The cyclic Deligne conjecture for spaces, chain complexes and Hopf algebras\textsuperscript{1}

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\textsuperscript{1}joint work with Michael Batanin (Sydney)
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Hochschild cochains

Definition
For a (unital associative) $K$-algebra $A$ and $A$-bimodule $M$, the *Hochschild cochain complex* of $A$ with coefficients in $M$ is given by

$$C^n(A; M) = \text{Hom}_K(A \otimes^n, M), \quad n \geq 0,$$

where for $f \in C^n(A; M)$,

$$(\partial_i f)(a_1, \ldots, a_{n+1}) = \begin{cases} 
  a_1 f(a_2, \ldots, a_n) & i = 0; \\
  f(a_1, \ldots, a_i a_{i+1}, \ldots, a_n) & i = 1, \ldots, n; \\
  f(a_1, \ldots, a_n) a_{n+1} & i = n + 1.
\end{cases}$$

$$(s_i f)(a_1, \ldots, a_{n-1}) = f(a_1, \ldots, a_i, 1_A, a_{i+1}, \ldots, a_{n-1}).$$

The Hochschild cohomology $HH^{\bullet}(A; M)$ is the cohomology of the cochain complex of the cosimplicial $K$-module $C^{\bullet}(A; M)$. 
There is a cup product

\[ \cup : C^m(A; A) \otimes_K C^n(A; A) \to C^{m+n}(A; A) \]

\[ (f \cup g)(a_1, \ldots, a_{m+n}) = f(a_1, \ldots, a_m)g(a_{m+1}, \ldots, a_{m+n}) \]

and a brace operation

\[ \{-\} : C^m(A; A) \otimes_K C^n(A; A) \to C^{m+n-1}(A; A) \]

where \( f \{ g \}(a_1, \ldots, a_{m+n-1}) \) is defined by

\[
\sum_{1 \leq i \leq m} (-1)^{(i-1)(n-1)} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+n-1}), a_{i+n}, \ldots, a_{m+n-1}).
\]

The bracket \( \{ f, g \} = f \{ g \} - (-1)^{|f|-1(|g|-1)} g \{ f \} \) induces a Lie bracket of degree \(-1\) on \( HH^\bullet(A; A) \).
Gerstenhaber structure

Definition
A Gerstenhaber $K$-algebra $(H, \cup, \{-, -\})$ is a graded-commutative $K$-algebra with Lie bracket of degree $-1$ such that

$$\{f, g \cup h\} = \{f, g\} \cup h + (-1)^{|g|(|f|-1)} g \cup \{f, h\}.$$  

Proposition (Gerstenhaber '63)
For any algebra $A$, the Hochschild cohomology $HH^\bullet(A; A)$ is a Gerstenhaber algebra.

Theorem (F. Cohen '72)
For any field $K$, the homology $H_\bullet(D_2; K)$ of the little disks operad is the operad for Gerstenhaber $K$-algebras.

Corollary
For any based space $(X, \ast)$, the homology $H_\bullet(\Omega^2 X; K)$ is a Gerstenhaber $K$-algebra.
Connes’ coboundary on $C^\bullet(A; A^*)$

For $A^* = \text{Hom}_K(A, K)$, the adjunction

$$\text{Hom}_K(A \otimes^n, A^*) \cong \text{Hom}_K(A \otimes^{n+1}, K)$$

induces a cyclic operator $\tau_n$ on $C^n(A; A^*)$ of order $n + 1$. These cyclic operators are compatible with the simplicial operators:

$$\tau_{n+1} \partial_i = \partial_{i-1} \tau_n \quad i > 0, \quad \tau_{n-1} s_i = s_{i-1} \tau_n \quad i > 0.$$

It results a covariant functor on Connes’ cyclic category

$$\Delta C \to \text{Mod}_K : [n] \mapsto C^n(A; A^*).$$

In particular, $C^\bullet(A; A^*)$ is a mixed complex

$$C^0(A; A^*) \leftrightarrow C^1(A, A^*) \leftrightarrow C^2(A; A^*) \leftrightarrow \cdots$$

and $HH^\bullet(A; A^*)$ has a differential $\Delta$ of degree $-1$:

$$\Delta^n : HH^n(A, A^*) \to HH^{n-1}(A; A^*).$$
Batalin-Vilkovisky structure

Definition

A **Batalin-Vilkovisky algebra** is a Gerstenhaber algebra \((H, \cup, \{-, -\})\) with a differential \(\Delta\) of degree \(-1\) such that

\[
(-1)^{|f|}\{f, g\} = \Delta(f \cup g) - (\Delta f \cup g) - (-1)^{|f|}(f \cup \Delta g).
\]

A **symmetric** \(K\)-**algebra** \(A\) is a \(K\)-algebra equipped with an isomorphism of \(A\)-bimodules \(A \cong A^*\), i.e. a symmetric exact pairing \(<-,->: A \otimes_K A \to K\) such that \(<ab, c>=<a, bc>\).

**Proposition (Menichi '04)**

For any symmetric algebra \(A\), the Hochschild cohomology \(HH^{\bullet}(A, A)\) is a Batalin-Vilkovisky algebra.

**Theorem (Getzler '94)**

For any field \(K\), the homology \(H_{\bullet}(fD_2, K)\) of the framed little disks operad is the operad for Batalin-Vilkovisky \(K\)-algebras.
The dg- Deligne conjecture

Theorem (MS '02, KS '02, Vo '02, Ta '04, BF '04)
The Hochschild cochain complex of an algebra $A$ admits a $C_\bullet(D_2)$-action inducing the Gerstenhaber structure on $HH^\bullet(A; A)$.

Theorem (KS '06, TZ '06, Ka '07, BB '09)
The Hochschild cochain complex of a symmetric algebra $A$ admits a $C_\bullet(fD_2)$-action inducing the BV-structure on $HH^\bullet(A; A)$.

Proposition (Gerstenhaber-Voronov '95, Menichi '04)
The Hochschild cochain complex of $A$ is isomorphic to the deformation complex of the endomorphism operad $\text{End}_A$ of $A$.
If $A$ is symmetric, then $\text{End}_A$ is multiplicative cyclic.

Proof.
$C^n(A; A) = \text{Hom}(A^\otimes n, A) = \text{End}_A(n) \ni \mu_n$. For $f \in \text{End}_A(n)$, $\partial_0 f = \mu_2 \circ_1 f$, $\partial_n f = \mu_2 \circ_0 f$, $\partial_i f = f \circ_i \mu_2$ if $0 < i < n$.
If $A$ is symmetric then $\text{End}_A$ is cyclic and $\tau_n(\mu_n) = \mu_n$. \qed
Multiplicative operads

Definition
A multiplicative (cyclic) operad is a non-symmetric (cyclic) operad \( \mathcal{O} \) equipped with a map of (cyclic) operads \( \text{Ass} \to \mathcal{O} \). A multiplicative (cyclic) operad \( \mathcal{O} \) has an underlying cosimplicial (cocyclic) object \( \mathcal{O}^\bullet \). In a closed monoidal category \( \mathcal{E} \) equipped with \( \delta : \Delta \to \mathcal{E} \), the deformation complex of \( \mathcal{O} \) is \( \text{Hom}_\Delta(\delta^\bullet, \mathcal{O}^\bullet) \).

Example
For \( \mathcal{E} = \text{Ch}(\mathbb{Z}) \) and \( \delta_\mathbb{Z} : \Delta \to \text{Ch}(\mathbb{Z}) : [n] \mapsto N_* (\Delta [n]; \mathbb{Z}) \) we get

\[
C^\bullet (A; A) = \text{Hom}_\Delta (\delta^\bullet_\mathbb{Z}, \text{End}_A^\bullet).
\]

Theorem (Kaufmann '07, BB '09)
For any multiplicative chain operad \( \mathcal{O} \), the deformation complex of \( \mathcal{O} \) admits a \( C_\bullet (D_2) \)-action. If \( \mathcal{O} \) is multiplicative cyclic, this action extends to a \( C_\bullet (fD_2) \)-action.
The coloured operad for multiplicative operads

Let $\mathcal{L}_2(n_1, \ldots, n_k; n)$ be the set of iso-classes of planar rooted trees with $n$ leaves and a bipartite vertex-set such that:

1. one part of the vertex-set is in bijection with $\{1, \ldots, k\}$;
2. the vertex with label $i$ has arity $n_i$;
3. each edge has at least one labelled extremity;
4. unlabelled vertices have arity $\neq 1$.

Let $C[n] = \mathbb{Z}/(n + 1)\mathbb{Z}$ and put

$$\mathcal{L}^{cyc}_2(n_1, \ldots, n_k; n) = \mathcal{L}_2(n_1, \ldots, n_k; n) \times C[n_1] \times \cdots \times C[n_k].$$

$\mathcal{L}_2$ and $\mathcal{L}^{cyc}_2$ are $\mathbb{N}$-coloured operads for an evident substitution of trees into trees; in $\mathcal{L}^{cyc}_2$, the cyclic permutations distinguish for each labelled vertex one of its incident edges, the neutral element stands for the edge closest to the root of the tree.

**Lemma**

$\mathcal{L}_2$-algebras are multiplicative operads; $\mathcal{L}^{cyc}_2$-algebras are multiplicative cyclic operads. The category of unary operations of $\mathcal{L}_2$ (resp. $\mathcal{L}^{cyc}_2$) is $\Delta$ (resp. $\Delta C$).
Condensation of coloured operads

Unary operations of a coloured operad act covariantly on inputs and contravariantly on the output; therefore:

\[ \mathcal{L}_2(\cdot, \cdots, \cdot; \cdot): \Delta^{\text{op}} \times \cdots \times \Delta^{\text{op}} \times \Delta \to \text{Sets}. \]

Given \( \delta_{\mathbb{Z}}: \Delta \to \text{Ch}(\mathbb{Z}) \) we can realize multisimplicially, and totalize the resulting cosimplicial chain complex. This yields

\[ \xi(\mathcal{L}_2, \delta_{\mathbb{Z}})(k) := \text{Hom}_{\Delta}(\delta^\bullet, \left| \mathcal{L}_2(-, \cdots, -; \cdot) \right|_{\delta_{\mathbb{Z}}^\otimes k}), \quad k \geq 0. \]

**Proposition (Day-Street '03, McClure-Smith '04, BB '09)**

\( \xi(\mathcal{L}_2, \delta_{\mathbb{Z}}) \) (resp. \( \xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\mathbb{Z}}^{\text{cyc}}) \)) is a chain operad acting on the deformation complex of any multiplicative (cyclic) operad.

**Theorem (BB '09)**

As chain operads we have \( \xi(\mathcal{L}_2, \delta_{\mathbb{Z}}) \sim C_\bullet(D_2) \) and \( \xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\mathbb{Z}}^{\text{cyc}}) \sim C_\bullet(fD_2). \)
The cobar complex of a bialgebra

**Theorem (cf. Gerstenhaber-Schack ’92, Menichi ’04)**

The cobar complex $\Omega A$ of a bialgebra (resp. involutive Hopf algebra) $A$ has an action by $\xi(L_2, \delta_Z)$ (resp. $\xi(L_2^{cyc}, \delta_Z^{cyc})$). Its homology $H_\bullet(\Omega A; \mathbb{Z})$ is a Gerstenhaber (resp. BV-) algebra.

**Proof.**

The bialgebra $A$ is a comonoid in the monoidal category of $A$-modules. Therefore: $(\Omega A)_n = A^\otimes n \cong \text{Hom}_A(A, A^\otimes n)$. This $\mathbb{Z}$-linear operad is multiplicative via the diagonal of $A$. If $A$ has an involutive antipode then the operad is multiplicative cyclic. □

**Remark**

(a) $\Omega C_\bullet(\Omega X; \mathbb{Z}) \sim C_\bullet(\Omega^2 X; \mathbb{Z})$ (Adams). The $\xi(L_2, \delta_Z)$-action on $\Omega C_\bullet(\Omega X; \mathbb{Z})$ corresponds to the $C_\bullet(D_2)$-action on $C_\bullet(\Omega^2 X; \mathbb{Z})$.

(b) If $A$ is involutive, the $\xi(L_2^{cyc}, \delta_Z^{cyc})$-action induces a cocyclic structure on $\Omega A$ yielding $HC^\bullet(A)$ of Connes-Moscovici ’99.

(c) $\xi(L_2, \delta_Z)$ contains the second filtration stage of the surjection operad of MS ’03, BF ’04 as a suboperad. Cyclic extension?
The topological Deligne conjecture

There is a cosimplicial resp. cocomplex space

\[ \delta_{\text{top}} : \Delta \to \text{Top} : [n] \mapsto \Delta^n \text{ resp. } \delta_{\text{cyc}}^{\text{top}} : \Delta C \to \text{Top} : [n] \mapsto \Delta^n \times S^1. \]

**Theorem (McClure-Smith '04, Salvatore '09, BB '09)**

The operad \( \xi(\mathcal{L}_2, \delta_{\text{top}}) \) is weakly equivalent to \( D_2 \) and acts on the deformation complex of multiplicative operads in spaces.

The operad \( \xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\text{cyc}}^{\text{top}}) \) is weakly equivalent to \( fD_2 \) and acts on the deformation complex of multiplicative cyclic operads in spaces.

**Remark (cf. Markl '99, Salvatore-Wahl '03, Salvatore '09)**

\[ fD_2(k) \cong D_2(k) \times (S^1)^k, \quad \xi(\mathcal{L}_2^{\text{cyc}}, \delta_{\text{cyc}}^{\text{top}})(k) \cong \xi(\mathcal{L}_2, \delta_{\text{top}})(k) \times (S^1)^k. \]

For \( n = 1 \):

\[ fD(1) \cong D(1) \times S^1, \quad \text{Hom}_{\Delta C}(\delta_{\text{cyc}}^{\text{top}}, \delta_{\text{cyc}}^{\text{top}}) \cong \text{Hom}_{\Delta}(\delta_{\text{top}}, \delta_{\text{top}}) \boxtimes S^1. \]

**Proposition (Sinha '06)**

The simplicial 2-sphere \( S^2 = \Delta[2]/\partial \Delta[2] \) is an \( \mathcal{L}_2 \)-coalgebra in finite pointed sets. For a based space \((X, \ast)\), \( \Omega^2 X \) is the deformation complex of the multiplicative operad \((X, \ast)(S^2, \ast)\).
Braid and ribbon-braid groups

$\mathcal{S}_k$ denotes the *permutation group* on $k$ letters. $\mathcal{S}_k^\pm$ denotes the *signed permutation group* on $k$ letters. 

$\mathcal{S}_k^\pm = \mathcal{S}_k \wr \mathcal{S}_2 = \mathcal{S}_k \ltimes (\mathcal{S}_2)^k$ acts on $fD_2(k) = D_2(k) \times (S^1)^k$.

**Definition (Braid and ribbon-braid groups on $k$ strands)**

\[
\begin{align*}
B_k &= \pi_1(D_2(k)/\mathcal{S}_k) \\
RB_k &= \pi_1(fD_2(k)/\mathcal{S}_k^\pm) \\
PB_k &= \pi_1(D_2(k)) \\
PRB_k &= \pi_1(fD_2(k))
\end{align*}
\]

**Proposition (Asphericity of $D_2(k)$ and $fD_2(k)$)**

\[
\begin{align*}
D_2(k)/\mathcal{S}_k &= K(B_k, 1) \\
fD_2(k)/\mathcal{S}_k^\pm &= K(RB_k, 1) \\
D_2(k) &= K(PB_k, 1) \\
fD_2(k) &= K(PRB_k, 1)
\end{align*}
\]

**Corollary**

The coverings $D_2(k) \to D_2(k)/\mathcal{S}_k$ and $fD_2(k) \to fD_2(k)/\mathcal{S}_k^\pm$ are classified by the short exact sequences $1 \to PB_k \to B_k \to \mathcal{S}_k \to 1$ and $1 \to PRB_k \to RB_k \to \mathcal{S}_k^\pm \to 1$.

**Problem**

Describe the operad structure of $D_2$ (resp. $fD_2$) in terms of the pure braid (resp. ribbon-braid) groups.
Coxeter geometry of permutation groups

The braid group $B_k$ is an Artin group with presentation 
$< s_1, \ldots, s_{k-1} | s_isj = sjsi$ if $|i - j| > 1$ and $s_isi+1si = si+1sisi+1 >$. The pure Artin group $PB_k = \text{Ker}(B_k \to \mathfrak{S}_k) \cong \pi_1(C^k - \mathcal{A}_{\mathfrak{S}_k})$ where $\mathcal{A}_{\mathfrak{S}_k}$ is the complexified braid arrangement.

The Salvetti complex $Sal_{\mathfrak{S}_k}$ is a partially ordered set of the same equivariant homotopy type as $C^k - \mathcal{A}_{\mathfrak{S}_k}$.

$Sal_- : (\text{Coxeter groups}) \to (\text{posets})$

is a functor commuting with finite products. Thus, $(PB_k)_{k \geq 0}$ is a categorical operad. Similarly, $(PRB_k)_{k \geq 0}$ is a categorical operad.

**Proposition**

$D_2 \sim K(PB, 1)$ and $fD_2 \sim K(PRB, 1)$ as operads. Moreover, $PB$-algebras are braided strict monoidal categories; $PRB$-algebras are ribbon-braided (i.e. balanced) strict monoidal categories.

**Corollary (B ’98, Salvatore-Wahl ’03)**

The nerve of a braided (resp. ribbon-braided) strict monoidal category is $E_2$ (resp. framed $E_2$).
The categorical Deligne conjecture

Consider the cosimplicial category

$$\delta_{\text{Cat}} : \Delta \to \text{Cat} : [n] \mapsto [n][n]^{-1}$$

**Proposition**

There are weak equivalences of categorical operads

$$PB \sim \xi(L_2, \delta_{\text{Cat}}) \quad \text{and} \quad PRB \sim \xi(L_{2, \text{cyc}}, \delta_{\text{cyc}}).$$

**Definition**

A central element of a monoidal category $\mathcal{E}$ is a pair $(A, c_A)$ where $c_{A,-} : A \otimes - \cong - \otimes A$ and $c_{A,B \otimes C} = (1_B \otimes c_{A,C}) \circ (c_{A,B} \otimes 1_C)$. The center $\mathcal{Z}\mathcal{E}$ is the category of central elements.

**Proposition**

For $\mathcal{E} = \text{Mod}_H$, $\mathcal{Z}\mathcal{E} \simeq \text{Mod}_{DH}$ where $DH$ is the Drinfeld double of the Hopf algebra $H$. 
The Drinfeld double of a Hopf algebra

**Proposition (Street ’04)**
\[ Z \mathcal{E} = \text{Hom}_{\Delta}(\delta_{\text{Cat}}, \text{End}\mathcal{E}) \]

**Corollary**
The center of a monoidal category is braided monoidal; in particular, the Drinfeld double of a Hopf algebra is “braided”.

**Definition**
An involutive category is a closed monoidal category $\mathcal{E}$ such that the duality functor $(\_)^* = \text{Hom}(\_ , I)$ is self-adjoint. A Hopf algebra $H$ is called quasi-involutive if $\text{Mod}_H$ is involutive.

**Proposition**
The category $\mathcal{E}_f$ of symmetric duality objects of an involutive category $\mathcal{E}$ has a multiplicative cyclic endomorphism-operad $\text{End}\mathcal{E}_f$.

**Corollary**
The center of $\mathcal{E}_f$ is ribbon-braided; in particular, the Drinfeld double of a quasi-involutive Hopf algebra is “ribbon-braided”.