On a certain filtration of the universal bundle of a finite Coxeter group

Clemens Berger

Université Côte d'Azur

Algebraic Combinatorics of
Symmetric Groups and Coxeter Groups
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A *hyperplane arrangement* $\mathcal{A}$ in euclidean space V is a finite family $(H_\alpha)_{\alpha\in\mathcal{A}}$ of hyperplanes of V containing the origin. The arrangement is *essential* if its center $\bigcap_{\alpha\in\mathcal{A}} H_\alpha$ is trivial.

The *complement* $\mathcal{M}(\mathcal{A}) = V\backslash(\bigcup_{\alpha\in\mathcal{A}} H_\alpha)$ decomposes into path components, called *chambers*: $\mathcal{C}_\mathcal{A} = \pi_0(\mathcal{M}(\mathcal{A}))$.

Denote by s_α the *orthogonal symmetry* with respect to H_α. If $(H_\alpha)_{\alpha\in\mathcal{A}}$ is stable under s_β for all $\beta\in\mathcal{A}$, the arrangement is called a *Coxeter arrangement*. We write $\mathcal{A} = \mathcal{A}_W$ where W is the subgroup $W =< s_\alpha, \alpha\in\mathcal{A} >$ of $O_n(\mathbb{R})$. This is justified by

Proposition (Coxeter,Tits)

There is a one-to-one correspondence between essential *Coxeter arrangements* $\mathcal{A}_W$ and finite *Coxeter groups* W.

The Coxeter group W acts simply transitively on $\mathcal{C}_{\mathcal{A}_W}$.

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Definition (higher complements)

The *k-th complement* of a hyperplane arrangement $\mathcal{A}$ is

$$\mathcal{M}_k(\mathcal{A}) = V^k \setminus \bigcup_{\alpha \in \mathcal{A}} (H_\alpha)^k.$$

Example (braid arrangement)

$V = \mathbb{R}^n$, $\mathcal{A} = (H_{ij})_{1 \le i < j \le n}$ where $H_{ij} = \{x \in \mathbb{R}^n | x_i = x_j\}$. This is the Coxeter arrangement $\mathcal{A}_{\mathfrak{S}_n}$ for the *symmetric group* $\mathfrak{S}_n$.

The higher complements of $\mathcal{A}_{\mathfrak{S}_n}$ are *configuration spaces*: $\mathcal{M}_k(\mathcal{A}_{\mathfrak{S}_n}) = F(\mathbb{R}^k, n) = \{(x_1, \ldots, x_n) \in \mathbb{R}^{kn} | x_i \neq x_j\}$.

Theorem (Brieskorn '71, Deligne '72)

For any Coxeter arrangement $\mathcal{A}$, $\mathcal{M}_2(\mathcal{A})$ is aspherical. In particular, $\pi_1(\mathcal{M}_2(\mathcal{A}_W)/W)$ is the *Artin group* of W.

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Purpose of the talk

Construct *simplicial models* for the homotopy type of $\mathcal{M}_k(\mathcal{A})$.

- Fox-Neuwirth '62 and Milgram '66 construct poset models for $\mathcal{M}_k(\mathcal{A}_{\mathfrak{S}_n})$ for any k and any n.
- Salvetti '87 constructs poset models for $\mathcal{M}_2(\mathcal{A})$ for any hyperplane arrangement $\mathcal{A}$.
- Smith '89 constructs simplicial models for $\mathcal{M}_k(\mathcal{A}_{\mathfrak{S}_n})$ for any k and n.
- This induces simplicial models for E_n*-operads* for $1 \leq n \leq \infty$, cf. Barratt-Eccles, B. '96.

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$$sgn_x(\alpha) = \begin{cases} 0 & \text{if } x \in H_\alpha; \\ \pm & \text{if } x \in H_\alpha^\pm \backslash H_\alpha. \end{cases}$$

The *face monoid* $\mathcal{F}_\mathcal{A} \subset \{0, \pm\}^\mathcal{A}$ is the set of sign vectors $P \in \{0, \pm\}^\mathcal{A}$ such that there exists $x \in V$ with $sgn_x = P$. For $P, Q \in \mathcal{F}_\mathcal{A}$ the product $PQ \in \mathcal{F}_\mathcal{A}$ is defined by

$$(PQ)(\alpha) = \begin{cases} P(\alpha) & \text{if } P(\alpha) \neq 0; \\ Q(\alpha) & \text{if } P(\alpha) = 0. \end{cases}$$

The *facets* $c_P = \{x \in V | sgn_x = P\}$ are convex subsets of V.

Lemma (Green order of left regular band $\mathcal{F}_\mathcal{A}$)

$$\overline{c}_P \subseteq \overline{c}_Q \overset{dfn}{\iff} P \leq Q \iff PQ = Q$$

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The chamber system $\mathcal{C}_{\mathcal{A}}$ is the *discrete* subposet of $\mathcal{F}_{\mathcal{A}}$ consisting of the *maximal* facets. In particular, $|\mathcal{C}_{\mathcal{A}}| \simeq \mathcal{M}(\mathcal{A})$.

$\mathcal{F}_{\mathcal{A}} \times \mathcal{F}_{\mathcal{A}} = \mathcal{F}_{\mathcal{A} \oplus \mathcal{A}}$ where $\mathcal{A} \oplus \mathcal{A} = (\mathcal{A} \times V) \cup (V \times \mathcal{A})$ in $V \times V$.

Definition (Orlik '91)

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$(P, Q) \notin \mathcal{C}_{\mathcal{A}}^{(2)}$ iff $\exists \alpha \in \mathcal{A} : P(\alpha) = Q(\alpha) = 0$.

For subcomplexes K_1, K_2 of a simplicial complex L sth. $Vert(L) = Vert(K_1) \sqcup Vert(K_2)$, one has: $|L| \backslash |K_1| \simeq |K_2|$. Thus,

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For subcomplexes K_1, K_2 of a simplicial complex L sth. $Vert(L) = Vert(K_1) \sqcup Vert(K_2)$, one has: $|L|\backslash|K_1| \simeq |K_2|$. Thus,

Proposition (Orlik '91)

$|\mathcal{C}^{(2)}_{\mathcal{A}}| \simeq \mathcal{M}_2(\mathcal{A})$

Definition (Salvetti '87)

$\mathcal{S}^{(2)}_{\mathcal{A}} = \{(P, C) \in \mathcal{F}_{\mathcal{A}} \times \mathcal{C}_{\mathcal{A}} \mid P \leq C\}$
$(P, C) \geq (P', C')$ iff $P \leq P'$ and $P'C = C'$.

Definition (Higher Orlik and Salvetti complexes)

$\mathcal{C}^{(k)}_{\mathcal{A}} = \{(P_1, \ldots, P_k) \in (\mathcal{F}_{\mathcal{A}})^k \mid P_1 \cdots P_k \in \mathcal{C}_{\mathcal{A}}\}^{\mathrm{op}}$
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Theorem (cf. Mori-Salvetti '11)

$|\mathcal{C}^{(k)}_{\mathcal{A}}| \simeq \mathcal{M}_k(\mathcal{A})$ and $(P_1, \ldots, P_k) \mapsto (P_1, P_1P_2, \ldots, P_1P_2 \cdots P_k)$ defines a homotopy equivalence of posets $\mathcal{C}^{(k)}_{\mathcal{A}} \xrightarrow{\simeq} \mathcal{S}^{(k)}_{\mathcal{A}}$.

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Definition

The *universal bundle* EG of a group G is the simplicial set with d-simplices $(g_0, \dots, g_n) \in G^{n+1}$ and diagonal G-action.
The *classifying space* BG is the quotient EG/G.

Proposition

$H_*(G;\mathbb{Z}) = H_*(|BG|;\mathbb{Z})$

Remark

The symmetric group $\mathfrak{S}_n$ embeds into a product $\prod_{1\leq i<j\leq n} \mathfrak{S}_{ij}$:

- $\mathfrak{S}_{ij} = \mathrm{Bij}(\{i,j\}) \cong \mathfrak{S}_2$.
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Lemma

$E\mathfrak{S}_2$ is an ∞-dimensional sphere, hemispherically decomposed, with the antipodal $\mathfrak{S}_2$-action. Convention: $E_d\mathfrak{S}_2 = S^{d-1}$

Corollary (Smith filtration '89)

$E\mathfrak{S}_n$ embeds into a product $\prod_{1\leq i<j\leq n} E\mathfrak{S}_{ij}$ and inherits a canonical filtration $E_d\mathfrak{S}_n$ by restriction of the product filtration.

Theorem (Smith '89, Kashiwabara '93, B. '96)

$|E_d(\mathfrak{S}_n)| \simeq \mathcal{M}_d(\mathcal{A}_{\mathfrak{S}_n}) \simeq F(\mathbb{R}^d, n)$

Corollary

The permutation operad $(\mathfrak{S}_n)_{n\geq 0}$ induces E_d-suboperads $(E_d(\mathfrak{S}_n))_{n\geq 0}$ of the Barratt-Eccles E_∞-operad $(E\mathfrak{S}_n)_{n\geq 0}$.

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For each $\mathcal{A}$, the *adjacency graph* $\mathcal{G}_\mathcal{A}$ has *vertex set* $\mathcal{C}_\mathcal{A}$ and *edge set* $\{(C, C') \in \mathcal{C}_\mathcal{A} \times \mathcal{C}_\mathcal{A} \,|\, \exists P \in \mathcal{F}_\mathcal{A} : P \prec C \text{ and } P \prec C'\}$. Since $P(\alpha) = 0$ for a unique $\alpha \in \mathcal{A}$, the edges of $\mathcal{G}_\mathcal{A}$ are labelled by $\mathcal{A}$.

Let $S(C, C') = \{\alpha \in \mathcal{A} \,|\, C(\alpha)C'(\alpha) = -1\}$. Then:

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Proposition (Björner-Edelman-Ziegler '90)

The face monoid $\mathcal{F}_\mathcal{A}$ is determined by the adjacency graph $\mathcal{G}_\mathcal{A}$.

Definition

Let $E_\mathcal{A}$ be the simplicial set whose n-simplices are $(n+1)$-tuples $(C_0, C_1, \ldots, C_n)$ of chambers. $(C_0, C_1, \ldots, C_n) \in E^{(d)}_\mathcal{A}$ iff $(S(C_0, C_1), \ldots, S(C_{n-1}, C_n))$ contains $< d$ times each $\alpha \in \mathcal{A}$.

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- There is a simplicial map $nerve(\mathcal{S}_{\mathcal{A}}^{(k)}) \to E_{\mathcal{A}}^{(k)}$ defined by projection onto the chamber component.
- $E_{\mathcal{A}\oplus\mathcal{B}} \cong E_{\mathcal{A}} \times E_{\mathcal{B}}$ compatible with filtrations.

Remark

$E_{\mathcal{A}_{\mathfrak{S}_n}}^{(d)} = E_d(\mathfrak{S}_n)$

Conjecture

For any finite Coxeter group W, one has $|E_{\mathcal{A}_W}^{(d)}| \simeq \mathcal{M}_d(\mathcal{A}_W)$.

This would extend the operad structure of the B/C/D-Coxeter groups to the higher complements of their Coxeter arrangement.

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- There is a simplicial map $nerve(\mathcal{S}_{\mathcal{A}}^{(k)}) \to E_{\mathcal{A}}^{(k)}$ defined by projection onto the chamber component.
- $E_{\mathcal{A}\oplus\mathcal{B}} \cong E_{\mathcal{A}} \times E_{\mathcal{B}}$ compatible with filtrations.

Remark

$E_{\mathcal{A}_{\mathfrak{S}_n}}^{(d)} = E_d(\mathfrak{S}_n)$

Conjecture

For any finite Coxeter group W, one has $|E_{\mathcal{A}_W}^{(d)}| \simeq \mathcal{M}_d(\mathcal{A}_W)$.

This would extend the operad structure of the B/C/D-Coxeter groups to the higher complements of their Coxeter arrangement.

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