operators \( \epsilon_n : [n - 1] \hookrightarrow [n] \).

Lemma (quotienting out inessential \( \mathcal{M} \)-maps)

By axiom (3), there is a locally pointed category \( \Xi_c = \mathcal{M}/\mathcal{M}_{iness} \).

Remark (description of \( \Xi_\Delta = \mathcal{M}/\mathcal{M}_{iness} \))

\[
\begin{array}{ccccccc}
\end{array}
\]
Remark (DK-category structure for $\Delta$)

Each epi $e : [m] \rightarrow [n]$ has a maximal section $e^* : [n] \rightarrow [m]$. The primitive $E$-projectors of $[n]$ are the $\eta_i^* \eta_i = \epsilon_i \eta_i$, $0 \leq i < n$.

Remark (essential $M$-maps of $\Delta$)

are precisely the “last” face operators $\epsilon_n^n : [n - 1] \hookrightarrow [n]$.

Lemma (quotienting out inessential $M$-maps)

By axiom (3), there is a locally pointed category $\Xi_C = M/M_{iness}$.

Remark (description of $\Xi_\Delta = M/M_{iness}$)

$$0 \rightarrow [1] \rightarrow [2] \rightarrow [3] \rightarrow [4] \cdots \sim [\Xi_{\Delta}^{\text{op}}, \text{Ab}]_* = \text{Ch}(\mathbb{Z})$$
Remark (DK-category structure for $\Delta$)

Each epi $e : [m] \to [n]$ has a maximal section $e^* : [n] \to [m]$. The primitive $E$-projectors of $[n]$ are the $\eta_i^* \eta_i = \epsilon_i \eta_i$, $0 \leq i < n$.

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$$
\begin{array}{c}
0 \\
\longrightarrow [1] \\
\longrightarrow [2] \\
\longrightarrow [3] \\
\longrightarrow [4] \cdots \\
\sim \Xi_{\Delta}^{op}, \text{Ab}_* = \text{Ch}(\mathbb{Z})
\end{array}
$$
Theorem (generalised Dold-Kan correspondence, BCW 2022)

For each Dold-Kan category $\mathcal{C} = (\mathcal{E}, \mathcal{M}, (\cdot)^*)$ and each abelian category $\mathcal{A}$ there is an adjoint equivalence

$$M_\mathcal{C} : [\mathcal{C}^{\text{op}}, \mathcal{A}] \simeq [\Xi^{\text{op}}_\mathcal{C}, \mathcal{A}]_* : K_\mathcal{C}$$

Remark (constructing $M_\mathcal{C}$ and $K_\mathcal{C}$ for general DK-categories $\mathcal{C}$)

Denote $j : \mathcal{M} \hookrightarrow \mathcal{C}$ and $q : \mathcal{M} \twoheadrightarrow \Xi_\mathcal{C} = \mathcal{M}/\mathcal{M}_{\text{iness}}$. Then

$$M_\mathcal{C} : [\mathcal{C}^{\text{op}}, \mathcal{A}] \overset{j^*}{\cong} [\mathcal{M}^{\text{op}}, \mathcal{A}] \overset{q_*}{\cong} [\Xi^{\text{op}}_\mathcal{C}, \mathcal{A}]_* : K_\mathcal{C}$$

Examples

- $\Gamma$ (Pirashvili 2000) and $\text{Fl}_{\mathbb{A}}$ (Church-Ellenberg-Farb 2015)
- $\Omega_{\text{planar}}$ (Gutierrez-Lukacs-Weiss 2011)
**Theorem (generalised Dold-Kan correspondence, BCW 2022)**

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**Examples**

- $\Gamma$ (Pirashvili 2000) and $\text{Fl}^\sharp$ (Church-Ellenberg-Farb 2015)
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\[
M_{\mathcal{C}} : [C^{\text{op}}, \mathcal{A}] \cong [\Xi_{\mathcal{C}}^{\text{op}}, \mathcal{A}]_* : K_{\mathcal{C}}
\]

Remark (constructing \( M_{\mathcal{C}} \) and \( K_{\mathcal{C}} \) for general DK-categories \( \mathcal{C} \))

Denote \( j : M \hookrightarrow \mathcal{C} \) and \( q : M \twoheadrightarrow \Xi_{\mathcal{C}} = M/M_{\text{iness}} \). Then

\[
M_{\mathcal{C}} : [C^{\text{op}}, \mathcal{A}] \xrightarrow{j^*} [M^{\text{op}}, \mathcal{A}] \xrightarrow{q_*} [\Xi_{\mathcal{C}}^{\text{op}}, \mathcal{A}]_* : K_{\mathcal{C}}
\]

Examples

- \( \Gamma \) (Pirashvili 2000) and \( \text{FI} \# \) (Church-Ellenberg-Farb 2015)
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Examples

- $\Gamma$ (Pirashvili 2000) and $Fl$ (Church-Ellenberg-Farb 2015)
- $\Omega_{planar}$ (Gutierrez-Lukacs-Weiss 2011)
**Theorem (generalised Dold-Kan correspondence, BCW 2022)**

For each Dold-Kan category $\mathcal{C} = (\mathcal{E}, \mathcal{M}, (-)^*)$ and each abelian category $\mathcal{A}$ there is an adjoint equivalence

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$$\mathcal{M}_\mathcal{C} : [C^{\text{op}}, \mathcal{A}] \xleftarrow{j^*} [\mathcal{M}^{\text{op}}, \mathcal{A}] \xrightarrow{j!} [\Xi_{\mathcal{C}}^{\text{op}}, \mathcal{A}]_* : \mathcal{K}_\mathcal{C}$$

**Examples**

- $\Gamma$ (Pirashvili 2000) and $\text{Fl}^!$ (Church-Ellenberg-Farb 2015)
- $\Omega_{\text{planar}}$ (Gutierrez-Lukacs-Weiss 2011)
**Theorem (generalised Dold-Kan correspondence, BCW 2022)**

For each Dold-Kan category $\mathcal{C} = (\mathcal{E}, \mathcal{M}, (-)^*)$ and each abelian category $\mathcal{A}$ there is an adjoint equivalence

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**Examples**

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For each Dold-Kan category $C = (\mathcal{E}, \mathcal{M}, (-)^\ast)$ and each abelian category $\mathcal{A}$ there is an adjoint equivalence

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Remark (constructing $M_C$ and $K_C$ for general DK-categories $C$)

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Examples

- $\Gamma$ (Pirashvili 2000) and $\text{FI}^{\mathbb{N}}$ (Church-Ellenberg-Farb 2015)
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### Definition (wreath product over $\Delta$)

For any small category $\mathcal{A}$ the category $\Delta \wr \mathcal{A}$ is defined by

- $\text{Ob}(\Delta \wr \mathcal{A}) = \bigsqcup_{n \geq 0} \mathcal{A}^n = \{(\begin{bmatrix} m \end{bmatrix}; A_1, \ldots, A_m)\}$
- $(\phi; \phi_{ij}): (\begin{bmatrix} m \end{bmatrix}, A_1, \ldots, A_m) \to (\begin{bmatrix} n \end{bmatrix}, B_1, \ldots, B_n)$ is given by $\phi: \begin{bmatrix} m \end{bmatrix} \to \begin{bmatrix} n \end{bmatrix}$ and $\phi_{ij}: A_i \to B_j$ whenever $\phi(i - 1) < j \leq \phi(i)$

### Definition (B 2007, cf. Joyal 1997)

Put $\Theta_1 = \Delta$ and $\Theta_n = \Delta \wr \Theta_{n-1}$ for $n > 1$.

### Theorem (Makkai-Zawadowski 2003, B 2003)

$\Theta_n$ fully embeds in $\mathcal{nCat}$, inducing a fully faithful nerve functor

$$\mathcal{nCat} \hookrightarrow \text{Sets}^{\Theta_n^{\text{op}}}$$
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![Diagram of 2-categorical structure](image)

Proposition (full embedding $\Theta_n \hookrightarrow n\text{Cat}$)

$\Theta_n(S, T) = n\text{Cat}(F_n(S_*), F_n(T_*))$

where $F_n : n\text{Grph} \rightarrow n\text{Cat}$ is left adjoint to the forgetful functor.
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$\tau_8 \rightarrow \tau_4 \rightarrow \tau_1 \downarrow \tau_6 \rightarrow \tau_2 \downarrow \tau_3 \rightarrow \tau_7 \rightarrow \tau_9 \rightarrow \tau_5 \rightarrow \tau_2 \downarrow \tau_8 \rightarrow \tau_3 \rightarrow \tau_1 \rightarrow \tau_7 \rightarrow \tau_3$

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**Definition (geometric DK-categories)**

A DK-category $\mathcal{C} = (\mathcal{E}, \mathcal{M}, (\cdot)^*)$ is called *geometric* if

1. the $\mathcal{E}$-quotients of any object form a *lattice*
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3. $\mathcal{C}$ has a terminal object without proper $\mathcal{M}$-faces

**Proposition (CW-realisation)**

Any presheaf $X : \mathcal{C}^{\text{op}} \to \text{Sets}$ on a geometric DK-category has *CW*-realisation $|X|$ whose chain complex $C^\text{cell}_*(|X|)$ is isomorphic to the “totalisation” of the Moore normalisation $M_C(\mathbb{Z}[X])$.

**Proposition (B 2007, Bergner-Rezk 2017, BCW 2022)**

If $\mathcal{A}$ is a geometric DK-category then so is $\Delta \wr \mathcal{A}$.

For instance, Joyal’s cell category $\Theta_n$ is a geometric DK-category.
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Proposition (CW-realisation)

Any presheaf $X : C^{\text{op}} \to \text{Sets}$ on a geometric DK-category has CW-realisation $|X|$ whose chain complex $C^\text{cell}_* (|X|)$ is isomorphic to the “totalisation” of the Moore normalisation $M_C(\mathbb{Z}[X])$.

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Dold-Kan categories & Catalan monoids
Joyal’s categories $\Theta_n$

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Theorem (Dold-Kan correspondence for $\Theta_n$)

$$M_{\Theta_n} : A^{\Theta_n^{op}} \simeq [\Xi_{\Theta_n}^{op}, A]_* : K_{\Theta_n}$$

Remark ($\Theta_n$-set model for Eilenberg-MacLane spaces)

- For each abelian group $A$ there is a strict $n$-category $B^n A$ with one $k$-cell for $0 \leq k < n$ and $A$ as endo-$n$-object;
- $|B^n A|_{\Theta_n}$ is a CW-complex of type $K(A, n)$;
- $C^\text{cell}_*(|B^n A|_{\Theta_n})$ is the “totalisation” of $M_{\Theta_n} (\mathbb{Z}[B^n A])$.

Example (# cells of $K(\mathbb{Z}/2\mathbb{Z}, n) = \text{generalised Fibonacci number}$)

<table>
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<tr>
<th># cells in dim</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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Dold-Kan categories & Catalan monoids
Joyal’s categories $\Theta_n$

**Theorem (Dold-Kan correspondence for $\Theta_n$)**

$$M_{\Theta_n} : A^{\Theta_n^{\text{op}}} \simeq [\Xi_{\Theta_n}^{\text{op}}, A]_* : K_{\Theta_n}$$

**Remark ($\Theta_n$-set model for Eilenberg-MacLane spaces)**

- For each abelian group $A$ there is a strict $n$-category $B^n A$
  with one $k$-cell for $0 \leq k < n$ and $A$ as endo-$n$-object;
- $|B^n A|_{\Theta_n}$ is a CW-complex of type $K(A, n)$;
- $C_{cell}^*(|B^n A|_{\Theta_n})$ is the “totalisation” of $M_{\Theta_n}(\mathbb{Z}[B^n A])$.

**Example (# cells of $K(\mathbb{Z}/2\mathbb{Z}, n)$ = generalised Fibonacci number)**

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$$\mathcal{M}_{\Theta_n} : \mathcal{A}^{\Theta_n^{\text{op}}} \simeq [\Xi_{\Theta_n}^{\text{op}}, \mathcal{A}]_* : K_{\Theta_n}$$

Remark ($\Theta_n$-set model for Eilenberg-MacLane spaces)

- For each abelian group $A$ there is a strict $n$-category $B^n A$ with one $k$-cell for $0 \leq k < n$ and $A$ as endo-$n$-object;
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\[
M_{\Theta_n} : \mathcal{A}^{\Theta_n^{\text{op}}} \cong [\Xi_{\Theta_n^{\text{op}}}, \mathcal{A}]_* : K_{\Theta_n}
\]

**Remark (\( \Theta_n \)-set model for Eilenberg-MacLane spaces)**

- For each abelian group \( A \) there is a strict \( n \)-category \( B^nA \)
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Example (action of $\Xi_{\Theta_2}$ on $C^\text{cell}_*(K(\mathbb{Z}/2, 2))$ for $2 \leq * \leq 6$)

\[
\begin{align*}
(1; 5) & \leftarrow (1; 4) \leftarrow (2; 3, 1) \\
(1; 1) & \leftarrow (1; 2) \leftarrow (1; 3) \leftarrow (2; 2, 1) \leftarrow (2; 2, 2) \\
(2; 1, 1) & \leftarrow (2; 1, 2) \leftarrow (2; 1, 3) \leftarrow (3; 1, 1, 1)
\end{align*}
\]

Theorem (Serre 1953)

\[H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[Sq^J(\iota_2), \ J \text{ admissible, } e(J) < n]\]

Each $Sq^J(\iota_2)$ is represented by an admissible cocycle of ht $n$. 
Example (action of $\Xi_{\Theta_2}$ on $C^\text{cell}_{*}(K(\mathbb{Z}/2, 2))$ for $2 \leq * \leq 6$)

(1; 5)

(1; 4) ← (2; 3, 1)

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$H^*(K(\mathbb{Z}/2\mathbb{Z}, n); \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[Sq^J(\nu_2), J \text{ admissible, } e(J) < n]$  

Each $Sq^J(\nu_2)$ is represented by an admissible cocycle of ht $n$. 
Example (action of $\Xi \Theta_2$ on $C^\text{cell}_\ast(K(\mathbb{Z}/2, 2))$ for $2 \leq \ast \leq 6$)

\[
\begin{align*}
(1; 5) & \searrow \quad (1; 4) \quad (2; 3, 1) \\
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(2; 1, 1) \quad (2; 1, 2) & \quad (2; 1, 3) \\
& \quad (3; 1, 1, 1)
\end{align*}
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Each $Sq^J(\nu_2)$ is represented by an admissible cocycle of ht $n$. 
Proposition

Let \((x_i)_{1 \leq i \leq n}\) be a family of projectors of an \(R\)-module \(X\) such that \(x_i x_j x_i = x_i x_j = x_j x_i x_j\) for \(i < j\). Then we get a direct sum decomposition \(X = N_X \oplus D_X := \bigcap_{1 \leq i \leq n} \ker(x_i) \oplus \sum_{1 \leq i \leq n} \text{im}(x_i)\).

Corollary

Let \(X : \mathcal{C}^{\text{op}} \to \mathcal{A}\) be a presheaf on a Dold-Kan category \(\mathcal{C}\) with \(\mathcal{A}\) abelian. Then, for each object \(A\) of \(\mathcal{C}\), we get

\[
X(A) = N_{X(A)} \oplus D_{X(A)} = \bigcap_{\phi \in \text{Prim}_\varepsilon(A)} \ker(X(\phi)) \oplus \sum_{\phi \in \text{Prim}_\varepsilon(A)} \text{im}(X(\phi))
\]

Proof.

If \(\phi, \psi, \psi \phi \in \text{Proj}_\varepsilon(A)\) then \(\psi \phi \psi = \psi \phi = \phi \psi \phi\).
Proposition

Let \((x_i)_{1 \leq i \leq n}\) be a family of projectors of an \(R\)-module \(X\) such that \(x_ix_jx_i = x_ix_j = x_jx_ix_j\) for \(i < j\). Then we get a direct sum decomposition \(X = N_X \oplus D_X := \bigcap_{1 \leq i \leq n} \ker(x_i) \oplus \sum_{1 \leq i \leq n} \text{im}(x_i)\).

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**Definition**

Let $\Gamma$ by a finite quiver with $V(\Gamma) = \{1, \ldots, n\}$ and edge set $E(\Gamma) \subset V(\Gamma) \times V(\Gamma)$ such that if $(i, j) \in E(\Gamma)$ then $i < j$. The **Catalan monoid** $C_\Gamma$ is generated by $x_i, i \in V(\Gamma)$, with relations:

- $x_i^2 = x_i$ for $i \in V(\Gamma)$;
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- $x_ix_j = x_jx_i$ if $(i, j) \not\in E(\Gamma)$ and $(j, i) \not\in E(\Gamma)$.

**Proposition (Kudryatseva-Mazorchuk 2009)**

Every Catalan monoid $C_\Gamma$ is finite and has $2^{\#V(\Gamma)}$ idempotents. The unit of $\mathbb{Q}[C_\Gamma]$ is a sum of $2^{\#V(\Gamma)}$ pairwise orth. idempotents:

$$
1 = \sum_{\{i_1, \ldots, i_k\} \sqcup \{j_1, \ldots, j_{n-k}\} = n} x_{i_k} \cdots x_{i_2}x_{i_1}(1 - x_{j_1})(1 - x_{j_2}) \cdots (1 - x_{j_{n-k}}).
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**Proposition (Kudryatseva-Mazorchuk 2009)**

Every Catalan monoid $C_\Gamma$ is finite and has $2^{\#V(\Gamma)}$ idempotents. The unit of $\mathbb{Q}[C_\Gamma]$ is a sum of $2^{\#V(\Gamma)}$ pairwise orth. idempotents:

$$1 = \sum_{\{i_1, \ldots, i_k\} \sqcup \{j_1, \ldots, j_{n-k}\} = n} x_{i_k} \cdots x_{i_2} x_{i_1} (1 - x_{j_1}) (1 - x_{j_2}) \cdots (1 - x_{j_{n-k}}).$$
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Remark (Catalan monoid rings are semi-perfect)
The idempotents of $C_\Gamma$ induce the simple modules while the decomposition of 1 induces the irreducible components of $\mathbb{Q}[C_\Gamma]$.

Example (Catalan monoids inside $\Delta$)
- The submonoid $C_{[n]} \subset \Delta([n], [n])$ generated by the primitive projectors $x_i = \epsilon_i \eta_i$ ($0 \leq i < n$) is the Catalan monoid $C_{L_n}$ of the linear quiver because $x_i x_j = x_j x_i$ if $|i - j| \geq 2$.
- $C_{[n]}$ consists of those $\phi : [n] \to [n]$ sth. $\phi(i) \geq i$ for all $i$.
- $\# C_{[n]} = \frac{1}{n+2} \binom{2n+2}{n+1}$

Remark (Kiselman monoids $C_{K_n}$)
The cardinalities of $C_{K_n}$ for the complete quivers $K_n$ are not known.
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