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ON THE GROUPS $J(X)$—IV

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§1. INTRODUCTION

FROM ONE POINT of view, the present paper is mainly concerned with specialising the results on the groups $J(X)$, given in previous papers of this series [3, 4, 5], to the case $X = S^n$. It can, however, be read independently of the previous papers in this series; because from another point of view, it is concerned with the use of extraordinary cohomology theories to define invariants of homotopy classes of maps; and this machinery can be set up independently of the previous papers in this series. We refer to them only for certain key results.

From a third point of view, this paper represents a very belated attempt to honour the following two sentences in an earlier paper [2]. "However, it appears to the author that one can obtain much better results on the J-homomorphism by using the methods, rather than the results, of the present paper. On these grounds, it seems best to postpone discussion of the J-homomorphism to a subsequent paper." I offer topologists in general my sincere apologies for my long delay in writing up results which mostly date from 1961/62.

I will now summarise the results which relate to the homotopy groups of spheres. For this one needs some notation. The stable group $\lim_{n\to\infty} \pi_{n+r}(S^n)$ will be written π_r^S. The stable J-homomorphism is thus a homomorphism

$$J: \pi_r(SO) \to \pi_r^S.$$

THEOREM 1.1. *If $r \equiv 0 \bmod 8$ and $r > 0$ (so that $\pi_r(SO) = Z_2$), then J is a monomorphism and its image is a direct summand in π_r^S.*

Before considering the case $r \equiv 1 \bmod 8$, we need a preliminary result. Suppose that $r \equiv 1$ or $2 \bmod 8$. Then any map $f: S^{q+r} \to S^q$ induces a homomorphism

$$f^*: \tilde{K}_R^q(S^q) \to \tilde{K}_R^q(S^{q+r}),$$

where the functor $\tilde{K}_R^*$ is that due to Grothendieck–Atiyah–Hirzebruch [10, 11, 2]. We have

$$\tilde{K}_R^q(S^q) = Z, \qquad \tilde{K}_R^q(S^{q+r}) = Z_2.$$

THEOREM 1.2. *Suppose that $r \equiv 1$ or $2 \bmod 8$ and $r > 0$. Then π_r^S contains an element μ_r, of order 2, such that any map $f: S^{q+r} \to S^q$ representing μ_r induces a non-zero homomorphism of $\tilde{K}_R^q$.*

The elements μ_r may be described more precisely than is done in this theorem. We have $\mu_1 = \eta$ and $\mu_2 = \eta\eta$, where η is (as usual) the generator of π_1^S. The elements μ_r constitute a systematic family of elements, generalising η and $\eta\eta$; they have interesting properties, which I hope to discuss on another occasion. I am indebted to M. G. Barratt for ideas about systematic families of elements.

THEOREM 1.3. *Suppose that $r \equiv 1 \bmod 8$ and $r > 1$ (so that $\pi_r(SO) = Z_2$). Then J is a monomorphism and π_r^S contains a direct summand $Z_2 + Z_2$, one summand being generated by μ_r and the other being* Im J.

The case $r = 1$ is exceptional, in that the two summands coincide.

THEOREM 1.4. *Suppose that $r \equiv 2 \bmod 8$ and $r > 0$. Then π_r^S contains a direct summand Z_2 generated by μ_r.*

THEOREM 1.5. *Suppose $r = 4s - 1 \equiv 3 \bmod 8$, so that $\pi_r(SO) = Z$. Then the image of J is a cyclic group of order $m(2s)$, and is a direct summand in π_r^S.*

In this theorem, $m(t)$ is the numerical function discussed in [4, §2]. More explicitly, let B_s be the sth Bernoulli number; then $m(2s)$ is the denominator of $B_s/4s$, when this fraction is expressed in its lowest terms.

The direct sum splitting will be accomplished by defining (§7) a homomorphism

$$e'_R : \pi_r^S \to Z_{m(2s)}$$

such that

$$e'_R J : \pi_r(SO) \to Z_{m(2s)}$$

is an epimorphism.

THEOREM 1.6. *Suppose $r = 4s - 1 \equiv 7 \bmod 8$, so that $\pi_r(SO) = Z$. Then the image of J is a cyclic group of order either $m(2s)$ or $2m(2s)$. Moreover, there is a homomorphism*

$$e'_R : \pi_r^S \to Z_{m(2s)}$$

such that

$$e'_R J : \pi_r(SO) \to Z_{m(2s)}$$

is an epimorphism.

It follows that if the order of Im J is $m(2s)$, then Im J is a direct summand; this happens (for example) if $r = 7$ or 15. In any event, the subgroup of elements of odd order in Im J is a direct summand in π_r^S.

It will not be proved in this paper, but by more delicate arguments one can show that even for $r \equiv 7 \bmod 8$, the group π_r^S splits as (Ker e'_R) + $Z_{m(2s)}$; however, I do not know how the subgroup Im J lies with respect to this splitting.

The invariants (such as e'_R) which we shall introduce have convenient properties, and lend themselves to a variety of calculations; examples will be given in §§11, 12. They are not restricted to maps between spheres. The following result provides rather a striking example. We take p to be an odd prime, $g : S^{2q-1} \to S^{2q-1}$ to be a map of degree p^f, and Y to be the Moore space $S^{q-1} \cup_g e^{2q}$. Thus $\tilde{K}_C(Y) = Z_{p^f}$. $S^{2r}Y$ will mean the $2r$-fold suspension of Y; we take $r = (p-1)p^{f-1}$.

THEOREM 1.7. *For suitable q there is a map*

$$A : S^{2r}Y \to Y$$

which induces an isomorphism

$$A^* : \tilde{K}_C(Y) \to \tilde{K}_C(S^{2r}Y).$$

Therefore the composite

$$A . S^{2r}A . S^{4r}A . \dots S^{2r(s-1)}A : S^{2rs}Y \to Y$$

induces an isomorphism of $\tilde{K}_C$, *and is essential for every s.*

For $f = 1$ this result is related to Toda's sequence of elements $\alpha_s \in \pi^S_{2(p-1)s-1}$ [16,17], as will be explained in §12.

From the point of view of history or motivation, the sequence of ideas in this paper may be ordered as follows. Suppose given a map $f: X \to Y$. We may form the mapping cone $Y \cup_f CX$; by studying the group $K_C(Y \cup_f CX)$ and the homomorphism

$$ch: K_C(Y \cup_f CX) \to H^*(Y \cup_f CX; Q)$$

we may sometimes succeed in distinguishing $Y \cup_f CX$ from $Y \vee SX$; thus we may sometimes show that f is essential. This method was presumably known to Atiyah and Hirzebruch (*ca.* 1960/61); it is given in [6] (for the case in which X and Y are spheres) and was published by Dyer [13]. See also [19]. We touch on it in §7 of this paper.

One next realises that in the preceding construction, the possible Chern characters that can arise are severely limited by the fact that $K_C(Y \cup_f CX)$ admits operations Ψ^k. This observation leads to a proof of the non-existence of elements of Hopf invariant one (mod 2 and mod p); this proof was given in [6], and was first published by Dyer [13]. We touch on it in §8 of this paper. It should be said, however, that the most elegant proof by K-theory of the non-existence of elements of Hopf invariant one is somewhat different; see [8].

One next realises that the essential phenomenon we have to study is the short exact sequence

$$\tilde{K}_C(Y) \leftarrow \tilde{K}_C(Y \cup_f CX) \leftarrow \tilde{K}_C(SX)$$

of groups admitting operations Ψ^k. The class of this short exact sequence yields an element of a suitable group

$$\mathrm{Ext}^1(\tilde{K}_C(Y), \tilde{K}_C(SX)).$$

This element gives an invariant of f. If $K_C(Y \cup_f CX)$ is torsion-free this approach is equivalent to that using the Chern character; if $K_C(Y \cup_f CX)$ has torsion this approach is better than that using the Chern character. We therefore adopt this as our basic approach. It has been sketched in [7], and will be fully explained in §3.

In the above, we can of course use $\tilde{K}_R$ instead of $\tilde{K}_C$. The use of $\tilde{K}_R$ and the use of spaces with torsion gives the extra power needed to prove results such as Theorems 1.1, 1.3.

Once we realise that our invariants should take values in suitable Ext^1 groups, certain properties of the invariants become very plausible. Our invariants carry composition products (of homotopy classes) into composition products (in Ext) (§3); they carry Toda brackets (in homotopy) into Massey products (in Ext) (§§4,5). These products enable one to perform many calculations.

The arrangement of the paper is as follows. Since we make constant use of cofibre sequences

$$X \xrightarrow{f} Y \to Y \cup_f CX \to SX \dots ,$$

we devote §2 to them. In §3 we define our invariants and give their basic properties. §§4, 5 are devoted to their properties on Toda brackets, as indicated above. So far the work has been done for a quite general cohomology theory; in §§6, 7 we specialise to the case of $\tilde{K}_C$ and $\tilde{K}_R$. §7 contains the main theorem about the cases in which X and Y are spheres and $\tilde{K}$ is torsion-free. §8 contains the relationship between the invariants of §7 and the classical Hopf invariant in the sense of Steenrod. §9 considers the case needed for Theorems 1.1, 1.3, in which X and Y are spheres but $\tilde{K}$ is not torsion-free. In §10 we discuss the value of our invariants on the image of J. In §11 we work out the general theory of §§4, 5 (about Toda brackets) for the special cases which most concern us. In §§12 we prove Theorem 1.7 and discuss related matters; since the same machinery serves to discuss certain 2-primary phenomena, we also prove Theorem 1.2 there. In §12 we also give a number of examples and applications; the reader's attention is particularly directed to these, since they provide essential motivation.

Since drafting the body of this paper, I have become aware of Toda's paper [19], which has a considerable overlap with the present paper. I am very grateful to Toda for a letter about his results.

Toda defines an invariant

$$CH^{n+k} : \pi_{2n+2k-1}(S^{2n}) \to Q/Z$$

which is presumably the same as the invariant e_C discussed in this paper. He also defines an invariant $CH_*{}^{4m+2h}$, which is presumably the same (up to a certain constant factor) as the invariant e'_R discussed in this paper.

To give Toda proper credit for his priority, I offer the following concordance of results. Corollary 7.7 of this paper is to be found in Toda's paper, and is the essential step in the proof of his Theorems 6.3, 6.5(i) and (ii) which give restrictions on the values that can be taken by his invariants (compare 7.14, 7.15 of this paper). Proposition 7.20 of this paper is Theorem 6.5 (iii) of [19]. Corollary 8.3 of this paper is Theorem 6.7 of [19]. The case $\Lambda = C$ of Theorem 11.1 of this paper is Theorem 6.4 of [19]. Theorem 12.11 of this paper is contained in 6.8 of [19].

§2. COFIBERINGS

As explained in the introduction, this paper will make much use of sequences of cofiberings. We shall therefore devote this section to summarising some material about cofibre sequences, following [15]. We need only deal with "good" spaces; for the applications, it would be sufficient to consider finite CW-complexes.

Let $f: X \to Y$ be a map. We can construct from it a cofibering

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX.$$

Here i is an injection map; and $Y \cup_f CX$ is the space obtained from Y by attaching CX, the cone on X, using f as attaching map.

Iterating this construction, we can construct

$$Y \xrightarrow{i} (Y \cup_f CX) \xrightarrow{j} (Y \cup_f CX) \cup_i CY$$

and (setting $Z = Y \cup_f CX$)

$$Z \xrightarrow{j} (Z \cup_i CY) \xrightarrow{k} (Z \cup_i CY) \cup_j CZ.$$

Now the space $(Y \cup_f CX) \cup_i CY$ is homotopy-equivalent to the suspension SX; and similarly, the space $(Z \cup_i CY) \cup_j CZ$ is homotopy-equivalent to SY. In order to avoid errors of sign in what follows, it is desirable to use the "same" homotopy equivalence in the two cases. If we do this, then the map

$$k : (Y \cup_f CX) \cup_i CY \to (Z \cup_i CY) \cup_j CZ$$

corresponds to

$$-Sf : SX \to SY.$$

(This is easy to check; or see [15, p. 309, Satz 4].) We shall therefore take the following as our basic cofibre sequence.

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{j} SX \xrightarrow{-Sf} SY \ldots$$

This construction has various obvious properties, which we record for use later.

PROPOSITION 2.1. *If $f \sim g$, then we can construct the following homotopy-commutative diagram, in which all the vertical arrows are homotopy equivalences.*

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & Y \cup_f CX & \xrightarrow{j} & SX & \xrightarrow{-Sf} & SY \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
X & \xrightarrow{g} & Y & \xrightarrow{i'} & Y \cup_g CX & \xrightarrow{j'} & SX & \xrightarrow{-Sg} & SY
\end{array}$$

PROPOSITION 2.2. *Given a commutative diagram*

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} \\
X' & \xrightarrow{f'} & Y'
\end{array}$$

we can construct the following commutative diagram.

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & Y \cup_f CX & \xrightarrow{j} & SX & \xrightarrow{-Sf} & SY \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle k} & & \downarrow & & \downarrow{\scriptstyle Sh} & & \downarrow{\scriptstyle Sk} \\
X' & \xrightarrow{f'} & Y' & \xrightarrow{i'} & Y' \cup_{f'} CX' & \xrightarrow{j'} & SX' & \xrightarrow{-Sf'} & SY'
\end{array}$$

These obvious and elementary propositions are special cases of the more general results proved in [15, pp. 311–316].

PROPOSITION 2.3. *Given*

$$X \xrightarrow{f} Y \xrightarrow{g} Z,$$

we can construct the following commutative diagram.

$$\begin{array}{ccccccccc}
X & \xrightarrow{f} & Y & \xrightarrow{i} & Y \cup_f CX & \xrightarrow{j} & SX & \xrightarrow{-Sf} & SY \\
\downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle Sg} \\
X & \xrightarrow{gf} & Z & \xrightarrow{i'} & Z \cup_{gf} CX & \xrightarrow{j'} & SX & \xrightarrow{-S(gf)} & SZ \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle Sf} & & \downarrow{\scriptstyle 1} \\
Y & \xrightarrow{g} & Z & \xrightarrow{i''} & Z \cup_g CY & \xrightarrow{j''} & SY & \xrightarrow{-Sg} & SZ
\end{array}$$

This follows from two applications of Proposition 2.2.

PROPOSITION 2.4. *For each r, we can construct the following homotopy-commutative diagram, in which all the vertical arrows are homotopy equivalences.*

$$\begin{array}{ccccc}
S^rY & \xrightarrow{S^ri} & S^r(Y \cup_f CX) & \xrightarrow{S^rj} & S^{r+1}X \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle (-1)^r} \\
S^rY & \xrightarrow{i'} & (S^rY) \cup_{S^r} C(S^rX) & \xrightarrow{j'} & S^{r+1}X
\end{array}$$

This proposition is easy to check, provided we use the "reduced" cone and suspension. The map $(-1)^r$ of $S^{r+1}X$ arises as a permutation of the suspension coordinates.

§3. DEFINITION AND ELEMENTARY PROPERTIES OF THE INVARIANTS *d*, *e*

In this section we shall define our basic invariants d and e. We shall also establish the elementary properties of these invariants.

We shall suppose given a half-exact functor in the sense of [12]. For example, the functor may be one component of a (reduced) extraordinary cohomology theory. More precisely, k is to be a contravariant functor defined on (say) the category of finite CW-complexes and homotopy classes of maps, and taking values in some abelian category [14], say A. If

$$X \xrightarrow{i} Y \xrightarrow{j} Z$$

is a cofibre sequence, then

$$k(X) \xleftarrow{i^*} k(Y) \xleftarrow{j^*} k(Z)$$

is to be an exact sequence in the abelian category A. It follows that we may identify $k(X \vee Y)$ with the direct sum $k(X) \oplus k(Y)$ in the category A; see [12, p. 1].

Now suppose given a map $f: X \to Y$ between (say) finite connected CW-complexes. We can consider the induced homomorphism

$$f^*: k(Y) \to k(X).$$

If we take $X = Y = S^n$ and take k to be $H^n(\ \ ; Z)$, then the invariant f^* gives us the degree of f. We therefore regard

$$f^*: k(Y) \to k(X)$$

as "the degree of f, measured by k-theory". We define

$$d(f) = f^* \in \mathrm{Hom}(k(Y), k(X)).$$

Here $\mathrm{Hom}(M, N)$ means the set of maps from M to N in the abelian category A.

The invariant $e(f)$ will be defined when $d(f) = 0$ and $d(Sf) = 0$. In this case we use the map $f: X \to Y$ to start the following cofibre sequence.

$$X \xrightarrow{f} Y \xrightarrow{i} Y \cup_f CX \xrightarrow{j} SX \xrightarrow{-Sf} SY$$

Since we assume that $f^* = 0$ and $(Sf)^* = 0$, the functor k yields the following short exact sequence in the abelian category A.

$$0 \leftarrow k(Y) \xleftarrow{i^*} k(Y \cup_f CX) \xleftarrow{j^*} k(SX) \leftarrow 0$$

In an abelian category we can define Ext^1 by classifying short exact sequences; therefore the short exact sequence above yields an element of

$$\mathrm{Ext}^1(k(Y), k(SX)).$$

We call this element $e(f)$. The letter e stands for "extension", and goes well with d.

For example, let us consider the case in which $k = \tilde{H}^*(\ \ ; Z_2)$ and A is the category of graded modules over the mod 2 Steenrod algebra. Let us take $X = S^{m+n-1}$, $Y = S^m$. Given a map $f: S^{m+n-1} \to S^m$, we are led to consider the following short exact sequence.

$$0 \leftarrow \tilde{H}^*(S^m; Z_2) \leftarrow \tilde{H}^*(S^m \cup_f e^{m+n}; Z_2) \leftarrow \tilde{H}^*(S^{m+n}; Z_2) \leftarrow 0$$

As an extension of modules over the Steenrod algebra, this is completely determined by the Steenrod square

$$Sq^n: H^m(S^m \cup_f e^{m+n}; Z_2) \to H^{m+n}(S^m \cup_f e^{m+n}; Z_2).$$

We therefore recover Steenrod's approach to the mod 2 Hopf invariant.

The invariant $e(f)$ may thus be regarded as a "Steenrod–Hopf invariant" in which ordinary cohomology has been replaced by k-theory.

We have just defined

$$d(f) \in \mathrm{Ext}^0(k(Y), k(X))$$

(if we interpret $\mathrm{Ext}^0(M, N)$ as meaning $\mathrm{Hom}(M, N)$), and

$$e(f) \in \mathrm{Ext}^1(k(Y), k(SX)).$$

One would naturally hope to construct a third invariant, which should be defined when suitable d and e invariants vanish, and should take values in

$$\mathrm{Ext}^2(k(Y), k(S^2X)).$$

Similarly for a fourth invariant, and so on. However, we will not pursue this line of thought any further here.

In later sections we will give examples and applications of the invariants d and e, and develop the resources to do practical calculations with them. For the moment we consider the elementary properties of these invariants.

PROPOSITION 3.1 (a). *If $f \sim g$, then $d(f) = d(g)$.*

(b) *If $f \sim g$ and $e(f)$ is defined, the $e(g)$ is defined and $e(f) = e(g)$.*

Proof. Part (a) is obvious. Part (b) is proved by applying the functor k to the diagram given in Proposition 2.1.

We now consider the situation in which we have two maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

We aim to show that the invariants d and e send composition products (in homotopy) into composition products, i.e. Yoneda products, in Ext groups.

PROPOSITION 3.2 (a). *We have*

$$d(gf) = d(f)d(g).$$

(b) *If* $e(f)$ *is defined then so is* $e(gf)$, *and we have*

$$e(gf) = e(f)d(g).$$

(c) *If* $e(g)$ *is defined then so is* $e(gf)$, *and we have*

$$e(gf) = d(Sf)e(g).$$

Here statements (b) and (c) use the pairing of Ext^0 and Ext^1 to Ext^1.

Proof. All the statements about invariants d are obvious. For the rest, we apply the functor k to the diagram given in Proposition 2.3, and we obtain the following commutative diagram.

$$\begin{array}{ccccc} k(Y) & \leftarrow & k(Y \cup_f CX) & \leftarrow & k(SX) \\ \uparrow{\scriptstyle g^*} & & \uparrow & & \uparrow{\scriptstyle 1} \\ k(Z) & \leftarrow & k(Z \cup_{gf} CX) & \leftarrow & k(SX) \\ \uparrow{\scriptstyle 1} & & \uparrow & & \uparrow{\scriptstyle (Sf)^*} \\ k(Z) & \leftarrow & k(Z \cup_g CY) & \leftarrow & k(SY) \end{array}$$

If $e(f)$ is defined, it is represented by the top row; similarly for $e(gf)$ and the middle row; similarly for $e(g)$ and the bottom row. By definition of the products in Ext, this shows that

$$e(gf) = e(f)\cdot g^*$$

in case (b), and

$$e(gf) = (Sf)^*\cdot e(g)$$

in case (c). This completes the proof.

For our next proposition, we assume that X is a co-H-space, for example, a suspension. That is, we are provided with a map

$$\Delta : X \to X \vee X$$

of type (1, 1). This allows us to define the sum of two (base-point-preserving) maps

$$f, g : X \to Y;$$

by definition, $f + g$ is the composite

$$X \xrightarrow{\Delta} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\mu} Y,$$

where μ is a map of type (1, 1) in the dual sense.

PROPOSITION 3.3 (a). *We have*

$$d(f + g) = d(f) + d(g).$$

(b) *If $e(f)$ and $e(g)$ are defined then so is $e(f+g)$, and*

$$e(f+g) = e(f) + e(g).$$

In part (b), the sum occurring on the right-hand side is, of course, the Baer sum in Ext^1.

Proof. All the statements about invariants d are obvious. For the rest, we may identify $k(Y \vee Y)$ with the direct sum $k(Y) \oplus k(Y)$, and $k(S(X \vee X))$ with $k(SX) \oplus k(SX)$. In this way we can identify the sequence

$$k(Y \vee Y) \leftarrow k((Y \vee Y) \cup_{f \vee g} C(X \vee X)) \leftarrow k(S(X \vee X))$$

with the direct sum of the sequences

$$k(Y) \leftarrow k(Y \cup_f CX) \leftarrow k(SX)$$

$$k(Y) \leftarrow k(Y \cup_g CX) \leftarrow k(SX).$$

That is: if $e(f)$ and $e(g)$ are defined, so is $e(f \vee g)$, and it can be identified with the "external" sum $e(f) \oplus e(g)$. According to Proposition 3.2, we have

$$\begin{aligned} e(f+g) &= e(\mu(f \vee g)\Delta) \\ &= (S\Delta)^* e(f \vee g)\mu^* \\ &= (S\Delta)^*(e(f) \oplus e(g))\mu^*. \end{aligned}$$

But with our identifications,

$$(S\Delta)^* : k(SX) \oplus k(SX) \to k(SX)$$

is a map of type (1, 1) in the category A, and

$$\mu^* : k(Y) \to k(Y) \oplus k(Y)$$

is a map of type (1, 1) in the dual sense. Thus the element

$$(S\Delta)^*(e(f) \oplus e(g))\mu^*$$

is the Baer sum of $e(f)$ and $e(g)$. This completes the proof.

We will now discuss the behaviour of our invariants under suspension. For this purpose we shall suppose that for some integer r, $k(S^r X)$ is known as a function of $k(X)$. For example, when we take $k(X) = \tilde{K}_C(X)$ [10, 11, 2], we shall take $r = 2$; when we take $k(X) = \tilde{K}_R(X)$ we shall take $r = 8$. If we took $k(X) = \tilde{H}^*(X; Z_2)$ we could take $r = 1$. More formally, we shall suppose given a functor T, from the abelian category A to itself, which preserves exact sequences; and we shall suppose given an isomorphism

$$k(S^r X) \cong Tk(X)$$

natural for maps of X. We shall allow ourselves to identify $k(S^r X)$ and $Tk(X)$ under this isomorphism.

Since the functor T preserves exact sequences, it defines a function

$$T : \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(TM, TN).$$

This function is actually a homomorphism.

PROPOSITION 3.4 (a). *We have*

$$d(S^r f) = Td(f).$$

(b) *If $e(f)$ is defined, then so is $e(S^r f)$, and we have*

$$e(S^r f) = (-1)^r Te(f).$$

Proof. All the statements about the invariant d are obvious. For the rest, we apply the functor k to the diagram given in Proposition 2.4 and use the fact that $kS^r = Tk$.

We now define stable track groups by

$$\mathrm{Map}_S(X, Y) = \operatorname*{Dir\,Lim}_{n\to\infty} \mathrm{Map}(S^{nr}X, S^{nr}Y).$$

We also define stabilised Hom groups in the abelian category A by iterating T and taking direct limits; thus,

$$\mathrm{Hom}_S(M, N) = \operatorname*{Dir\,Lim}_{n\to\infty} \mathrm{Hom}(T^n M, T^n N).$$

Similarly, we define stabilised Ext^1 groups by iterating the homomorphism $(-1)^r T$ and taking direct limits; thus,

$$\mathrm{Ext}^1_S(M, N) = \operatorname*{Dir\,Lim}_{n\to\infty} \mathrm{Ext}^1(T^n M, T^n N).$$

PROPOSITION 3.5 (a). *The invariant d defines a homomorphism from* $\mathrm{Map}_S(X, Y)$ *to* $\mathrm{Hom}_S(k(Y), k(X))$.

(b) *The invariant e defines a homomorphism from the subgroup* $\mathrm{Ker}\, d \cap \mathrm{Ker}(dS)$ *of* $\mathrm{Map}_S(X, Y)$ *to* $\mathrm{Ext}^1_S(k(Y), k(SX))$.

This follows immediately from Propositions 3.1, 3.3, 3.4.

The pairing of Ext groups used in Proposition 3.2 are evidently compatible with the operations T on Ext^0 and $(-1)^r T$ on Ext^1; therefore these pairings pass to the limit. With this interpretation, Proposition 3.2 continues to give the value of the invariants d, e on a composite gf of stable homotopy classes.

§4. MASSEY PRODUCTS IN HOMOLOGICAL ALGEBRA

In §3 we showed that the d and e invariants map composition products (in homotopy) into composition products (in homological algebra). In §5 we shall show that the d and e invariants map Toda brackets (in homotopy) into Massey products (in homological algebra). Of course it is necessary to begin by defining these Massey products, and that is the object of this section.

If we could work in a category containing sufficient projectives, so that we could use projective resolutions, the construction of Massey products would present no difficulty. Unfortunately, we have to work in a category which is not known to contain enough projectives. We have therefore to construct our Massey products without using projectives. In a work on homological algebra it would be desirable to show that if we accidentally have enough projectives, then the definitions which do not use projectives coincide (up to sign) with those which do use projectives. However, for present purposes we need not discuss this question; I hope that the definitions given below will commend themselves by their inherent plausibility and by the applications given in §5.

We shall suppose given four objects L, M, N and P of an abelian category, and three elements

$$\alpha \in \operatorname{Ext}^a(L, M)$$
$$\beta \in \operatorname{Ext}^b(M, N)$$
$$\gamma \in \operatorname{Ext}^c(N, P)$$

such that

$$\beta\alpha = 0 \qquad \text{in } \operatorname{Ext}^{a+b}(L, N)$$
$$\gamma\beta = 0 \qquad \text{in } \operatorname{Ext}^{b+c}(M, P).$$

Our object is to define the Massey product $\{\gamma, \beta, \alpha\}$, which should be an element of

$$\frac{\operatorname{Ext}^{a+b+c-1}(L, P)}{\gamma \operatorname{Ext}^{a+b-1}(L, N) + (\operatorname{Ext}^{b+c-1}(M, P))\alpha}.$$

Here the group $\operatorname{Ext}^{a+b-1}(L, N)$ is to be interpreted as zero if $a + b - 1 < 0$, and similarly for $\operatorname{Ext}^{b+c-1}(M, P)$. It is sufficient for us to consider the cases in which a, b, c and $a + b + c - 1$ are each either 0 or 1.

Case 1. $b = 1$. (Perhaps this should be counted as three cases.) In this case we can represent β by a short exact sequence, as follows.

$$0 \to N \xrightarrow{i} E \xrightarrow{j} M \to 0$$

This leads to the following exact sequences, in which the boundary maps coincide, up to sign, with multiplication by β.

$$\operatorname{Ext}^a(L, N) \xrightarrow{i} \operatorname{Ext}^a(L, E) \xrightarrow{j} \operatorname{Ext}^a(L, M) \xrightarrow{\delta} \operatorname{Ext}^{a+1}(L, N)$$
$$\operatorname{Ext}^c(M, P) \xrightarrow{j} \operatorname{Ext}^c(E, P) \xrightarrow{i} \operatorname{Ext}^c(N, P) \xrightarrow{\delta} \operatorname{Ext}^{c+1}(M, P)$$

Since $\beta\alpha = 0$ and $\gamma\beta = 0$, we can write α, γ in the form

$$\alpha = j\alpha', \qquad \gamma = \gamma' i$$

where

$$\alpha' \in \operatorname{Ext}^a(L, E), \qquad \gamma' \in \operatorname{Ext}^c(E, P).$$

We have only to take the element

$$\gamma'\alpha' \in \operatorname{Ext}^{a+c}(L, P).$$

It is easy to check that its indeterminacy is

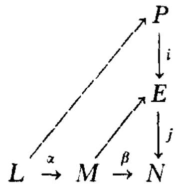
$$\gamma \operatorname{Ext}^a(L, N) + (\operatorname{Ext}^c(M, P))\alpha,$$

as given above.

Case 2. $b = 0$, $a + c = 1$. Perhaps this should be counted as two cases. They are somewhat special, because they are low-dimensional. Suppose first that $a = 0$, $c = 1$. Let

$$0 \to P \xrightarrow{i} E \xrightarrow{j} N \to 0$$

be an extension representing γ. Then the fact that $\gamma\beta = 0$ allows us to factor $\beta : M \to N$ through j; using also the fact that $\beta\alpha = 0$, we obtain the following diagram.



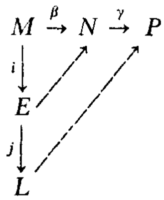
This yields an element of

$$\frac{\operatorname{Hom}(L, P)}{(\operatorname{Hom}(M, P))\alpha}.$$

The case $a = 1$, $c = 0$ is dual. Let

$$0 \to M \xrightarrow{i} E \xrightarrow{j} L \to 0$$

be an extension representing α; then we can construct the following diagram.



This yields an element of

$$\frac{\operatorname{Hom}(L, P)}{\gamma \operatorname{Hom}(L, N)}.$$

Case 3. $b = 0$, $a = c = 1$. Let

$$0 \to M \xrightarrow{i} E \xrightarrow{j} L \to 0$$
$$0 \to P \xrightarrow{i'} F \xrightarrow{j'} N \to 0$$

be extensions representing α, γ. The most convenient way to define an element of $\operatorname{Ext}^1(L, P)$ and check that it has the correct indeterminacy is to chase the element $\beta \in \operatorname{Ext}^0(M, N)$ back through the following diagram.

$$\begin{array}{ccccccc} & & \operatorname{Ext}^0(L, N) & \xrightarrow{j} & \operatorname{Ext}^0(E, N) & \xrightarrow{i} & \operatorname{Ext}^0(M, N) & \xrightarrow{\alpha} & \operatorname{Ext}^1(L, N) \\ & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} & & \\ \operatorname{Ext}^0(M, P) & \xrightarrow{\alpha} & \operatorname{Ext}^1(L, P) & \xrightarrow{j} & \operatorname{Ext}^1(E, P) & \xrightarrow{i} & \operatorname{Ext}^1(M, P) & & \end{array}$$

The reader may wonder why we do not place an equal emphasis on the following dual diagram.

$$\begin{array}{ccccccc} & & \operatorname{Ext}^0(M, P) & \xrightarrow{i'} & \operatorname{Ext}^0(M, F) & \xrightarrow{j'} & \operatorname{Ext}^0(M, N) & \xrightarrow{\gamma} & \operatorname{Ext}^1(M, P) \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha} & & \\ \operatorname{Ext}^0(L, N) & \xrightarrow{\gamma} & \operatorname{Ext}^1(L, P) & \xrightarrow{i'} & \operatorname{Ext}^1(L, F) & \xrightarrow{j'} & \operatorname{Ext}^1(L, N) & & \end{array}$$

The reason is that the element obtained from this diagram is the negative of that obtained from the first one. To prove this (and also for later use) it is convenient to give a direct construction of the required extension. Let us factor β in the form

$$\beta = j'\theta = \phi i;$$

we obtain the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \to & M & \xrightarrow{i} & E & \xrightarrow{j} & L & \to & 0 \\ & & \downarrow{\scriptstyle\theta} & & & & \downarrow{\scriptstyle\phi} & & \\ 0 & \to & P & \xrightarrow{i'} & F & \xrightarrow{j'} & N & \to & 0 \end{array}$$

We now form the maps

$$M \xrightarrow{(i,\theta)} E \oplus F \xrightarrow{(\phi, -j')} N$$

and define

$$G = \frac{\operatorname{Ker}(\phi, -j')}{\operatorname{Im}(i, \theta)}.$$

We check that we have an exact sequence

$$0 \to P \to G \to L \to 0,$$

yielding an element of $\operatorname{Ext}^1(L, P)$. By taking merely $\operatorname{Ker}(\phi, -j')$ or $\operatorname{Coker}(i, \theta)$, we obtain elements of $\operatorname{Ext}^1(E, P)$ and $\operatorname{Ext}^1(L, F)$. It is now easy to check that these are precisely the elements we want in chasing round the upper diagram, and their negatives are the elements we want in chasing round the lower diagram.

Finally, let us suppose given a functor T from our abelian category to itself, as in §3 above. Then it is clear that all the constructions above are compatible with T.

§5. TODA BRACKETS, I

In this section we shall show that the d and e invariants send Toda brackets (in homotopy) into Massey products (in homological algebra). For this purpose we shall generally suppose given four CW-complexes W, X, Y and Z, and three maps

$$W \xrightarrow{f} X \xrightarrow{g} Y \xrightarrow{h} Z$$

such that $hg \sim 0$, $gf \sim 0$.

First suppose given a specific homotopy

$$l: I \times W \to Y$$

such that $l(0, w)$ is constant and

$$l(1, w) = gfw.$$

Then we can define a map

$$G: X \cup_f CW \to Y$$

by

$$G(x) = g(x) \qquad (x \in X)$$
$$G(t, w) = l(t, w) \qquad (t \in I,\ w \in W).$$

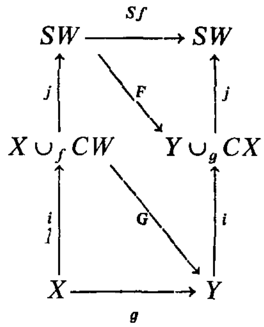
Again, we can define a map

$$F: SW \to Y \cup_g CX$$

by

$$F(t, w) = \begin{cases} (2t, fw) & (0 \leqq t \leqq \frac{1}{2}) \\ l(2 - 2t, w) & (\frac{1}{2} \leqq t \leqq 1). \end{cases}$$

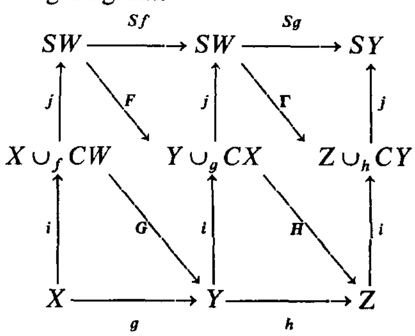
These maps figure in the following diagram.



Here the two triangles are homotopy-commutative, and the parallelogram becomes homotopy-commutative if one inserts the map

$$-1: SW \to SW.$$

If we suppose given also a specific homotopy $hg \sim 0$, we can construct similarly the right-hand half of the following diagram.



(5.1)

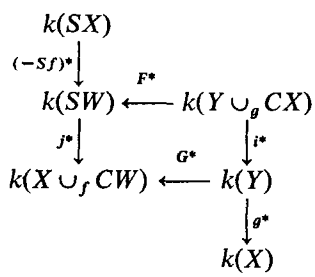
The Toda bracket $\{h, g, f\}$ is the composite

$$HF: SW \to Z.$$

LEMMA 5.2 (a). *Suppose that $e(Sf)$ and $e(g)$ are defined. Then a homotopy $gf \sim 0$ is such that $e(F)$ is defined, if and only if it is such that $e(G)$ is defined.*

(b) *Suppose that $e(Sg)$ and $e(h)$ are defined. Then a homotopy $hg \sim 0$ is such that $e(\Gamma)$ is defined if and only if it is such that $e(H)$ is defined.*

Proof. We have the following diagram, in which the columns are exact and $j^*F^* = -G^*i^*$.



According to the data, j^* is mono and i^* is epi. Therefore $F^* = 0$ if and only if $G^* = 0$. Similarly for $(SF)^*$ and $(SG)^*$. This proves part (a); substituting $X \xrightarrow{g} Y \xrightarrow{h} Z$ for $W \xrightarrow{f} X \xrightarrow{g} Y$, we obtain part (b).

We can now state the main result of this section.

THEOREM 5.3 (i). *Suppose that* $e(f)$ *is defined. Then*

$$d\{h, g, f\} \subset -\{e(f), d(g), d(h)\}.$$

(ii) *Suppose that* $e(g)$ *is defined. Then*

$$d\{h, g, f\} \subset \{d(Sf), e(g), d(h)\}.$$

(iii) *Suppose that* $e(h)$ *is defined. Then*

$$d\{h, g, f\} \subset -\{d(Sf), d(Sg), e(h)\}.$$

(iv) *Suppose that* $e(Sf)$ *and* $e(g)$ *are defined, and that we only consider homotopies* $gf \sim 0$ *such that* $e(F)$ *is defined* (*or equivalently, by Lemma* 5.2 (a), *such that* $e(G)$ *is defined*). *Then* $e\{h, g, f\}$ *is defined and*

$$e\{h, g, f\} \subset \{e(Sf), e(g), d(h)\}.$$

(v) *Suppose that* $e(Sf)$ *and* $e(h)$ *are defined. Then* $e\{h, g, f\}$ *is defined and*

$$e\{h, q, f\} \subset -\{e(Sf), d(Sg), e(h)\}.$$

(vi) *Suppose that* $e(Sg)$ *and* $e(h)$ *are defined, and that we only use homotopies* $hg \sim 0$ *such that* $e(H)$ *is defined* (*or equivalently, by Lemma* 5.2 (b), *such that* $e(\Gamma)$ *is defined*). *Then* $e\{h, g, f\}$ *is defined and*

$$e\{h, g, f\} \subset -\{d(S^2f), e(Sg), e(h)\}.$$

Proof. We tackle first the three cases in which the Massey product is defined by case (1) of §4, viz. the cases (ii), (iv) and (vi). For this purpose the objects L, M, E, N and P of §4 case (1) take the following values.

	L	M	E	N	P
Case (ii)	$k(Z)$	$k(Y)$	$k(Y \cup_g CX)$	$k(SX)$	$k(SW)$
Case (iv)	$k(Z)$	$k(Y)$	$k(Y \cup_g CX)$	$k(SX)$	$k(S^2W)$
Case (vi)	$k(Z)$	$k(SY)$	$k(S(Y \cup_g CX))$	$k(S^2X)$	$k(S^2W)$

It is to be noted that in case (vi), the invariant $e(Sg)$ is defined to be the short exact sequence

$$k(SY) \xleftarrow{i'^*} k(SY \cup_{Sg} CSX) \xleftarrow{j'^*} k(S^2X);$$

but by Proposition 2.4, this is the same as

$$k(SY) \xleftarrow{(Si)^*} k(S(Y \cup_g CX)) \xleftarrow{-(Sj)^*} k(S^2X).$$

We have now to construct

$$\alpha' \in \mathrm{Ext}^a(L, E), \qquad \gamma' \in \mathrm{Ext}^c(E, P)$$

as in §4 case (1). For this purpose we give the following values.

	α'	γ'
Case (ii)	$d(H)$	$d(F)$
Case (iv)	$d(H)$	$e(F)$
Case (vi)	$e(H)$	$-d(SF)$

The fact that these values have the required properties is proved by applying Propositions 3.1, 3.2 to the formulae

$$Hi \sim h, \qquad jF \sim Sf, \qquad (Sj)(SF) \sim S^2 f$$

where i, j are the maps appearing in

$$Y \xrightarrow{i} Y \cup_g CX \xrightarrow{j} SX.$$

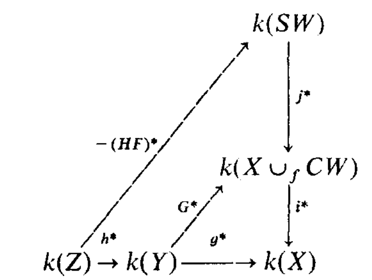
Using Proposition 3.2 again for the composite HF, we find the following results. In case (ii), $d(HF)$ represents the Massey product. In case (iv) $e(HF)$ is defined, and represents the Massey product. In case (vi) $e(HF)$ is defined, and $-e(HF)$ represents the Massey product. This completes cases (ii), (iv) and (vi).

We tackle next the two cases in which the Massey product is defined by case (2) of §4, viz. the cases (i) and (iii). For this purpose the objects L, M, E, N and P of §4 case (2) take the following values.

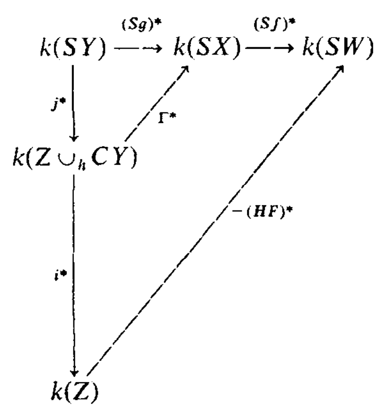
	L	M	E	N	P
Case (i)	$k(Z)$	$k(Y)$	$k(X \cup_f CW)$	$k(X)$	$k(SW)$
Case (iii)	$k(Z)$	$k(SY)$	$k(Z \cup_h CY)$	$k(SX)$	$k(SW)$

We have now to construct diagrams as in §4 case (2). The appropriate diagrams are obtained from Diagram 5.1, and are as follows.

Case (i)



Case (ii)



In both cases we see that $-d(HF)$ represents the Massey product. This completes cases (i) and (iii).

Finally, we tackle case (v). We first check that $e(HF)$ is defined. The fact that $(HF)^* = 0$, under the hypotheses given, follows by chasing round the following commutative diagram, in which the columns are exact.

$$\begin{array}{ccc}
k(SX) & \xleftarrow{\Gamma^*} & k(Z \cup_h CY) \\
\big\downarrow{\scriptstyle (Sf)^*} & & \big\downarrow{\scriptstyle i^*} \\
k(SW) & \xleftarrow{-(HF)^*} & k(Z) \\
\big\downarrow{\scriptstyle j^*} & & \big\downarrow{\scriptstyle h^*} \\
k(X \cup_f CW) & \xleftarrow{G^*} & k(Y)
\end{array}$$

Similarly for the fact that $(S(HF))^* = 0$.

We now recall that in case (v) the Massey product is defined by case (3) of §4. For this purpose the objects considered in §4 take the following values.

$$L = k(Z), \qquad M = k(SY), \qquad N = k(SX), \qquad P = k(S^2W),$$
$$E = k(Z \cup_h CY),$$
$$\alpha = e(h), \qquad \beta = (Sg)^*, \qquad \gamma = e(Sf).$$

We start with the element $-e(HF)$ in $\mathrm{Ext}^1(L, P)$. Its image in $\mathrm{Ext}^1(E, P)$ is $-e(HF)i^*$. By Proposition 3.2 this is

$$\begin{aligned} -e(iHF) &= e(\Gamma . Sf) \\ &= e(Sf) . \Gamma^* \\ &= \gamma . \Gamma^*. \end{aligned}$$

But the element Γ^* in $\mathrm{Ext}^0(E, N)$ projects to $(Sg)^* = \beta$ in $\mathrm{Ext}^0(M, N)$. Therefore $-e(HF)$ qualifies as a representative for the Massey product. This proves case (v), and completes the proof of Theorem 5.3.

Perhaps it should be pointed out that Theorem 5.3 is consistent with the behaviour of d, e under suspension S^r (as in §3), because of the behaviour of Toda brackets under suspension:

$$S^r\{h, g, f\} \subset (-1)^r\{S^r h, S^r g, S^r f\}.$$

In our applications r will always be even, so the signs $(-1)^r$ can be forgotten.

§6. AN ABELIAN CATEGORY

The construction of §3 requires a half-exact functor k taking values in an abelian category A. In the applications we shall take $k(K)$ to be the Grothendieck–Atiyah–Hirzebruch group $\tilde{K}_\Lambda(X)$ [10, 11] equipped with its operations Ψ^k [2]. We shall therefore need to consider $\tilde{K}_\Lambda(X)$ as an object in a suitable abelian category A. Actually the category A will depend on Λ, where $\Lambda = R$ or C; but we shall not display the symbol Λ in the notation. It is the object of this section to define the category A.

By definition, an object of the category A is to be a finitely-generated abelian group M provided with endomorphisms

$$\Psi^k : M \to M$$

(one for each integer k) and satisfying the following axioms.

$$\Psi^k . \Psi^l = \Psi^{kl} \tag{6.1}$$

$$\Psi^0 = 0, \qquad \Psi^1 = 1 \quad \text{and} \quad (\text{if } \Lambda = R)\ \Psi^{-1} = 1. \tag{6.2}$$

(6.3) For each $x \in M$ and $q \in Z$, the mod q value of $\Psi^k x$ is periodic in k with period q^e for some $e = e(x, q)$.

In this axiom, and below, the statement "$f(k)$ is periodic in k with period q^e" means simply "$k_1 \equiv k_2 \bmod q^e$ implies $f(k_1) = f(k_2)$". It is not asserted that q^e is the smallest possible period. In particular the condition is true for $q = 0$ in a trivial way.

By definition, a map in the category A is to be a homomorphism $\theta : M \to N$ of abelian groups which commutes with the operations Ψ^k.

EXAMPLE 6.4. *The functor $\tilde{K}_\Lambda$ associates to each finite connected CW-complex X an abelian group $\tilde{K}_\Lambda(X)$ provided with endomorphisms Ψ^k, and associates with each map $f : X \to Y$ an induced homomorphism*

$$f^* : \tilde{K}_\Lambda(Y) \to \tilde{K}_\Lambda(X).$$

The functor $\tilde{K}_\Lambda$ takes values in the category $A = A(\Lambda)$. In fact, axioms (6.1) and (6.2) are satisfied, according to [2 Theorem 5.1 (v), (vii)]; and axiom (6.3) is satisfied, according to [5 Theorem 5.1].

PROPOSITION 6.5. *The category A defined above is an abelian category, in the sense of* [14, Chapter IX].

The only point which requires detailed proof is the following.

LEMMA 6.6. *If M is an object in A, and N is a subgroup of M closed under the operations Ψ^k, then N satisfies axiom* (6.3).

Proof. This follows the lines of [5 Lemma 6.5]. Consider the subgroup S_t of elements x in M such that $q^t x \in N$. This an increasing sequence of Z-submodules in the finitely-generated Z-module M, therefore convergent. That is, there exists t such that $x \in M$, $q^{t+1} x \in N$ imply $q^t x \in N$. Now we use axiom (6.3) for M; given $y \in N$, there is an f such that the value of $\Psi^k y$ in $M/q^{t+1}M$ is periodic in k with period $q^{(t+1)f}$. That is, if $k \equiv l \bmod q^{(t+1)f}$, we have

$$\Psi^k y - \Psi^l y = q^{t+1} x$$

for some x in M. By our choice of t, this shows that

$$\Psi^k y - \Psi^l y \in qN.$$

We have only to take $e = (t + 1)f$. This completes the proof.

In §3, we assumed that $k(S^r X)$ could be calculated in terms of $k(X)$ by a functor T from A to A. It is clear what functor T we should take in the category A described above.

If M is an object in A, then the abelian group underlying TM is the same as that underlying M, but the operation Ψ^k in TM is $k^{\frac{1}{2}r}$ times that in M (where $r = 2$ if $\Lambda = C$ and $r = 8$ if $\Lambda = R$). It is clear that these new operations satisfy axioms (6.1), (6.2) and (6.3). Similarly, if $f: M \to N$ is a map in A, then Tf is to be the same homomorphism as f; this clearly commutes with the new operations.

It is now clear that we have an isomorphism

$$\tilde{K}_\Lambda(S^r X) \cong T\tilde{K}_\Lambda(X)$$

natural for maps of X; see [2 Corollary 5.3].

The theory given in §§3–5 can now be applied to the functors $k = \tilde{K}_R$ and $k = \tilde{K}_C$.

§7. AN INVARIANT DEFINED USING THE CHERN CHARACTER

We are now in a position to apply the theory given in §§3–6. To give applications, we shall begin by taking the spaces X and Y to be spheres of suitable dimension, so that we obtain information about stable homotopy groups of spheres. We shall write d_Λ, e_Λ for the invariants obtained by taking $k = \tilde{K}_\Lambda$, where $\Lambda = R$ or C.

We start with a preliminary discussion of the invariants d_Λ (7.1, 7.2). Next we show that the invariant e_C can be described in a more elementary way using the Chern character. As remarked in the introduction, there is considerable overlap at this point with work of Dyer [13]. We will discuss the relationship between e_C and the invariants d_R, e_R (7.14, 7.18). We will also give substantial information about the values taken by these invariants (e.g. 7.15, 7.16). There remain certain cases in which the invariant e_R is independent of e_C; we postpone these cases to §9.

We begin by considering the invariant d_Λ. Let θ be an element of π_r^S; choose a representative map

$$f: S^{q+r} \to S^q$$

for θ. *A priori,*

$$d_\Lambda(f) = f^*: \tilde{K}_\Lambda(S^q) \to \tilde{K}_\Lambda(S^{q+r})$$

depends on the residue class of q (mod 2 if $\Lambda = C$, mod 8 if $\Lambda = R$). I claim that it is sufficient to consider the case $q \equiv 0$ (mod 2 if $\Lambda = C$, mod 8 if $\Lambda = R$). In fact, suppose we know $d_\Lambda(f)$ in this case. Let P be a point; then $\tilde{K}_\Lambda^*(S^q)$ is a free module over $K_\Lambda^*(P)$, on one generator which lies in $\tilde{K}_\Lambda^0(S^q) = \tilde{K}_\Lambda(S^q)$. Therefore

$$d_\Lambda(f) = f^*: \tilde{K}_\Lambda^0(S^q) \to \tilde{K}_\Lambda^0(S^{q+r})$$

determines

$$f^*: \tilde{K}_\Lambda^{-t}(S^q) \to \tilde{K}_\Lambda^{-t}(S^{q+r}),$$

which is the same as

$$d_\Lambda(S^t f) = (S^t f)^*: \tilde{K}_\Lambda^0(S^{q+t}) \to \tilde{K}_\Lambda^0(S^{q+t+r}).$$

PROPOSITION 7.1. *d_Λ is zero on π_r^S for $r > 0$ unless $\Lambda = R$ and $r \equiv 1$ or 2 mod 8.*

First proof. By the above argument, d_Λ defines a homomorphism from π_r^S to G, where

$$G = \begin{cases} Z & \text{if } \Lambda = C \text{ and } r \equiv 0 \bmod 2 \\ & \text{or if } \Lambda = R \text{ and } r \equiv 0, 4 \bmod 8 \\ 0 & \text{if } \Lambda = C \text{ and } r \equiv 1 \bmod 2 \\ & \text{or if } \Lambda = R \text{ and } r \equiv 3, 5, 6, 7 \bmod 8. \end{cases}$$

Since π_r^S is a finite group, d_Λ must be zero.

Second proof. It is sufficient to consider the case of a map $f: S^{q+r} \to S^q$, where q, r are divisible by 2 if $\Lambda = C$, by 4 if $\Lambda = R$. Then the groups $\tilde{K}_\Lambda(S^q)$, $\tilde{K}_\Lambda(S^{q+r})$ are Z, and their operations Ψ^k are given by

$$\Psi^k x = k^{\frac{1}{2}q} x, \qquad \Psi^k x = k^{\frac{1}{2}(q+r)} x$$

respectively. If $r > 0$, the only homomorphism commuting with the operations is zero.

We now consider the case $\Lambda = R$, $r \equiv 1$ or $2 \bmod 8$. We take as our basic invariant the homomorphism

$$d_R : \pi_r^S \to Z_2$$

obtained by considering maps

$$f: S^{q+r} \to S^q$$

with $q \equiv 0 \bmod 8$. (It is understood, of course, that if we later wish to apply the theorems of §§3, 5 we shall still have to use the invariant d_R appropriate to spheres of the dimensions which actually arise).

THEOREM 7.2. *Assume $r \equiv 1$ or 2 mod 8 and $r > 0$. Then the invariant*

$$d_R : \pi_r^S \to Z_2$$

is an epimorphism; we have

$$\pi_r^S = Z_2 + \operatorname{Ker} d_R,$$

where the subgroup Z_2 is generated by μ_r.

This theorem includes Theorems 1.2 and 1.4. Its proof is deferred to §12.

We will now give an elementary construction, using the Chern character, for an invariant which we will later prove equivalent to e_C. This invariant has already been described in [6, 13]. See also [19].

Suppose given a map $f: S^{2n-1} \to S^{2q}$, where $n > q > 0$. If $\Lambda = R$ we assume that n and q are even. We use f to start the following cofibre sequence.

$$S^{2n-1} \xrightarrow{f} S^{2q} \xrightarrow{i} S^{2q} \cup_f e^{2n} \xrightarrow{j} S^{2n} \xrightarrow{-Sf} S^{2q+1}$$

Applying $\tilde{K}_\Lambda$, we obtain the following exact sequence.

$$0 \leftarrow \underset{\cong Z}{\tilde{K}_\Lambda(S^{2q})} \xleftarrow{i^*} \tilde{K}_\Lambda(S^{2q} \cup_f e^{2n}) \xleftarrow{j^*} \underset{\cong Z}{\tilde{K}_\Lambda(S^{2n})} \leftarrow 0$$

The group $\tilde{K}_\Lambda(S^{2q} \cup_f e^{2n})$ is therefore $Z + Z$; we can choose generators ξ, η so that ξ projects to the generator of $\tilde{K}_\Lambda(S^{2q})$, and η is the image of the generator in $\tilde{K}_\Lambda(S^{2n})$.

As in [4], we write ch_C for the Chern character

$$ch: K_C(X) \to H^*(X; Q),$$

and ch_R for the composite

$$K_R(X) \xrightarrow{c} K_C(X) \xrightarrow{ch} H^*(X; Q).$$

Let

$$h^{2q} \in H^{2q}(S^{2q} \cup_f e^{2n}; Z)$$
$$h^{2n} \in H^{2n}(S^{2q} \cup_f e^{2n}; Z)$$

be cohomology generators, corresponding under i^*, j^* to the generators in $H^{2q}(S^{2q}; Z)$, $H^{2n}(S^{2n}; Z)$. Then in $H^*(S^{2q} \cup_f e^{2n}; Q)$ we must have formulae of the following form.

$$\begin{aligned} ch_\Lambda \xi &= a_{2q}h^{2q} + \lambda a_{2n}h^{2n} \\ ch_\Lambda \eta &= \qquad\qquad a_{2n}h^{2n} \end{aligned} \tag{7.3}$$

Here we have

$$a_r = \begin{cases} 1 \text{ if } \Lambda = C \text{ and } r \equiv 0 \bmod 2 \\ 1 \text{ if } \Lambda = R \text{ and } r \equiv 0 \bmod 8 \\ 2 \text{ if } \Lambda = R \text{ and } r \equiv 4 \bmod 8. \end{cases}$$

(The coefficient a_{2n} is introduced into the term $\lambda a_{2n}h^{2n}$ for technical convenience). The coefficient $\lambda = \lambda(f)$ is some rational number. Of course, λ depends on the choice of ξ; we can replace ξ by $\xi + N\eta$, where N is any integer; this replaces λ by $\lambda + N$. To obtain an invariant of f we have therefore to consider the coset $\{\lambda(f)\}$ of $\lambda(f)$ in Q/Z, the rationals mod 1.

EXAMPLE 7.4. *Take $\Lambda = C$ and take f to be the Hopf map from S^3 to S^2. Then $S^2 \cup_f e^4$ is CP^2, the complex projective plane. We may take ξ to be the canonical line bundle minus the trivial line bundle. Then*

$$ch\xi = e^x - 1 = x + \tfrac{1}{2}x^2,$$

where x is the cohomology generator. Thus we have $\lambda = \frac{1}{2}$ and $\{\lambda(f)\} = \frac{1}{2}$ mod 1.

It is easy to establish the properties of the invariant $\{\lambda(f)\}$ directly, by following the pattern of §3; but in fact this is not necessary, as we will establish that the invariant $\{\lambda(f)\}$ is equivalent to the invariant $e(f)$ introduced in §3 (see Proposition 7.8). We will first show that the invariant $\{\lambda(f)\}$ determines $e(f)$, by using the Chern character to compute the operations Ψ^k in $S^{2q} \cup_f e^{2n}$.

PROPOSITION 7.5. *With the notation introduced above, the operations Ψ^k in $\tilde{K}_\Lambda(S^{2q} \cup_f e^{2q})$ are given by the following formulae.*

$$\begin{aligned} \Psi^k \xi &= k^q \xi + \lambda(k^n - k^q)\eta \\ \Psi^k \eta &= \qquad\qquad\quad k^n \eta. \end{aligned} \tag{7.6}$$

Proof. Since

$$ch_\Lambda : \tilde{K}_\Lambda(S^{2q} \cup_f e^{2n}) \to H^*(S^{2q} \cup_f e^{2n}; Q)$$

is monomorphic, the formulae can be checked by applying ch_Λ to both sides, using (7.3). To evaluate $ch_\Lambda \Psi^k \xi$ one uses [2 Theorem 5.1 (vi)].

COROLLARY 7.7. *The rational number λ has the form z/h, where $z \in Z$ and h is the highest common factor of the expressions $k^n - k^q$ as k runs over Z.*

This follows immediately, since the coefficients $\lambda(k^n - h^q)$ must be integers.

In order to discuss the invariant $e(f)$ we must now compute the appropriate Ext group. We write M, N for the objects $\tilde{K}_\Lambda(S^{2q})$, $\tilde{K}_\Lambda(S^{2n})$ of the abelian category A; thus the abelian group underlying M is Z and its operations are given by

$$\Psi^k x = k^q x;$$

similarly for N, in which

$$\Psi^k x = k^n x.$$

The following proposition computes $\mathrm{Ext}^1(M, N)$.

PROPOSITION 7.8. *There is a monomorphism*

$$\theta : \mathrm{Ext}^1(M, N) \to Q/Z$$

such that for any map

$$f : S^{2n-1} \to S^{2q}$$

we have

$$\theta(e(f)) = \{\lambda(f)\}.$$

The image of θ is the subgroup of cosets $\{z/h\}$, where z, h are as in Corollary 7.7.

The following proposition computes $\mathrm{Ext}^1_S(M, N)$.

PROPOSITION 7.9. *There is a monomorphism*

$$\theta_S : \mathrm{Ext}^1_S(M, N) \to Q/Z$$

such that for any map

$$f : S^{2n-1} \to S^{2q}$$

we have

$$\theta_S(e(f)) = \{\lambda(f)\}.$$

The image of θ_S is the subgroup of cosets $\{z/m(t)\}$, where $z \in Z$, $t = n - q$, and the numerical function $m(t)$ is as in [4 §2].

The explicit definitions of θ, θ_S will be given during the course of the proof. We begin by explaining the use of factor sets in studying our extensions.

Suppose given an extension

$$0 \leftarrow M \leftarrow E \leftarrow N \leftarrow 0$$

in the category A, where N, M are as above. Then we can choose generators ξ, η in E so that ξ projects to the generator in M and η is the image of the generator in N. The operations in E must be given by formulae of the following form.

$$\begin{aligned} \Psi^k \xi &= k^q \xi + c(k)\eta \\ \Psi^k \eta &= k^n \eta \end{aligned} \tag{7.10}$$

The integers $c(k)$ constitute a "factor set" describing the operations Ψ^k in the extension E.

LEMMA 7.11. *This factor set has the form*

$$c(k) = \lambda(k^n - k^q) \tag{7.12}$$

for some $\lambda \in Q$.

This lemma shows that the "abstract" algebraic extensions are described by the same formulae that we have already found in the "concrete" topological situation.

Proof of Lemma 7.11. By Axiom (6.1) we have in E the relation $\Psi^k\Psi^l = \Psi^{kl}$. This yields

$$c(kl) = c(k)l^q + c(l)k^n.$$

Interchanging k and l, we find

$$c(kl) = c(l)k^q + c(k)l^n.$$

Choosing l so that $l^n - l^q \neq 0$, we find

$$c(k) = \frac{c(l)(k^n - k^q)}{l^n - l^q}.$$

That is,

$$c(k) = \lambda(k^n - k^q)$$

for some rational λ. This proves the lemma.

If we replace ξ by $\xi + N\eta$, we replace the factor set $c(k)$ by $c(k) + N(k^n - k^q)$. This replaces λ by $\lambda + N$.

It is now clear how to define

$$\theta : \mathrm{Ext}^1(M, N) \to Q/Z;$$

by definition, the function θ will assign to any extension E the coset $\{\lambda\}$ in Q/Z given by formulae (7.10) and (7.12). The equation

$$\theta(e(f)) = \{\lambda(f)\}$$

follows immediately by comparing formulae (7.6), (7.10) and (7.12).

We have to remark that θ is a homomorphism; in fact, it is not hard to check that the Baer sum in $\mathrm{Ext}^1(M, N)$ corresponds to addition of factor sets, i.e. to addition in Q/Z. It is also clear that θ is a monomorphism.

It remains to discuss the image of θ. It is clear that in Lemma 7.11 the rational number λ has the form z/h, as in Corollary 7.7. We require the converse result.

LEMMA 7.13. *Each rational λ of the form z/h arises by formulae* (7.10), (7.12) *from some extension E and some choice of ξ.*

Proof. We use the formulae (7.10) and (7.12) to define operations Ψ^k on the free abelian group generated by ξ and η. We easily check that these operations satisfy axioms (6.1) to (6.3). This gives the extension E required.

This completes the proof of Proposition 7.8. It remains to check that our proceedings are compatible with suspension. We easily check from our formulae that if M and N are as above, then the following diagram is commutative.

$$\begin{array}{ccc} \mathrm{Ext}^1(M, N) & & \\ \downarrow T & \searrow^{\theta} & \\ & & Q/Z \\ & \nearrow_{\theta} & \\ \mathrm{Ext}^1(TM, TN) & & \end{array}$$

Therefore θ passes to the limit and defines a monomorphism

$$\theta_S : \mathrm{Ext}^1_S(M, N) \to Q/Z$$

such that $\theta_S(e(f)) = \{\lambda(f)\}$, as required. It remains to discuss the image of θ_S. Let n and q

tend to infinity so that their difference $n - q = t$ remains constant. Then according to [4 §2], the integer h increases, and ultimately attains a constant value, namely $m(t)$. This completes the proof of Proposition 7.9.

We shall now regard our invariant $e(f)$ as taking values in the rationals mod 1, in the case under discussion. We repeat that this is the case $X = S^{2n-1}$, $Y = S^{2q}$, where n and q are even if $\Lambda = R$.

At this point we possess a choice of invariants defined on the r-stem π_r^S for $r \equiv 3 \bmod 4$. In fact, by considering $e_R(f)$ for maps $f: S^{2q+r} \to S^{2q}$ with $2q \equiv 0 \bmod 8$ we obtain one invariant, say e'_R; by considering $e_R(f)$ for maps $f: S^{2q+r} \to S^{2q}$ with $2q \equiv 4 \bmod 8$ we obtain another invariant, say e''_R. We also have the invariant $e_C(f)$ for maps $f: S^{2q+r} \to S^{2q}$. We must discuss the relations between these invariants.

PROPOSITION 7.14. *If* $r \equiv 7$ *mod* 8 *then*

$$e_C = e'_R = e''_R : \pi_r^S \to Q/Z.$$

If $r \equiv 3$ *mod* 8 *then*

$$e_C = 2e'_R : \pi_r^S \to Q/Z$$

and

$$e''_R = 2e_C = 4e'_R : \pi_r^S \to Q/Z.$$

Proof. Consider the following diagram.

$$\begin{array}{ccccccc} 0 \leftarrow & \tilde{K}_R(S^{2q}) & \leftarrow & \tilde{K}_R(S^{2q} \cup_f e^{2n}) & \leftarrow & \tilde{K}_R(S^{2n}) & \leftarrow 0 \\ & \downarrow{c'} & & \downarrow{c} & & \downarrow{c''} & \\ 0 \leftarrow & \tilde{K}_C(S^{2q}) & \leftarrow & \tilde{K}_C(S^{2q} \cup_f e^{2n}) & \leftarrow & \tilde{K}_C(S^{2n}) & \leftarrow 0 \end{array}$$

Let us identify $\tilde{K}_\Lambda(S^{2q})$ with Z; then the map c' is multiplication by 1 if $2q \equiv 0 \bmod 8$, by 2 if $2q \equiv 4 \bmod 8$. Similarly for c''. So if $2q \equiv 0 \bmod 8$ we have

$$e_C(f) = c''.e_R(f);$$

if $2n \equiv 0 \bmod 8$ we have

$$e_R(f) = e_C(f).c'.$$

Similarly, consider the following diagram.

$$\begin{array}{ccccccc} 0 \leftarrow & \tilde{K}_C(S^{2q}) & \leftarrow & \tilde{K}_C(S^{2q} \cup_f e^{2n}) & \leftarrow & \tilde{K}_C(S^{2n}) & \leftarrow 0 \\ & \downarrow{r'} & & \downarrow{r} & & \downarrow{r''} & \\ 0 \leftarrow & \tilde{K}_R(S^{2q}) & \leftarrow & \tilde{K}_R(S^{2q} \cup_f e^{2n}) & \leftarrow & \tilde{K}_R(S^{2n}) & \leftarrow 0 \end{array}$$

This is a diagram in the category A, since r commutes with Ψ^k [9]. The map r' is multiplication by 2 if $2q \equiv 0 \bmod 8$, by 1 if $2q \equiv 4 \bmod 8$. Similarly for r''. So if $2q \equiv 4 \bmod 8$ we have

$$e_R(f) = r''e_c(f);$$

if $2n \equiv 4 \bmod 8$ we have

$$e_C(f) = e_R(f).r'.$$

This yields the results stated; actually it gives two proofs for each.

We will now describe the values taken by the invariants considered in Proposition 7.14.

THEOREM 7.15. *If $r = 4s - 1$, then the image of*

$$e'_R : \pi_r^S \to Q/Z$$

is precisely the subgroup of cosets $\{z/m(2s)\}$ $(z \in Z)$; that is it is, a cyclic group of order $m(2s)$.

It follows from Proposition 7.9 that the image of e'_R is contained in the subgroup indicated. In order to prove that the image of e'_R is the whole of this subgroup, we compute e'_R on the image of the J-homomorphism.

THEOREM 7.16. *If $r = 4s - 1$, then the value of the composite*

$$e'_R J : \pi_r(SO) \to Q/Z$$

on a suitable generator of $\pi_r(SO)$ is $\frac{1}{2}\alpha_{2s}$ mod 1.

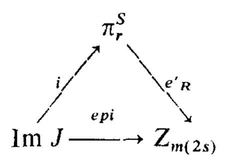
In these theorems, the numerical function $m(t)$ and the rational number $\frac{1}{2}\alpha_{2s}$ are as in [4 §2]. That is,

$$\tfrac{1}{2}\alpha_{2s} = (-1)^{s-1}\frac{B_s}{4s},$$

where B_s is the s^{th} Bernoulli number. Theorem 7.16 thus reproves the result of Milnor and Kervaire, as improved by Atiyah and Hirzebruch [10]. The value of $\frac{1}{2}\alpha_{2s}$ mod 1 is explicitly given by [4 Theorem 2.5], which was proved for this purpose. We recall from [4 Theorem 2.6] that the denominator of $\frac{1}{2}\alpha_{2s}$, when this fraction is expressed in its lowest terms, is precisely $m(2s)$. Theorem 7.15 will therefore follow from Theorem 7.16. The proof of Theorem 7.16 is deferred to §10.

I believe that Theorem 7.16 was known to earlier workers, for example, Atiyah (ca. 1960/61); see also Dyer [13, Theorem 1 and formulae on p.370].

Theorems 1.5, 1.6 will follow immediately from Theorems 7.15, 7.16 and [4 Theorem 3.7]. Suppose for example that $r = 4s - 1 \equiv 3 \bmod 8$. Then by Theorems 7.15, 7.16 we have the following diagram.



But by [4 Theorem 3.7] the image of J is cyclic of order dividing $m(2s)$. Therefore the diagram provides a direct sum splitting. Similarly for the case $r \equiv 7 \bmod 8$, except that [4 Theorem 3.7] only states that the order of Im J divides $2m(2s)$.

We will substitute a few small numbers in Theorem 7.16 in order to provide examples. For $r \equiv -1 \bmod 4$, let us take the generator in $\pi_r(SO)$, and let its image under $J : \pi_r(SO) \to \pi_r^S$ be j_r. Then we have:

EXAMPLE 7.17.

$$e'_R j_3 = \tfrac{1}{24}$$
$$e'_R j_7 = -\tfrac{1}{240}$$
$$e'_R j_{11} = \tfrac{1}{504}$$
$$e'_R j_{15} = -\tfrac{1}{480}$$
$$e'_R j_{19} = \tfrac{1}{264}$$

Inspecting Toda's tables [18, pp.186–188] we see that

$$e'_R : \pi_3^S \to Z_{24},$$

$$e'_R : \pi_7^S \to Z_{240}$$

and

$$e'_R : \pi_{11}^S \to Z_{504}$$

are isomorphisms, while

$$e'_R : \pi_{15}^S \to Z_{480}$$

and

$$e'_R : \pi_{19}^S \to Z_{264}$$

are epimorphisms with kernel Z_2. Toda gives the elements $\eta\kappa$ in π_{15}^S and $\bar{\sigma} \in \{\nu, \bar{\nu} + \varepsilon, \sigma\}$ in π_{19}^S as generating Z_2 summands; these elements are annihilated by e'_R, as we see using Proposition 3.2 and Theorem 5.3 (v).

We have still to describe the invariant e_C on the r-stem for $r \equiv 1 \bmod 4$. In this case the integer $m(t)$ occurring in Proposition 7.9 is 2, and so e_C gives a homomorphism from π_r^S to Z_2. We have already remarked that if $r \equiv 1 \bmod 8$ the invariant d_R gives a homomorphism from π_r^S to Z_2.

THEOREM 7.18. *If $r \equiv 1$ mod 8 we have*

$$e_C = d_R : \pi_r^S \to Z_2.$$

The behaviour of d_R has been described in Theorem 7.2. The proof of this theorem is deferred to §12.

For completeness we describe the value of this invariant on the image of the J-homomorphism.

PROPOSITION 7.19. *Suppose $r \equiv 1$ mod 8. Then the composite*

$$e_C J = d_R J : \pi_r(SO) \to Z_2$$

is an isomorphism for $r = 1$ and is zero for $r > 1$.

For $r = 1$ the J-homomorphism becomes an isomorphism from $\pi_1(SO) = Z_2$ to $\pi_1^S = Z_2$. The value of d_R on π_1^S is well known, and the value of e_C is given by Example 7.4. For $r > 1$ the proof of this proposition is deferred to §10.

PROPOSITION 7.20. *If $r \equiv 5$ mod 8 we have*

$$e_C = 0 : \pi_r^S \to Z_2.$$

This will follow immediately from Proposition 7.1, by using the following lemma.

LEMMA 7.21. *Suppose given $f: S^{2q+r} \to S^{2q}$ with $2q \equiv 0$ mod 8 and $r \equiv 1$ mod 4. If $d_R(f) \equiv 0$, then $e_C(f) = 0$.*

Proof. Consider the following diagram.

$$\begin{array}{ccccccc}
\tilde{K}_R(S^{2q+r}) & \xleftarrow{f^*} & \tilde{K}_R(S^{2q}) & \leftarrow & \tilde{K}_R(S^{2q} \cup_f e^{2q+r+1}) & \leftarrow & \tilde{K}_R(S^{2q+r+1}) = Z_2 \text{ or } 0 \\
\downarrow & & \downarrow \cong & & \downarrow & & \downarrow 0 \\
\tilde{K}_C(S^{2q+r}) & \leftarrow & \tilde{K}_C(S^{2q}) & \leftarrow & \tilde{K}_C(S^{2q} \cup_f e^{2q+r+1}) & \leftarrow & \tilde{K}_C(S^{2q+r+1}) = Z
\end{array}$$

If $f^* = 0$, the diagram provides a splitting of the extension $e_C(f)$.

§8. RELATION WITH THE HOPF INVARIANT

In this section we shall establish the relation between the invariant e_C discussed in §7 and the Hopf invariant (mod 2 or mod p) in the sense of Steenrod. As mentioned in the introduction, this leads to a proof, first published by Dyer [13], of the non-existence of elements of Hopf invariant one (mod 2 or mod p).

We first recall the definition of the Hopf invariant in the sense of Steenrod. As in §7, we take a map $f: S^{2n-1} \to S^{2q}$ and form $S^{2q} \cup_f e^{2n}$. Let p be a prime; and suppose that $n - q = k(p-1)$, where k is an integer. Then in $H^*(S^{2q} \cup_f e^{2n}; Z_p)$ we have a formula of the following form.

(8.1) $$P^k \rho h^{2q} = \mu \rho h^{2n}.$$

Here P^k is the Steenrod reduced power (interpreted as Sq^{2k} if $p = 2$); the homomorphism

$$\rho: H^*(X; Z) \to H^*(X; Z_p)$$

is induced by the quotient map $Z \to Z_p$ of coefficients; the classes h^{2q} and h^{2n} are generators in $H^*(S^{2a} \cup_f e^{2n}; Z)$, as in §7; and μ is some element of Z_p.

It is easy to see that μ is an invariant of f. We will now show that the value of μ is determined by $e_C(f)$. For this purpose we define Q'_p to be the additive group of rationals with denominators prime to p; then we have a unique homomorphism $\rho' : Q'_p \to Z_p$ extending the quotient map $Z \to Z_p$.

PROPOSITION 8.2. *We have*

$$p^k e_C(f) \in Q'_p$$
$$\mu = -\rho'(p^k e_C(f)).$$

Proof. Formula (7.3) states that

$$ch\xi = h^{2q} + \lambda h^{2n},$$

where $e_C(f) = \{\lambda\}$. We now appeal to [1, Theorems 1, 2]. The statements of this paper involve a further numerical function; we set

$$M(r) = \prod_p p^{[r/p-1]}.$$

(This function is written $m(r)$ in [1], but it is different from the function $m(t)$ of [4 §2].) In our application, we take the integer "r" of [1] to be $k(p-1)$. Theorem 1 of [1] now states that the class $M(r)\lambda h^{2n}$ is integral; that is, $M(r)\lambda \in Z$; thus $p^k\lambda \in Q'_p$. Moreover, in [1, Theorem 2], the class "$ch_{q,0}\xi$" must be h^{2q}, and the class "$ch_{q,r}\xi$" must be $M(r)\lambda h^{2n}$. Thus [1, Theorem 2 part (5)] gives

$$\rho(M(r)\lambda h^{2n}) = \frac{M(r)}{p^k} \chi(P^k)\rho h^{2q}.$$

Here χ means the canonical anti-automorphism of the Steenrod algebra. But in the complex $S^{2q} \cup_f e^{2n}$ decomposable Steenrod operations are zero; thus

$$\chi(P^k)\rho h^{2q} = -P^k \rho h^{2q}.$$

Since $M(r)/p^k$ is an integer prime to p, this leads at once to the result given.

COROLLARY 8.3. *The Hopf invariant in the sense of Steenrod is zero except in the following cases;*

(a) $p = 2$, $k = 1, 2$ *or* 4;

(b) p *is odd*, $k = 1$.

It is (of course) classical that non-zero values can occur in the exceptional cases given.

Proof. According to Proposition 7.9, we have $e_C(f) = \{z/m(t)\}$ where $z \in Z$ and $t = k(p-1)$. We have only to check that $m(t)$ contains the prime p to the power $(k-1)$ at most—except in the exceptional cases. This follows from the explicit definition of $m(t)$ given in [4, §2].

COROLLARY 8.4. *The stable group* π^S_{2p-3} *contains an element* α *with* $p\alpha = 0$ *and* $e_C(\alpha) = -1/p \bmod 1$.

In fact, the p-component of π^S_{2p-3} is known to be Z_p; and it is known that we can choose a generator α whose Hopf invariant is 1 mod p.

The same argument shows that we can find elements in the 2-components of π^S_1, π^S_3 and π^S_7 whose e_C-invariants are $\frac{1}{2}$ mod 1, $\frac{1}{4}$ mod $\frac{1}{2}$ and $\frac{1}{16}$ mod $\frac{1}{8}$.

§9. THE INVARIANT e_R ON THE r-STEM FOR $r \equiv 0, 1 \bmod 8$

In this section we will add to the discussion of §7 by discussing the invariant e_R as it applied to maps $f: S^{2q+r} \to S^{2q}$ with $r \equiv 0$ or 1 mod 8 and $2q \equiv 0$ mod 8. The results are stated in Theorems 9.4, 9.5.

There are of course other possibilities for the dimensions of the spheres; one of them will actually arise in the proof of Proposition 12.17. The earnest student may consider the e_R-invariants of maps $f: S^{n-1} \to S^t$, where n, t run over the congruence classes 0, 1, 2 and 4 mod 8, so obtaining 16 cases. He will find that all the resulting invariants are determined by those we consider in this paper.

We will begin by computing the Ext groups which arise in our case. As before, let M be the object of the abelian category A in which the underlying group is Z and the operations are given by

$$\Psi^k x = k^q x.$$

Let N be the similar object in which the underlying group is Z and the operations are given by

$$\Psi^k x = k^n x.$$

Let N' be the quotient object N/vN, where v is some positive integer; thus the abelian group underlying N' is Z_v. We shall consider only the case $\Lambda = R$, and so we assume that q and n are even.

We have already computed $\mathrm{Ext}^1_S(M, N)$, which is a cyclic group (Proposition 7.9). The next result computes $\mathrm{Ext}^1_S(M, N')$.

PROPOSITION 9.1. *The quotient map $N \to N'$ induces an isomorphism*

$$\mathrm{Ext}^1_S(M, N)/\nu\, \mathrm{Ext}^1_S(M, N) \xrightarrow{\cong} \mathrm{Ext}^1_S(M, N').$$

It follows that we may represent $Ext^1_S(M, N')$ as the group of rationals $z/m(t)$ modulo 1 and $\nu/m(t)$, where $z \in Z$ and $t = n - q$.

Proof. The exact sequence

$$O \to N \xrightarrow{\nu} N \to N' \to O$$

induces an exact sequence

$$\mathrm{Ext}^1(M, N) \xrightarrow{\nu} \mathrm{Ext}^1(M, N) \to \mathrm{Ext}^1(M, N')$$

and so (passing to direct limits) an exact sequence

$$\mathrm{Ext}^1_S(M, N) \xrightarrow{\nu} \mathrm{Ext}^1_S(M, N) \to \mathrm{Ext}^1_S(M, N').$$

All that is required is to show that the map

$$\mathrm{Ext}^1_S(M, N) \to \mathrm{Ext}^1_S(M, N')$$

is epi. By splitting N' into p-components, we see that it is sufficient to consider the case $\nu = p^f$.

Suppose then that $\nu = p^f$, and suppose given an exact sequence

$$0 \leftarrow M \leftarrow E \leftarrow N' \leftarrow 0$$

in the category A. We may choose in E an element ξ projecting to the generator in M; we may write η for the image in E of the generator in N'. The operations Ψ^k in E must be given by formulae of the following form.

$$\begin{aligned} \Psi^k\xi &= k^q\xi + c(k)\eta \\ \Psi^k\eta &= \qquad\; k^n\eta \end{aligned} \tag{9.2}$$

Here the coefficients $c(k)$ lie in Z_ν, and constitute a "factor set".

We now invoke Axiom 6.3, which shows that the value of $c(k)$ modulo $\nu = p^f$ is periodic in k with period p^{ef} for some e. Now the multiplicative group G of residue classes prime to p, modulo p^{ef}, is cyclic if p is odd; let l be a generator for G, or for $G/\{\pm 1\}$ if $p = 2$. From the equation $\Psi^{kl} = \Psi^k\Psi^l$, we find

$$c(kl) = l^q c(k) + k^n c(l) \qquad \bmod \nu. \tag{9.3}$$

(Compare the proof of Lemma 7.11.) By induction over r, we find that

$$c(l^r) = \frac{l^{rn} - l^{rq}}{l^n - l^q}\, c(l) \qquad \bmod \nu.$$

Since we have assumed we are in the case $\Lambda = R$, we have $\Psi^{-k} = \Psi^k$, and thus

$$c(-l^r) = \frac{l^{rn} - l^{rq}}{l^n - l^q}\, c(l) \qquad \bmod \nu.$$

(Recall now that n and q are even.) We have thus shown that

$$c(k) = (k^n - k^q)\mu \qquad \bmod \nu$$

for all k prime to p, where μ is the rational number $c(l)/(l^n - l^q)$. It is now easy to see that we have

$$c(k) = (k^n - k^q)\lambda \qquad \bmod v$$

for all k prime to p, where λ is a rational number whose denominator is a power of p.

Next recall that the class of E in $\mathrm{Ext}^1_S(M, N')$ is not affected by applying the "eight-fold suspension operator" T (see §3). Suppose we do this t times; then the equation

$$c(k) = (k^n - k^q)\lambda \qquad \bmod v$$

(valid for k prime to p) becomes

$$k^{4t}c(k) = (k^{n+4t} - k^{q+4t})\lambda \qquad \bmod v$$

(for k prime to p). We can easily choose t large enough to satisfy the following two conditions.

(i) $k^{4t}c(k) = 0 \bmod v$ wherever k is divisible by p.

(ii) $(k^{n+4t} - k^{q+4t})\lambda$ is integral and divisible by v whenever k is divisible by p.

The equation

$$k^{4t}c(k) = (k^{n+4t} - k^{q+4t})\lambda \qquad \bmod v$$

will thus be true for all k. We have shown that the factor set $k^{4t}c(k)$ has the form considered in §7; thus E represents an element in the image of

$$\mathrm{Ext}^1_S(M, N) \to \mathrm{Ext}^1_S(M, N').$$

This completes the proof.

As a particular case of Proposition 9.1, we may put $v = 2$. Then the operations Ψ^k in N' are independent of n, being given by

$$\Psi^k x = \begin{cases} x & (k \text{ odd}) \\ 0 & (k \text{ even}). \end{cases}$$

We have

$$\mathrm{Ext}^1_S(M, N') \cong Z_2.$$

In this case the proof given above specialises a little. Equation (9.3) shows that the factor set $c(k)$ gives a homomorphism from G, the multiplicative group of odd numbers modulo 2^{ef}, to the additive group Z_2. We arrive at two factor sets; the zero factor set, and that given by

$$c(k) = \begin{cases} 0 & \text{for } k \equiv \pm 1 \bmod 8 \\ 1 & \text{for } k \equiv \pm 3 \bmod 8. \end{cases}$$

The latter represents the non-zero element of $\mathrm{Ext}^1_S(M, N')$.

Next, let $f: S^{2q+r} \to S^{2q}$ be a map with $r \equiv 0$ or $1 \bmod 8$ and $2q \equiv 0 \bmod 8$. Then we have

$$\tilde{K}_R(S^{2q}) = M, \qquad \tilde{K}_R(S^{2q+r+1}) = N'$$

and so

$$e_R(f) \in \mathrm{Ext}^1_S(M, N') = Z_2.$$

Thus e_R gives a homomorphism from $\mathrm{Ker}\, d_R \subset \pi_S$ to Z_2.

THEOREM 9.4. *If* $r \equiv 0$ *or* 1 *mod* 8 *and* $r > 1$ *then* e_R *maps* Ker d_R *onto* Z_2, *and* Ker e_R *is a direct summand in* Ker d_R.

We note that if $r \equiv 0 \bmod 8$ then Ker $d_R = \pi_r^S$, by Proposition 7.1. If $r \equiv 1 \bmod 8$ then Ker d_R is a direct summand in π_r^S, by Theorem 7.17, and therefore Ker e_R is a direct summand in π_r^S.

Theorem 9.4 will follow immediately from the following result.

THEOREM 9.5. *If* $r \equiv 0$ *or* 1 *mod* 8 *and* $r > 1$ *then the composite*

$$e_R J : \pi_r(SO) \to Z_2$$

is an isomorphism.

(Note that e_R is defined on Im J, by Proposition 7.19.)

We see that Theorem 1.1 will follow immediately from Theorem 9.5; also Theorem 1.3 will follow immediately from Theorems 7.2 and 9.5. The proof of Theorem 9.5 will be given in §10.

§10. THE VALUES OF THE INVARIANTS ON THE IMAGE OF J

In §§7, 9 we have introduced certain invariants; in this section we shall compute the values which they take on the image of the stable J-homomorphism

$$J : \pi_r(SO) \to \pi_r^S.$$

Our main object, then, is to prove Theorems 7.16, 7.19 and 9.5.

We will first show that if we use an element in the image of the J-homomorphism as an attaching map, then the resulting two-cell complex is, in fact, a Thom complex. More precisely, suppose given a map $\varphi : S^r \to SO(q)$. We can apply the "Hopf construction" J to φ; we obtain the map

$$J\phi : S^{q+r} \to S^q$$

and the two-cell complex

$$X = S^q \cup_{J\phi} e^{q+r+1}.$$

On the other hand, we can use φ to define an E^q bundle over S^{r+1}, and so obtain a Thom complex, which actually has the form

$$Y = S^q \cup e^{q+r+1}.$$

LEMMA 10.1. *The complexes* X *and* Y *are homotopy-equivalent. With suitable choices of sign in the constructions given above, we can choose the equivalence to have degree* $+1$ *on both cells.*

I believe that this lemma was known to earlier workers, for example, Atiyah (ca. 1960); see also [13, p.370].

Proof. We first discuss the Thom complex Y. The E^q-bundle over S^{r+1} can be obtained from $E^q \cup (E^{r+1} \times E^q)$ by identifying each point (x, y) in $S^r \times E^q$ with the point $(\varphi x)y$ in E^q. (Here $SO(q)$ acts on E^q in the usual way.) We can now obtain the Thom complex by further identifying $S^{q-1} \cup (E^{r+1} \times S^{q-1})$ to a single point.

We now discuss the Hopf construction. To construct the map $J\varphi$, we realise S^{q+r} as the boundary of $E^{r+1} \times E^q$. We map $S^r \times S^{q-1}$ to S^{q-1} by

$$(J\phi)(x, y) = (\phi x)y \qquad (y \in S^{q-1});$$

we extend to a map from $S^r \times E^q$ to the upper hemisphere E^q_+ of S^q, say

$$(J\phi)(x, y) = (\phi x)y \qquad (y \in E^q);$$

we also extend it to a map from $E^{r+1} \times S^{q-1}$ to the lower hemisphere E^q_- of S^q. (Actually this construction differs in sign from the one the author would usually prefer.)

The complex X is now

$$S^q \cup_{J\phi} (E^{r+1} \times S^q).$$

It will not alter its homotopy type if we identify E^q_- to a point. By doing this we obtain precisely the description given above for Y. This completes the proof.

Proof of Theorem 7.16. We may start from a real bundle β over S^{r+1}, where $r = 4s$, such that β represents a generator of $\tilde{K}_R(S^{r+1})$. With the notation of §7, this is expressed by the equation

$$ch_{2s} c\beta = a_{4s} h^{4s}.$$

We may suppose that the structural group of β is Spin(q), where q is divisible by 8.

We now consider the Thom complex $S^q \cup e^{q+4s}$ corresponding to β, and we make use of the Thom isomorphism φ_K [4 §4]. In $\tilde{K}_R(S^q \cup e^{q+4s})$ we have the element $\varphi_K 1$; moreover, with the notation of [4 §§2, 5] we have

$$\varphi_H^{-1} ch_R \varphi_K 1 = 1 + \tfrac{1}{2}\alpha_{2s} a_{4s} h^{4s}$$

[4, Proposition 5.2]. That is, we have

$$ch_R \varphi_K 1 = h^q + \tfrac{1}{2}\alpha_{2s} a_{q+4s} h^{q+4s}.$$

We may take $\varphi_K 1$ for our generator ξ. This yields

$$e'_R(J\beta) = \tfrac{1}{2}\alpha_{2s} \qquad \text{mod } 1,$$

which proves Theorem 7.16.

Proof of Theorem 7.19, for the case $r > 1$. As in the previous proof, we may start with a real bundle β over S^{r+1}, with structural group Spin(q), where q is divisible by 8. As above, we obtain a generator $\varphi_K 1$ in $\tilde{K}_R(S^q \cup_{J\beta} e^{q+r+1})$, which restricts to the generator in $\tilde{K}_R(S^q)$. Therefore the generator in $\tilde{K}_R(S^q)$ is annihilated by $(J\beta)^*$; that is, $d_R(J\beta) = 0$. Lemma 7.21 now shows that $e_C J\beta = 0$.

First proof of Theorem 9.5. As in the two previous proofs, we may start from a real bundle β over S^{r+1} such that β represents a generator of $\tilde{K}_R(S^{r+1})$, and we may obtain a generator $\varphi_K 1$ in $\tilde{K}_R(S^q \cup_{J\beta} e^{q+r+1})$. We now wish to calculate $\Psi^k \varphi_K 1$ (at least for k odd). By [4 §5, especially Theorem 5.15], we have

$$\begin{aligned} \varphi_K^{-1} \Psi^k \varphi_K 1 &= \rho^k \beta \\ &= \begin{cases} 1 & \text{if } k \equiv \pm 1 \text{ mod } 8 \\ 1 + \beta & \text{if } k \equiv \pm 3 \text{ mod } 8. \end{cases} \end{aligned}$$

With the notation of §9, this gives

$$\Psi^k\xi = \begin{cases} \xi & \text{if } k \equiv \pm 1 \bmod 8 \\ \xi + \eta & \text{if } k \equiv \pm 3 \bmod 8. \end{cases}$$

If we recall the description of $\mathrm{Ext}^1_{\mathfrak{S}}(M, N)$ given in §9, this shows that $e_R J\beta$ is non-zero.

A second proof of Theorem 9.5 will be given in §12.

§11. TODA BRACKETS, II

The main purpose of this section is to show explicitly how the theorems of §5 apply to the invariants of §7. The spaces we shall deal with will thus be spheres; and we shall stay in those dimensions where the invariants e_Λ take values in Q/Z, the rationals mod 1.

We will begin by stating the main results, without proofs. The following result, which is typical, will be obtained by specialising Theorem 5.3 (v).

THEOREM 11.1. *Suppose given integers $a > b > c > 0$, which are even if $\Lambda = R$. Suppose given $f: S^{2a-2} \to S^{2b-1}$, $h: S^{2b-1} \to S^{2c}$ and $q \in Z$ such that $h(q\iota) \sim 0$ and $(q\iota)f \sim 0$. Then*

$$e_\Lambda\{h, q\iota, f\} = -qe\ (Sf)e_\Lambda(h) \qquad \textit{mod}\ 1.$$

We pause to check that both sides of this equation are well-defined as rationals mod 1. The indeterminacy of $\{h, q\iota, f\}$ is

$$h\pi_{2a-1}(S^{2b-1}) + \pi_{2b}(S^{2c})Sf,$$

and therefore (using (3.2) and (7.1)) $e_\Lambda\{h, q\iota, f\}$ is well-defined as a rational mod 1. If we change the fraction representing $e_\Lambda(Sf)$ by 1, we change $qe_\Lambda(Sf)\, e_\Lambda(h)$ by $qe_\Lambda(h)$, which is an integer since $h(q\iota) \sim 0$; similarly if we change the fraction representing $e_\Lambda(h)$ by 1. Thus $-qe_\Lambda(Sf)e_\Lambda(h)$ is well-defined mod 1.

In applying Theorem 11.1 in the case $\Lambda = R$, we have to distinguish when the invariant e_R means e'_R, and when it means e''_R, according to the dimensions of the spheres concerned.

Examples on Theorem 11.1. With the notation of Example 7.17, we have

$$\{j_3, 24, j_3\} = 40j_7$$

$$\{j_3, 24, 10j_7\} = 21j_{11}$$

$$\{j_3, 24, 21j_{11}\} = 80j_{15} \qquad \text{mod } \eta\kappa$$

$$\{j_7, 240, j_7\} = 2j_{15} \qquad \text{mod } \eta\kappa$$

etc.

In order to state the results obtained by specialising Theorem 5.3 (iv) and (vi) we need a little number theory. The numerical function $m(t)$ will be as in [4 §2]; as we shall need the explicit definition in our proofs, we recall it now. We write $v_p(n)$ for the exponent to which the prime p occurs in n, so that

$$n = 2^{v_2(n)}3^{v_3(n)}5^{v_5(n)}\ldots.$$

For odd primes p we set

$$v_p(m(t)) = \begin{cases} 0 & \text{if } t \not\equiv 0 \bmod (p-1) \\ 1 + v_p(t) & \text{if } t \equiv 0 \bmod (p-1). \end{cases}$$

For $p = 2$ we set

$$\nu_2(m(t)) = \begin{cases} 1 & \text{if } t \not\equiv 0 \bmod 2 \\ 2 + \nu_2(t) & \text{if } t \equiv 0 \bmod 2. \end{cases}$$

In order to avoid worrying about signs in what follows, we remark that this definition is equally valid if t is negative; only the case $t = 0$ need be excluded. Thus we have $m(-t) = m(t)$.

We shall suppose given two even integers u, v; the cases $u = 0$, $v = 0$ and $u = v$ are excluded.

LEMMA 11.2. *There exists a rational number $\delta(u, v)$ such that for sufficiently large t (depending on u and v), and for all $k \in Z$ we have*

$$(k^{t+u} - k^t) - \delta(u, v)(k^{t+v} - k^t) \equiv 0 \qquad \textit{mod } m(u)m(v-u).$$

The congruence is to be interpreted as meaning that the left-hand side is an integer multiple of $m(u)\, m(v - u)$.

We shall not only prove that $\delta(u, v)$ exists; we will give a definition for $\delta(u, v)$ which allows one to compute it easily. I am indebted to Dr. B. J. Birch for conversations about an earlier version of this lemma.

We recall from [4, §2] that for sufficiently large t, the highest common factor of the numbers $(k^{t+v} - k^t)$ (as k runs over Z) is $m(v)$. This shows that the property stated in Lemma 11.2 characterises $\delta(u, v)$ up to an integer multiple of $m(u)\, m(v - u)/m(v)$.

We shall need to refer to the following further properties of $\delta(u, v)$.

LEMMA 11.3.

(i) $\delta(-u, -v) \equiv \delta(u, v)$ *mod* $m(u)\, m(v - u)/m(v)$.

(ii) $\delta(u, v) + \delta(v - u, v) \equiv 1$ *mod* $m(u)\, m(v - u)/m(v)$.

(iii) $\delta(u, v) = \dfrac{\gamma m(u)}{m(v)}$

for some integer $\gamma = \gamma(u, v)$.

(iv) $\delta(u, v) = 1 + \dfrac{\gamma' m(v - u)}{m(v)}$

for some integer $\gamma' = \gamma'(u, v)$.

The following result may be obtained by specialising Theorem 5.3 (iv).

THEOREM 11.4. *Suppose given even integers $a > b > c > 0$. Suppose given $f: S^{2a-2} \to S^{2b-1}$, $g: S^{2b-1} \to S^{2c}$ and $q \in Z$ such that $(q\iota)g \sim 0$ and $gf \sim 0$. Then*

$$e_\Lambda\{q\iota, g, f\} = -q\delta e_\Lambda(Sf)e_\Lambda(g) \qquad \textit{mod } 1 \textit{ and } q/m(a-c)$$

where $\delta = \delta(a - b, a - c)$ (with the notation of Lemma 11.2).

As for Theorem 11.1, we have to check that both sides are well-defined modulo 1 and $q/m(a - c)$. For the left-hand side this is easy. Altering δ by $m(a - b)\, m(b - c)/m(a - c)$ alters the right-hand side by an integer multiple of $q/m(a - c)$, since $m(a - b)\, e_\Lambda(Sf)$ and

$m(b-c)\, e_\Lambda(g)$ are integers. Altering $e_\Lambda(Sf)$ by 1 alters the right-hand side by

$$q\left(1+\gamma' \frac{m(b-c)}{m(a-c)}\right)e_\Lambda(g)$$

(using Lemma 11.3 (iv)); since $qe_\Lambda(g)$ and $m(b-c)e_\Lambda(g)$ are integers this is zero mod 1 and $q/m(a-c)$. Altering $e_\Lambda(g)$ by 1 alters the right-hand side by

$$q\gamma \frac{m(a-b)}{m(a-c)} e_\Lambda(Sf)$$

(using Lemma 11.3 (iii)); since $m(a-b)\, e_\Lambda(Sf)$ is an integer, this is zero mod $q/(ma-c)$.

The following result may be obtained by specialising Theorem 5.3 (vi).

THEOREM 11.5. *Suppose given even integers* $a > b > c > 0$. *Suppose given* $g : S^{2a-2} \to S^{2b-1}$, $h : S^{2b-1} \to S^{2c}$ *and* $q \in Z$ *such that* $hg \sim 0$ *and* $g(q\iota) \sim 0$. *Then*

$$e_\Lambda\{h, g, q\iota\} = -q\delta e_\Lambda(Sg)e_\Lambda(h) \qquad \textit{mod } 1 \textit{ and } q/m(a-c)$$

where $\delta = \delta(b-c, a-c)$ (*with the notation of Lemma* 11.2).

As before, we have to check that both sides are well-defined modulo 1 and $q/m(a-c)$. This is done exactly as for Theorem 11.4.

In applying Theorems 11.4 and 11.5, we have again to distinguish when the invariant e_R means e'_R, and when it means e''_R.

Examples on Theorem 11.5. With the notation of Example 7.17, we have

$$\begin{aligned}
\{j_3, 2j_3, 12\} &= 0 && \text{mod } 12j_7\\
\{j_7, j_3, 24\} &= -j_{11} && \text{mod } 24j_{11}\\
\{j_3, j_7, 240\} &= 7j_{11} && \text{mod } 24j_{11}\\
\{j_{11}, j_3, 24\} &= -4j_{15} && \text{mod } 24j_{15} \text{ and } \eta\kappa\\
\{j_7, 2j_7, 120\} &= j_{15} && \text{mod } 120j_{15} \text{ and } \eta\kappa\\
\{j_3, j_{11}, 504\} &= -4j_{15} && \text{mod } 24j_{15} \text{ and } \eta\kappa
\end{aligned}$$

etc.

The calculation of these examples requires a knowledge of the coefficients δ, which will be provided later in this section.

Theorems 11.4 and 11.5 are equivalent. In fact, if α and β belong to odd-dimensional stable groups, then we have

$$\{\alpha, \beta, q\iota\} = \{q\iota, \beta, \alpha\}$$

by a theorem of Toda [18, p.26 (3.4) (i) or p.33 (3.9) (i)]. The reader is warned not to suppose that this remark makes the equivalence completely obvious; in the case $\Lambda = R$ we still have to distinguish when e_R means the invariant e'_R, and when it means e''_R; we have then to use Proposition 7.14. However, these details lead to the required result. It will therefore be sufficient to prove one of these theorems and deduce the other. Similarly, we will state corollaries of only one of these theorems.

Other checks on our work are provided by the identities

$$\{\alpha, q\iota, \beta\} = \{\beta, q\iota, \alpha\}$$

and

$$\{\alpha, \beta, q\iota\} - \{\beta, q\iota, \alpha\} + \{q\iota, \alpha, \beta\} = 0$$

[18, p.26 (3.4) (ii) or p.33 (3.9) (ii)]. The first is consistent with Theorem 11.1; the second is consistent with Theorems 11.1, 11.4 and 11.5, as we see using Lemma 11.3 (ii).

We will now state two corollaries of Theorem 11.5 which are useful in dealing with p-components of stable homotopy groups. We retain the notation and assumptions of Theorem 11.5.

COROLLARY 11.6. *Let p be an odd prime such that $a - b$ and $b - c$ are divisible by $p - 1$. Then we have*

$$e_\Lambda\{h, g, q\iota\} = -q\,\frac{b-c}{a-c}\,e_\Lambda(Sg)e_\Lambda(h)$$

as an equation in the p-adic numbers modulo 1 and $q/m(a-c)$.

The case in which $a - b$ and $b - c$ are divisible by $(p - 1)$ is, of course, the only case of interest if we are studying p-components.

COROLLARY 11.7. *Let $p = 2$. Then we have*

$$e_\Lambda\{h, g, q\iota\} = -q\,\frac{b-c}{a-c}(1 + \omega 2^g)e_\Lambda(Sg)e_\Lambda(h)$$

(where ω is any odd number and $g = 1 + v_2(a-b)$) as an equation in the 2-adic numbers modulo 1 and $q/m(a-c)$.

It is no great surprise that the case $p = 2$ is exceptional.

In both corollaries, the phrase "modulo 1 and $q/m(a-c)$" refers to multiples of 1 and $q/m(a-c)$ by p-adic integers. The use of p-adic numbers is not essential, but it is convenient; it allows us to invert numbers prime to p, modulo a high power of p, without stating exactly which high power of p is required.

A further check on our work is now provided by the following observation. Suppose given a generator $\gamma \in \pi_{4r-1}(SO)$, a map $\theta : S^{4r+4s-2} \to S^{4r-1}$ and an integer q such that $\gamma\theta \sim 0$, $\theta(q\iota) \sim 0$. Then we can form in $\pi_{4r+4s-1}(SO)$ the Toda bracket $\{\gamma, \theta, q\iota\}$. Let $\gamma' \in \pi_{4r+4s-1}(SO)$ be a generator; then we have

$$\{\gamma, \theta, q\iota\} = -qe_R(S\theta)\gamma' \qquad \text{mod } q.$$

(Whether $e_R(S\theta)$ is an invariant e'_R or e''_R depends on the parity of r). In $\pi^S_{4r+4s-1}$ we shall have

$$J\{\gamma, \theta, q\} \subset \{J\gamma, \theta, q\},$$

that is,

$$-qe_R(S\theta)J\gamma' \subset \{J\gamma, \theta, q\}.$$

We may now apply e'_R to both sides, using Theorem 7.16 and [4, Theorem 2.5]. The results should agree modulo 1 and $q/m(2r+2s)$. Calculating in the p-adic numbers, both sides yield

$$-qe_R(S\theta)\frac{p-1}{4p(r+s)},$$

providing that $2r$ and $2s$ are divisible by $p-1$; otherwise 0. Calculating in the 2-adic numbers, both sides yield

$$-qe_R(S\theta)\left(\tfrac{1}{2}+\frac{1}{8(r+s)}\right)$$

provided $r \geqq 3$.

The remainder of this section is organised as follows. We begin with the number theory, leading up to the proofs of Lemmas 11.2 and 11.3. Then we prove Theorems 11.1, 11.4 and 11.5. Corollaries 11.6 and 11.7 then follow easily.

LEMMA 11.8. *Let p be an odd prime, let k be an integer prime to p, and let a, b, c be integers divisible by $(p-1)$. Then we have*

$$(a-b)k^c+(b-c)k^a+(c-a)k^b \equiv 0 \qquad \textit{mod } p^{h+2},$$

where $h = v_p(a-b)+v_p(b-c)+v_p(c-a)$.

LEMMA 11.9. *Let $p=2$, let k be an odd integer, and let a, b, c be even integers. Then we have*

$$(a-b)k^c+(b-c)k^a+(c-a)k^b \equiv \varepsilon 2^{h+3} \qquad \textit{mod } 2^{h+4}$$

where

$$\varepsilon = \begin{cases} 0 & \textit{if } k \equiv \pm 1 \textit{ mod } 8 \\ 1 & \textit{if } k \equiv \pm 3 \textit{ mod } 8 \end{cases}$$

and $h = v_2(a-b)+v_2(b-c)+v_2(c-a)$.

We prove Lemma 11.9; the proof of Lemma 11.8 is similar but slightly simpler.

Without loss of generality we may assume that

$$v_2(a-b) = f$$
$$v_2(b-c) = f+g$$
$$v_2(c-a) = f$$

where $f \geqq 1$, $g \geqq 1$. Thus $h = 3f+g$. Set $d = 2^f$; by adding a constant to a, b and c we may assume they are all divisible by d. Set $K = k^d$; then $K \equiv 1 \bmod 2^{f+2}$. Hence

$$\frac{K^{a/d}-K^{b/d}}{K-1} \equiv (a-b)/d \qquad \bmod 2^{f+2}.$$

(Without loss of generality we may assume $a > b$; expand the left-hand side in powers of K.) Thus

$$k^a - k^b \equiv (K-1)(a-b)/d \qquad \bmod 2^{2f+4}$$

and

$$(b-c)(k^a-k^b) \equiv (K-1)(a-b)(b-c)/d \qquad \bmod 2^{3f+g+4}.$$

We now consider the sum of 2^g consecutive powers of K. I claim we have

$$K^{e+1}+K^{e+2}+\ldots+K^{e+2^g} \equiv 2^g+\varepsilon 2^{f+g+1} \qquad \bmod 2^{f+g+2}$$

where ε is as above. In fact, suppose that $K \equiv 1 \bmod 2^{\phi+2}$ but $K \not\equiv 1 \bmod 2^{\phi+3}$, where $\phi > f$ if $k \equiv \pm 1 \bmod 8$ and $\phi = f$ if $k \equiv \pm 3 \bmod 8$. Then the 2^g numbers

$$K^{e+1},\ K^{e+2},\ \ldots,\ K^{e+2^g}$$

give the 2^g residue classes $1 + q2^{\phi+2} \bmod 2^{\phi+g+2}$. Hence their sum is

$$2^g + \tfrac{1}{2}(2^g)(2^{g+1})2^{\phi+2} \qquad \bmod 2^{\phi+g+2}.$$

This proves the assertion.

Arguing as above, we find

$$\begin{aligned} \frac{K^{b/d} - K^{c/d}}{K-1} &\equiv (2^g + \varepsilon 2^{f+g+1})\frac{b-c}{2^g d} \qquad \bmod 2^{f+g+2} \\ &\equiv (b-c)/d + \varepsilon 2^{f+g+1} \qquad \bmod 2^{f+g+2}. \end{aligned}$$

Thus

$$k^b - k^c \equiv (K-1)(b-c)/d + \varepsilon 2^{2f+g+3} \qquad \bmod 2^{2f+g+4}$$

and

$$(a-b)(k^b - k^c) \equiv (K-1)(a-b)(b-c)/d + \varepsilon 2^{3f+g+3} \qquad \bmod 2^{3f+g+4}.$$

Thus

$$(b-c)(k^a - k^b) - (a-b)(k^b - k^c) \equiv \varepsilon 2^{h+3} \qquad \bmod 2^{h+4},$$

which proves the lemma.

We now define $\delta(u, v)$. As above, let u, v be two even integers; the cases $u = 0$, $v = 0$ and $u = v$ are excluded. We propose to define the rational number $\delta(u, v)$ modulo $m(u)\, m(v-u)/m(v)$ by giving a finite number of congruences. Each congruence will be written as a congruence in the p-adic integers, holding mod p^f where

$$f = v_p m(u) + v_p m(v-u) - v_p m(v).$$

The primes p to be considered are those which divide $m(u)$, $m(v)$ or $m(v-u)$. We stipulate that the denominator of $\delta(u, v)$ is to contain no other primes; thus the definition given for $\delta(u, v)$ amounts to defining an integer (namely the numerator of $\delta(u, v)$) by a finite set of congruences modulo powers of different primes. This is always legitimate.

We now give the congruences.

Case (i). p is odd; $(p-1)$ does not divide u or v, but divides $v-u$. Take

(11.10) $$\delta(u, v) \equiv 1 \qquad \bmod p^f.$$

Case (ii). p is odd; $(p-1)$ divides just one of u, v and therefore does not divide $(v-u)$. Take

(11.11) $$\delta(u, v) \equiv 0 \qquad \bmod p^f.$$

Case (iii). p is odd; $(p-1)$ divides both of u, v and therefore divides $v-u$. Take

(11.12) $$\delta(u, v) \equiv u/v \qquad \bmod p^f.$$

Case (iv). $p = 2$. Take

(11.13) $$\delta(u, v) = (1 + \omega 2^g)u/v \qquad \bmod 2^f$$

where ω is any odd number and $g = 1 + v_2(v-u)$. (Note that altering ω by 2 does not affect the result mod 2^f.)

Proof of Lemma 11.2. It is sufficient to verify the congruence in the p-adic numbers for a finite number of primes p, namely those mentioned above. For each prime p the congruence will be true for k divisible by p providing we choose t large enough; we may therefore restrict attention to the case $k \not\equiv 0 \bmod p$. In all cases we have given definitions of the form $\delta \equiv \delta' \bmod p^f$, where

$$f = \nu_p m(u) + \nu_p m(v-u) - \nu_p(v);$$

and we have

$$k^{t+v} - k^t \equiv 0 \qquad \bmod p^{\nu_p(v)}.$$

Thus we have

$$\delta(k^{t+v} - k^t) \equiv \delta'(k^{t+v} - k^t) \qquad \bmod p^h$$

where $h = \nu_p m(u) + \nu_p m(v-u)$. We may therefore replace δ by δ' in checking the congruence.

Case (i). p is odd; $(p-1)$ does not divide u or v, but divides $v-u$. We have

$$k^{t+v} - k^{t+u} \equiv 0 \qquad \bmod p^{\nu_p m(v-u)}$$

i.e.

$$k^{t+u} - k^t \equiv k^{t+v} - k^t \qquad \bmod p^{\nu_p m(v-u)}.$$

Since $\delta' = 1$ and $\nu_p m(u) = 0$ in this case, this is the result required.

Case (ii). p is odd; $(p-1)$ divides just one of u, v and therefore does not divide $(v-u)$. We have

$$k^{t+u} - k^t \equiv 0 \qquad \bmod p^{\nu_p m(u)}.$$

Since $\delta' = 0$ and $\nu_p m(v-u) = 0$ in this case, this is the result required.

Case (iii). p is odd; $(p-1)$ divides both of u, v and therefore divides $(v-u)$. Lemma 11.8 gives

$$v(k^{t+u} - k^t) \equiv u(k^{t+v} - k^t) \qquad \bmod p^{h+2},$$

where $h = \nu_p(u) + \nu_p(v) + \nu_p(v-u)$. This gives

$$k^{t+u} - k^t \equiv \frac{u}{v}(k^{t+v} - k^t) \qquad \bmod p^l,$$

where $l = \nu_p m(u) + \nu_p m(v-u)$. Since $\delta' = u/v$ in this case, this is the result required.

Case (iv). $p = 2$. Lemma 11.9 gives

$$v(k^{t+u} - k^t) - u(k^{t+v} - k^t) \equiv \varepsilon 2^{h+3} \qquad \bmod 2^{h+4}$$

where $h = \nu_2(u) + \nu_2(v) + \nu_2(v-u)$. We have

$$u(k^{t+v} - k^t) \equiv \varepsilon 2^r \qquad \bmod 2^{r+1}$$

where $r = \nu_2(u) + \nu_2(v) + 2$. Thus we have

$$v(k^{t+u} - k^t) - u(1 + \omega 2^g)(k^{t+v} - k^t) \equiv 0 \qquad \bmod 2^{h+4}$$

where ω is any odd number and $g = 1 + \nu_2(v-u)$. This gives

$$k^{t+u} - k^t \equiv \frac{u}{v}(1 + \omega 2^g)(k^{t+v} - k^t) \qquad \bmod 2^l$$

where $l = \nu_2(m(u)) + \nu_2(m(v-u))$. Since $\delta' = \frac{u}{v}(1+\omega 2^g)$ in this case, this is the result required. This completes the proof.

Proof of Lemma 11.3.

(i) The congruence

$$\delta(-u, -v) \equiv \delta(u, v) \qquad \text{mod } m(u)m(v-u)/m(v)$$

follows immediately by inspecting the congruence (11.10) to (11.13).

(ii) For sufficiently large t we have

$$k^{t+u} - k^t \equiv \delta(u, v)(k^{t+v} - k^t)$$
$$k^{t+u} - k^{t+v} \equiv \delta(u-v, -v)(k^t - k^{t+v})$$

mod $m(u)\, m(v-u)$. Subtracting, we obtain

$$k^{t+v} - k^t \equiv (\delta(u, v) + \delta(u-v, -v))(k^{t+v} - k^t)$$

mod $m(u)\, m(v-u)$. Since the highest common factor of the expressions $(k^{t+v} - k^t)$ is $m(v)$, we find

$$\delta(u, v) + \delta(u-v, -v) \equiv 1 \qquad \text{mod } m(u)m(v-u)/m(v).$$

The result now follows by part (i).

Alternatively, we can check part (ii) from the congruences (11.10) to (11.13).

(iii) For sufficiently large t we have

$$(k^{t+u} - k^t) \equiv \delta(u, v)(k^{t+v} - k^t)$$

mod $m(v)\, m(v-u)$. For sufficiently large t, the highest common factor of the expressions $(k^{t+u} - k^t)$ is $m(u)$ and that of the expressions $(k^{t+v} - k^t)$ is $m(v)$. Taking linear combinations, we find

$$Nm(u) \equiv \delta(u, v)m(v)$$

mod $m(u)\, m(v-u)$, for some integer N. Hence the result.

Alternatively, we can check part (iii) from the congruences (11.10) to (11.13).

(iv) This follows immediately from (ii) and (iii).

This completes the proof of Lemma 11.3.

Proof of Theorem 11.1. We have to evaluate the Massey product $\{e_\Lambda(Sf), q, e_\Lambda(h)\}$ according to the definition of §4, Case 3. In that section we have objects L, M, N and P in our abelian category; in the present application they all have the underlying group Z, and they have operations given by $\Psi^k x = k^c x$, $\Psi^k x = k^b x$, $\Psi^k x = k^b x$ and $\Psi^k x = k^a x$ respectively. We write λ, μ, ν, π for their respective generators. We also have in mind two extensions

$$0 \leftarrow L \leftarrow E \leftarrow M \leftarrow 0$$
$$0 \leftarrow N \leftarrow F \leftarrow P \leftarrow 0$$

given by the following formulae.

$$\Psi^k\lambda' = k^c\lambda' + e'(k^b - k^c)\mu$$

$$\Psi^k\nu' = k^b\nu' + e''(k^a - k^b)\pi.$$

Here λ', ν' are elements lifting λ, ν and e', e'' are rationals representing $e_\Lambda(h)$, $e_\Lambda(Sf)$. According to §4, Case 3 we have to construct maps

$$\theta: M \to F, \qquad \phi: E \to N;$$

we do so by the following formulae.

$$\theta(\mu) = q\nu' - qe''\pi$$

$$\phi(\lambda') = qe'\nu$$

$$\phi(\mu) = q\nu.$$

(Note that qe' and qe'' are integers.) According to §4, Case 3 we have to consider an extension G; in it we construct a lifting λ'' of λ by

$$\lambda'' = (\lambda', qe'\nu').$$

We then compute in G the formula

$$\Psi\lambda'' = k^c\lambda'' + qe'e''(k^a - k^c)\pi.$$

We conclude that in this case the Massey product in Ext^1 is given by

$$\{e_\Lambda(Sf), q, e_\Lambda(h)\} = qe'e''.$$

Theorem 11.1 thus follows from Theorem 5.3 (v).

We have given this proof of Theorem 11.1 because it seems in keeping. However, it is possible to give an *ad hoc* proof using an intermediate space $S^{2b-1} \cup_q e^{2b}$, on the lines to be explained in §12 [cf. 6, 7. In Proposition 6 of 7 a minus sign has been left out by mistake.] If one defines e_Λ using the Chern character, it is not necessary to use the operations Ψ^k in proving Theorem 11.1. By contrast, in proving Theorems 11.4, 11.5 it seems essential to use the operations Ψ^k and number-theory. In fact, the number-theory we have given may be interpreted as an investigation of what limitations the Ψ^k impose on the Chern characters in a 3-cell complex

$$S^{2t} \cup e^{2(t+u)} \cup e^{2(t+v)}.$$

This gives a partial answer to questions raised by Dyer [13, second paragraph on p.371].

Proof of Theorem 11.5. We have first to verify the conditions of Theorem 5.3 (vi). With the notation of §5, we have to show that for any choice of homotopy $hg \sim 0$, the invariant $e_\Lambda(H)$ is defined. In our case we have

$$H: S^{2b-1} \cup_g e^{2a-1} \to S^{2c}.$$

Since a and b are even, the exact sequence

$$\tilde{K}_\Lambda(S^{2b-1}) \leftarrow \tilde{K}_\Lambda(S^{2b-1} \cup_g e^{2a-1}) \leftarrow \tilde{K}_\Lambda(S^{2a-1})$$

shows that

$$\tilde{K}_\Lambda(S^{2b-1} \cup_g e^{2a-1}) = 0.$$

Hence $d_\Lambda(H) = 0$. It follows that $d_\Lambda(SH) = 0$.

We have now to evaluate the Massey product $\{q, e_\Lambda(Sg), e_\Lambda(h)\}$ according to the definition of §4, Case 1. In that section we have objects L, M, N and P in our abelian category; in the present application they all have the underlying group Z, and they have operations given by $\Psi^k x = k^c x$, $\Psi^k x = k^b x$, $\Psi^k x = k^a x$ and $\Psi^k x = k^a x$ respectively. We write λ, μ, ν, π for their respective generators. We also have in mind two extensions $\alpha \in \mathrm{Ext}^1(L, M)$, $\beta \in \mathrm{Ext}^1(M, N)$ given by the following formulae.

$$\Psi^k\lambda' = k^c\lambda' + e'(k^b - k^c)\mu$$
$$\Psi^k\mu' = k^b\mu' + e''(k^a - k^b)\nu.$$

Here λ', μ' are elements lifting λ, μ and e', e'' are rationals representing $e_\Lambda(h)$, $e_\Lambda(Sg)$. We also have in mind a homomorphism $\gamma \in \mathrm{Ext}^0(N, P)$ given by

$$\gamma(\nu) = q\pi.$$

We have next to construct $\alpha' \in \mathrm{Ext}^1(L, E)$, $\gamma' \in \mathrm{Ext}^0(E, P)$ lifting α, γ (where E is the extension representing β.)

A suitable extension α' is defined by the following formula.

$$\Psi^k\lambda'' = k^c\lambda'' + e'(k^b - k^c)\mu' + e'e''(\delta(k^a - k^c) - (k^b - k^c))\nu.$$

Here λ'' is a lifting of λ, and $\delta = \delta(b - c, a - c)$ (with the notation of Lemma 11.2). Lemma 11.2 plays a crucial role; it shows that the coefficient of ν is an integer. (We may suppose that c is sufficiently large, because the result is not affected by suspension—provided of course that the number of suspensions is divisible by 2 or 8.) It is necessary, of course, to check that the formula satisfies Axioms (6.1), (6.2) and (6.3). The need to satisfy Axiom (6.1) accounts for the formula given.

A suitable homomorphism γ' is defined by the following formulae.

$$\gamma'(\mu') = qe''\pi$$
$$\gamma'(\nu) = q\pi.$$

(Note that qe'' is an integer). It is necessary, of course, to check that γ' commutes with Ψ^k. The need to do this accounts for the formula given for $\gamma'(\mu')$.

We have next to compute the extension

$$\gamma'\alpha' \in \mathrm{Ext}^1(L, P).$$

This is characterised by the following formula.

$$\Psi^k\lambda'' = k^c\lambda'' + qe'e''\delta(k^a - k^c)\pi.$$

We conclude that in this case the Massey product in Ext^1 is given by

$$\{q, e_\Lambda(Sg), e_\Lambda(h)\} = qe'e''\delta$$

modulo the indeterminacy of the Massey product; that is, modulo 1 and $q/m(a - c)$. Theorem 11.5 thus follows from Theorem 5.3 (vi).

Theorem 11.4 may be deduced from Theorem 11.5 (as remarked above), or proved similarly. For the convenience of any reader who wishes to do the latter, we record formulae for

$$\alpha', \in \mathrm{Ext}^0(L,E), \qquad \gamma' \in \mathrm{Ext}^1(E,P)$$

with a notation similar to that used above.

$$\alpha'\lambda = q\mu' - qe'\nu$$

$$\Psi^k\mu'' = k^c\mu'' + e'(k^b - k^c)\nu' + e'e''((k^a - k^b) - \delta(k^a - k^c))\pi$$

$$\Psi^k\nu' = k^b\nu' + e''(k^a - k^b)\pi.$$

Proof of Corollaries 11.6, 11.7. These follow from Theorem 11.5 by applying the homomorphism from the rationals to the p-adic numbers and using (11.12), (11.13).

§12. EXAMPLES

In this section we will give various examples and illustrations of our general methods, and prove certain results whose proof was deferred in earlier sections. To begin with, our work is directed towards proving Theorem 1.7.

We can actually make Theorem 1.7 a little more complete. As in §1, let p be an odd prime, let $g : S^{2q-1} \to S^{2q-1}$ be a map of degree p^f, and let Y be the Moore space $S^{2q-1} \cup_g e^{2q}$. Thus $\tilde{K}_C(Y) = Z_{p^f}$.

THEOREM 12.1. *There is a map*

$$A : S^{2r}Y \to Y$$

(for suitable q) such that the image of

$$A^* : \tilde{K}_C(Y) \to \tilde{K}_C(S^{2r}Y)$$

is Z_{p^t} (where $1 \leq t \leq f$), if and only if r is divisible by $(p-1)p^{t-1}$.

It is clear that this includes Theorem 1.7 (take $t = f$). We will show how to deduce Theorem 12.1 from Theorem 1.7.

First, suppose that there is a map $A : S^{2r}Y \to Y$ such that the image of A^* is Z_{p^t}. Then A^* commutes with the operations Ψ^k, which are given in Y and $S^{2r}Y$ by the formulae

$$\Psi^k x = k^q x, \qquad \Psi^k x = k^{q+r}x.$$

Therefore we have $k^{q+r} \equiv k^q \bmod p^t$; so r is divisible by $(p-1)p^{t-1}$.

Secondly, suppose that r is divisible by $(p-1)p^{t-1}$ and Theorem 1.7 is true. Set $Y' = S^{2q-1} \cup_h e^{2q}$, where h is a map of degree p^t. Then by Theorem 1.7 there is a map

$$A' : S^{2r}Y' \to Y'$$

inducing an isomorphism of $\tilde{K}_C$. We have only to take A to be the composite

$$S^{2r}Y \xrightarrow{S^{2r}i} S^{2r}Y' \xrightarrow{A'} Y' \xrightarrow{j} Y$$

where i, j are obvious maps such that $j^* : \tilde{K}_C(Y) \to \tilde{K}_C(Y')$ is an epimorphism and $i^* : \tilde{K}_C(Y') \to \tilde{K}_C(Y')$ is a monomorphism.

This completes the deduction of Theorem 12.1 from Theorem 1.7. We proceed with lemmas needed for the proof of Theorem 1.7. First we consider the cofibering

$$S^{2n-1} \xrightarrow{f} S^{2n-1} \xrightarrow{i} S^{2n-1} \cup_f e^{2n},$$

where f is a map of degree m. If $\Lambda = R$, we assume that n is even; thus we shall certainly have $d_R i = 0$, $d_R(Si) = 0$.

PROPOSITION 12.2. *$e_\Lambda i$ is the class of the extension*

$$0 \leftarrow Z_m \leftarrow Z \xleftarrow{-m} Z \leftarrow 0,$$

in which all the abelian groups have operations Ψ^k defined by

$$\Psi^k x = k^n x.$$

Proof. If we continue the cofibre sequence, it becomes

$$S^{2n-1} \cup_f e^{2n} \xrightarrow{j} S^{2n} \xrightarrow{-Sf} S^{2n};$$

we have only to apply $\tilde{K}_\Lambda$.

For the next proposition, we suppose given a diagram of the following form,

$$\begin{array}{ccc} & S^{2n-1} \cup_m e^{2n} & \\ {}^{i}\nearrow & & \searrow^{G} \\ S^{2n-1} & \xrightarrow{\quad g \quad} & S^{2q} \end{array}$$

(Here we have written $S^{2n-1} \cup_m e^{2n}$ instead of $S^{2n-1} \cup_f e^{2n}$, where f is a map of degree m.) If $\Lambda = R$, we assume that n and q are even. Thus $\tilde{K}_\Lambda(S^{2q}) = Z$ and $\tilde{K}_\Lambda(S^{2n-1} \cup_m e^{2n}) = Z_m$; we can regard $d_\Lambda(G)$ as an integer mod m. We can also regard $e_\Lambda(g)$ as a rational mod 1; since $mg \sim 0$, $me_\Lambda(g)$ is an integer mod m.

PROPOSITION 12.3. *We have*

$$d_\Lambda(G) = -me_\Lambda(g) \qquad \textit{mod } m$$

or equivalently

$$e_\Lambda(g) = -\frac{1}{m} d_\Lambda(G) \qquad \textit{mod } 1.$$

Proof. This proposition is a special case of Proposition 3.2 (b), which states that

$$e(Gi) = e(i)\, d(G).$$

The element $e(i)$ has been given in Proposition 12.2; one has only to compute the product $e(i)\, d(G)$, which is an easy exercise in homological algebra.

LEMMA 12.4. *Let p be an odd prime, $m = p^f$, and $r = (p-1)p^f$. Then there is an element $\alpha \in \pi^S_{2r-1}$ satisfying the following conditions.*

(i) $m\alpha = 0$.

(ii) $e_C \alpha = -\dfrac{1}{m}$.

(iii) *The Toda bracket $\{m, \alpha, m\}$ is zero mod $m\pi^S_{2r}$.*

Proof. For $f = 1$ the result is easy; we have only to take α to be an element of Hopf invariant one mod p in π^S_{2p-3}. Then (i), (ii) are given by Corollary 8.4 and (iii) follows from the fact that the p-component of π^S_{2p-2} is zero.

For any f we can take α to be a suitable element in $\operatorname{Im} J$, using Theorem 1.5 or 1.6 to obtain (i), (ii). Condition (iii) follows from the fact that $\{m, \alpha, m\}$ is an element of order 2 [18, p.26 (2.4) (i), p.33 (3.9) (i)].

Lemma 12.4 supplies the data for the following lemma, which we shall also use with $m = 2$.

LEMMA 12.5. *Suppose given $\alpha \in \pi^S_{2r-1}$ and $m \in Z$ such that*

(i) $m\alpha = 0$,

(ii) $e_C\alpha = -\dfrac{1}{m}$,

(iii) $\{m, \alpha, m\} = 0 \bmod m\pi^S_{2r}$.

Then for suitably large q there exist maps A which make the following diagram homotopy-commutative; and for any such A we have $d_C(A) = 1$.

$$\begin{array}{ccc} S^{2q+2r-1} \cup_m e^{2q+2r} & \xrightarrow{A} & S^{2q-1} \cup_m e^{2q} \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle j} \\ S^{2q+2r-1} & \xrightarrow{\alpha} & S^{2q} \end{array}$$

Proof. Conditions (i), (iii) enable one to construct the diagram. By Proposition 12.3 and condition (ii) we have $d_C(jA) = 1$. Hence $d_C(A) = 1$.

Theorem 1.7 follows immediately from Lemmas 12.4, 12.5. Since A induces an isomorphism of $\tilde{K}_C$, so does the composite

$$A . S^{2r}A . S^{4r}A . \dots S^{2r(s-1)}A : S^{2rs}Y \to Y;$$

Indeed we have

$$d_C(A . S^{2r}A . S^{4r}A . \dots S^{2r(s-1)}A) = 1.$$

Therefore this composite is essential for every s.

Under the assumptions of Lemma 12.5, we construct a map

$$\alpha_s : S^{2q+2rs-1} \to S^{2q}$$

by the following diagram.

$$\begin{array}{ccc} S^{2q+2rs-1} \cup_m e^{2q+2rs} & \xrightarrow{A \cdot S^{2r}A \dots S^{2r(s-1)}A} & S^{2q-1} \cup_m e^{2q} \\ {\scriptstyle i}\uparrow & & \downarrow{\scriptstyle j} \\ S^{2q+2rs-1} & \xrightarrow{\alpha_s} & S^{2q} \end{array}$$

We have $\alpha_1 = \alpha$. The map α_s has order dividing m, since it can be extended over $S^{2q+2rs-1} \cup_m e^{2q+2rs}$. The maps α_s satisfy the equation

(12.6) $$\alpha_{s+t} \in \{\alpha_s, m, \alpha_t\}.$$

The case in which m is an odd prime p and $r = p - 1$ has been studied by Toda [16, 17].

PROPOSITION 12.7. *Under the assumptions of Lemma 12.5, the maps α_s are all essential; indeed we have*

$$e_C(\alpha_s) = -\frac{1}{m} \quad \bmod 1.$$

This improves and generalises a result of Toda [17]. Presumably the present proof is related to Toda's proof; however, it is hoped that the presentation given here may be found more conceptual.

Proofs. (i) Apply Proposition 12.3 to the diagram which defines α_s. (ii) Alternatively, apply Theorem 11.1 to equation (12.6) and use induction.

EXAMPLE 12.8. *We note that in* [16, 17] *Toda's elements* α_s *depend on the choice of* α_1, *which Toda does not fix; similarly, there is a choice for his element* α'_p. *However, we may take the choices so that*

$$e_C(\alpha_s) = -\frac{1}{p}, \qquad e_C(\alpha'_{rp}) = -\frac{1}{p^2}.$$

Then the coefficient δ *in Corollary* 11.6 *explains the coefficients which arise in Toda's formulae for*

$$\{\alpha_t, \alpha_s, p\} \quad and \quad \{\alpha'_{rp}, \alpha_s, p\}$$

[16, Theorem 4.17 (ii)].

We will now show how the invariant e_C applies to maps $f: S^{2r-1}Y \to Y$, where $Y = S^{2q-1} \cup_p e^{2q}$ for some odd prime p. We must first calculate the appropriate Ext groups. As in §9, let M be the object in A whose underlying group is Z and whose operations are given by $\Psi^k x = k^q x$; and let M' be the quotient object M/pM, whose underlying group is Z_p. Similarly for N', with q replaced by $q + r$.

PROPOSITION 12.9. *We have*

$$\mathrm{Ext}^1_S(M', N') = \begin{cases} Z_p + Z_p & \text{if } r \equiv 0 \bmod (p-1) \\ 0 & \text{if } r \not\equiv 0 \bmod (p-1). \end{cases}$$

Proof. The exact sequence

$$O \to M \xrightarrow{p} M \to M' \to O$$

induces the following exact sequence.

$$\mathrm{Hom}(M, N') \xrightarrow{p} \mathrm{Hom}(M, N') \to \mathrm{Ext}^1(M', N') \to \mathrm{Ext}^1(M, N') \xrightarrow{p} \mathrm{Ext}^1(M, N')$$

Passing to direct limits, we obtain the following exact sequence.

$$\mathrm{Hom}_S(M, N') \xrightarrow{p} \mathrm{Hom}_S(M, N') \to \mathrm{Ext}^1_S(M', N') \to \mathrm{Ext}^1_S(M, N') \xrightarrow{p} \mathrm{Ext}^1_S(M, N')$$

The group $\mathrm{Ext}^1_S(M, N')$ has been computed in Proposition 9.1; it is Z_p if $r \equiv 0 \bmod (p-1)$, 0 otherwise. (In §9 we assumed $\Lambda = R$; but this is not necessary if v is odd). The group $\mathrm{Hom}_S(M, N')$ is easy to compute; it is Z_p if $r \equiv 0 \bmod (p-1)$, 0 otherwise. This completes the proof in the case $r \not\equiv 0 \bmod (p-1)$. If $r \equiv 0 \bmod (p-1)$, we consider the functor from A to the category of abelian groups defined by forgetting the operations Ψ^k; this gives the following diagram.

$$\begin{array}{ccccccc} 0 \to & \mathrm{Hom}_S(M, N') & \to & \mathrm{Ext}^1_S(M', N') & \to & \mathrm{Ext}^1_S(M, N') & \to 0 \\ & \downarrow \cong & & \downarrow & & \downarrow & \\ 0 \to & \mathrm{Hom}(Z, Z_p) & \xrightarrow{\cong} & \mathrm{Ext}^1(Z_p, Z_p) & \longrightarrow & 0 & \end{array}$$

This shows that the exact sequence for $\mathrm{Ext}^1_S(M', N')$ splits.

COROLLARY 12.10. (of the proof). *If $r \equiv 0 \bmod (p-1)$, $\mathrm{Ext}^1_S(M', N')$ has a base consisting of the following two elements.*

(i) *An extension with underlying group Z_{p^2} and operations $\Psi^k x = k^q x$.*

(ii) *An extension with underlying group $Z_p + Z_p$ and operations*

$$\Psi^k \xi = k^q \xi + \lambda(k^{q+r} - k^q)\eta$$

for $\lambda = 1/m(r)$.

In fact, the element (i) represents a generator coming from $\mathrm{Hom}_S(M, N)'$, while the element (ii) maps to zero in $\mathrm{Ext}^1(Z_p, Z_p)$ and to a generator in $\mathrm{Ext}^1(M, N')$.

As above, let $Y = S^{2q-1} \cup_p e^{2q}$ for some odd prime p.

THEOREM 12.11. *If $r \equiv 0 \bmod (p-1)$ then the stable track group $Map^S(S^{2r-1}Y, Y)$ contains a direct summand $Z_p + Z_p$.*

Proof. Let $\beta : S^{-1}Y \to Y$ be the map which appears in the cofibre sequence

$$S^{-1}Y \xrightarrow{\beta} Y \to S^{2q-1} \cup_{p^2} e^{2q} \to Y.$$

Then $e_C(\beta)$ is the extension mentioned in Corollary 12.10 (i). Let $A : S^{2r}Y \to Y$ be a map with $d_C(A) = 1$, as above. Then by Proposition 3.2 (c) we have

$$e_C(\beta . S^{-1}A) = d_C(A) . e_C(\beta),$$

which is again the extension mentioned in Corollary 12.10 (i).

To construct the other generator, let $\gamma : S^{2t+2r+1} \to S^{2t}$ be an element in Im J such that

$$m(r)e_C(\gamma) \equiv 1 \qquad \mathrm{mod}\ p.$$

Then we can form the map

$$1 \wedge \gamma : Y \wedge S^{2t+2r-1} \to Y \wedge S^{2t},$$

where $A \wedge B$ is the "smash product" $A \times B / A \vee B$. If $e_C(\gamma)$ is represented by an extension E, then $e_C(1 \wedge \gamma)$ is represented (up to sign) by the extension E/pE; this is the extension mentioned in Corollary 12.10 (ii).

Since all elements of $\mathrm{Map}^S(S^{2r-1}Y, Y)$ have order dividing p, this proves Theorem 12.11.

Remark 12.12. In proving Theorem 12.11, we could have used $A . S^{2r}\beta$ instead of $\beta . S^{-1}A$. By Proposition 3.2 (b) we would then have

$$e_C(A . S^{2r}\beta) = e_C(S^{2r}\beta) . d_C(A),$$

giving an extension with underlying group Z_{p^2} and operations $\Psi^k x = k^{q+r}x$. Thus the invariant e_C serves to distinguish between $A . S^{2r}\beta$ and $\beta . S^{-1}A$ if $r \not\equiv 0 \bmod p(p-1)$, but not if $r \equiv 0 \bmod p(p-1)$. It might be interesting to know if these two elements are equal for $r \equiv 0 \bmod p(p-1)$. The groups $\mathrm{Map}^S_*(Y, Y)$ would perhaps repay study, since phenomena which in spheres appear as Toda brackets appear in $\mathrm{Map}^S_*(Y, Y)$ as compositions.

One could presumably obtain the analogue of Theorem 12.11 for Moore spaces $S^{2q-1} \cup_{p^f} e^{2q}$, or $S^{2q-1} \cup_2 e^{2q}$. In the latter case one would need to use $\tilde{K}_R$.

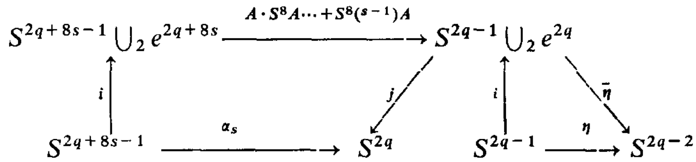
We now pass on to study 2-primary phenomena. To begin with we prove the following result.

THEOREM 12.13. *For each $s \geq 0$ there is an element μ_{8s+1} of order 2 in π^S_{8s+1} such that $e_C(\mu_{8s+1}) = \frac{1}{2}$ mod 1.*

Proof. Let α be the element of order 2 in π^S_7. Since $e'_R : \pi^S_7 \rightarrow Z_{240}$ is an isomorphism, we have $e_C(\alpha) = \frac{1}{2}$ mod 1. Also, by a delicate result of Toda [18, p.31 Corollary 3.7] we have

$$\begin{aligned} \{2, \alpha, 2\} &= \alpha\eta \quad \text{mod } 2 \\ &= 0 \quad \text{mod } 2, \end{aligned}$$

since α is divisible by 2 and $2\eta = 0$. Thus we can apply Lemma 12.5 to construct a map A. Now we have the following diagram.



We define μ_{8s+1} to be the composite

$$\bar{\eta} . A . S^8 A . \ldots S^{8(s-1)} A . i.$$

We have $\mu_1 = \eta$. The map μ_{8s+1} has order dividing 2, since it can be extended over $S^{2q+8s-1} \cup_2 e^{2q+8s}$. Since $e_C(\eta) = \frac{1}{2}$ mod 1, Proposition 12.3 shows that $d_C(\bar{\eta}) = 1$ mod 2. Hence

$$d_C(\bar{\eta} . A . S^8 A . \ldots S^{8(s-1)} A) = 1 \qquad \text{mod } 2.$$

A second application of Proposition 12.3 now yields

$$e_C(\mu_{8s+1}) = \tfrac{1}{2} \qquad \text{mod } 1.$$

Alternatively, we can obtain the same result by applying Theorem 11.1 to the equation

$$\mu_{8s+1} \in \{\eta, 2, \alpha_s\},$$

in which $e_C(\alpha_s) = \frac{1}{2}$ mod 1 by Proposition 12.7.

Proof of Theorem 7.18. Suppose $r \equiv 1$ mod 8. Then by Theorem 12.13 the homomorphism

$$e_C : \pi^S_r \rightarrow Z_2$$

is an epimorphism. But we also have

$$d_R : \pi^S_r \rightarrow Z_2$$

and Ker $d_R \subset$ Ker e_C by Lemma 7.21. Therefore $d_R = e_C$. This proves Theorem 7.18.

We have just shown that

$$d_R \mu_{8s+1} \neq 0.$$

(It is possible to show this directly from the construction of μ_{8s+1}, but this is unnecessary.)

PROPOSITION 12.14. *If $r \equiv 1$ mod 8 and $s \equiv 1$ mod 8 then the composite $\mu_r \mu_s$ is non-zero; indeed*

$$d_R(\mu_r \mu_s) \neq 0.$$

This proposition generalises the behaviour of the composite $\eta\eta$. The proof is immediate.

Proof of Theorem 7.2. Let us define μ_{8s+2} to be one of the composites considered in Proposition 12.14, for example, $\eta\mu_{8s+1}$. Then we have shown that for $r \equiv 1, 2 \bmod 8$ and $r > 0$ we have $d_R\mu_r \neq 0$. Thus d_R is an epimorphism; and since μ_r is of order 2, π_r^S splits as a direct sum $Z_2 + \operatorname{Ker} d_R$, where the subgroup Z_2 is generated by μ_r.

EXAMPLE 12.15. *Suppose that $\theta \in \pi_{8t-1}^S$ is an element such that $m(4t)e_R(\theta)$ is odd. Then for $r \equiv 1, 2$ mod 8 the composite $\theta\mu_r$ is essential; indeed*

$$e_R(\theta\mu_r) \neq 0.$$

Proof. By Theorem 3.2 (c) we have

$$e_R(\theta\mu_r) = d_R(\mu_r)e_R(\theta).$$

Let us use the notation of §9; then $e_R(\theta)$ is a generator of the 2-component of $\operatorname{Ext}_S^1(M, N)$ and the homomorphism $d_R(\mu_r)$ may be identified with the quotient map $N \to N'$. So according to the discussion in §9, $d_R(\mu_r) . e_R(\theta)$ represents a generator of $\operatorname{Ext}_S^1(M, N')$.

This example provides a second proof for Theorem 9.5. In fact, let γ be a generator for $\pi_{8u-1}(SO)$ $(u > 0)$. Then the generators for $\pi_{8u}(SO)$, $\pi_{8u+1}(SO)$ can be written as composites $\gamma\eta$, $\gamma\eta\eta$; and we have

$$J(\gamma\eta) = J(\gamma)\eta$$
$$J(\gamma\eta\eta) = J(\gamma)\eta\eta.$$

Thus Theorem 9.5 follows from Example 12.15.

EXAMPLE 12.16. *If $r \equiv 1$ mod 8 then $\{2, \mu_r, 2\}$ is non-zero; indeed $d_R\{2, \mu_r, 2\} \neq 0$.*

This example generalises the behaviour of $\{2, \eta, 2\}$. The reader will find that it is an easy application of Theorem 5.3 (i). Alternatively, of course, one can quote [18, p.31 Corollary 3.7] to show that $\{2, \mu_r, 2\} = \mu_r\eta$ mod 2 and use Proposition 12.14.

PROPOSITION 12.17. If $r \equiv 2$ *mod* 8 *and* $s \equiv 1$ mod 8 *then the composition $\mu_r\mu_s$ is non-zero; indeed*

$$e'_R(\mu_r\mu_s) = \tfrac{1}{2} \qquad \textit{mod } 1.$$

This proposition generalises the behaviour of the composite $\eta\eta\eta$.

Proof. Let

$$f : S^{2n-1} \to S^{2t}, \qquad g : S^{2t} \to S^{2q}$$

be maps representing μ_s, μ_r, where $2q \equiv 0 \bmod 8$, $2t \equiv 2 \bmod 8$, $2n - 1 \equiv 3 \bmod 8$. We have to consider the invariant $e_R(f)$. We have the following diagram.

$$\begin{array}{ccccc}
Z_2 = \tilde{K}_R(S^{2t}) & \leftarrow & \tilde{K}_R(S^{2t} \cup_f e^{2n}) & \leftarrow & \tilde{K}_R(S^{2n}) = Z \\
{\scriptstyle epi}\uparrow & & \uparrow{\scriptstyle r} & & \uparrow{\scriptstyle iso} \\
Z = \tilde{K}_C(S^{2t}) & \leftarrow & \tilde{K}_C(S^{2t} \cup_f e^{2n}) & \leftarrow & \tilde{K}_C(S^{2n}) = Z
\end{array}$$

Let ξ, η be generators in $\tilde{K}_C(S^{2t} \cup_f e^{2n})$. Then since $e_C(f) = \frac{1}{2}$ mod 1 we have (for a suitable choice of ξ)

$$\begin{aligned}
\Psi^{-1}\xi &= (-1)^t\xi + \tfrac{1}{2}((-1)^n - (-1)^t)\eta \\
&= -\xi + \eta.
\end{aligned}$$

Now in $\tilde{K}_R(S^{2t} \cup_f e^{2n})$ we have $r\Psi^{-1} = r$; thus we have $2r\xi = r\eta$. Thus $e_R(f)$ is the non-trivial extension

$$0 \leftarrow Z_2 \leftarrow Z \overset{2}{\leftarrow} Z \leftarrow 0$$

in which all the groups are given operations Ψ^k by the formula $\Psi^k x = k^n x$.

We must now compute the product $e_R(f)\, d_R(g)$, where

$$d_R(g) : \tilde{K}_R(S^{2q}) \to \tilde{K}_R(S^{2t})$$

is the epimorphism $Z \to Z_2$. We easily find that $e_R(f)\, d_R(g)$ is the extension corresponding to the rational $\frac{1}{2}$ mod 1.

PROPOSITION 12.18. *If $r \equiv 1 \bmod 8$ and $s \equiv 1 \bmod 8$ then any representative of the Toda bracket $\{\mu_r, 2, \mu_s\}$ is an element of order 4; indeed $e'_R\{\mu_r, 2, \mu_s\} = \frac{1}{4} \bmod \frac{1}{2}$.*

This proposition generalises the behaviour of $\{\eta, 2, \eta\}$.

Proof. We have just shown that the indeterminacy of $\{\mu_r, 2, \mu_s\}$ consists at least of the integers $\frac{1}{2}$ mod 1. By Theorem 11.1 we have

$$\begin{aligned} e_C\{\mu_r, 2, \mu_s\} &= -\tfrac{1}{2}.2.\tfrac{1}{2} && \bmod 1 \\ &= \tfrac{1}{2} && \bmod 1. \end{aligned}$$

By Proposition 7.14 this is equivalent to

$$e'_R\{\mu_r, 2, \mu_s\} = \tfrac{1}{4} \qquad \bmod \tfrac{1}{2}.$$

On the other hand, we have

$$2\{\mu_r, 2, \mu_s\} = \{2, \mu_r, 2\}\mu_s \qquad \bmod 0.$$

This actually gives $\eta\mu_r\, \mu_s$; but at all events it is an element of order 2 at most, so $\{\mu_r, 2, \mu_s\}$ has order dividing 4. This completes the proof.

EXAMPLE 12.19. *Suppose given an even integer m and an element $\theta \in \pi_r^S$ (where $r \equiv -1 \bmod 8$) such that $m\theta = 0$ and $me_R(\theta)$ is odd. Then for $s \equiv 1$ or $2 \bmod 8$ we have $\{\theta, m, \mu_s\} \neq 0$; indeed*

$$d_R\{\theta, m, \mu_s\} \neq 0.$$

Proof. If $s \equiv 1$ mod 8 we can make an easy calculation using Theorem 11.1:

$$\begin{aligned} e_C\{\theta, m, \mu_s\} &= -me_C(\theta)e_C(\mu_s) \\ &= \tfrac{1}{2} \qquad \bmod 1. \end{aligned}$$

If $s \equiv 2$ mod 8 then $d_R\{\theta, m, \mu_s\}$ depends only on $e_R(\theta)$, m and $d_R(\mu_s)$, by Theorem 5.3 (iii); so we may substitue $\mu_{s-1}\eta$ for μ_s, and then

$$\{\theta, m, \mu_s\} = \{\theta, m, \mu_{s-1}\}\eta.$$

So the result follows from the case $s \equiv 1$ mod 8.

Our final example is of interest in connection with certain rather technical manipulations with Toda brackets; this is perhaps not the place to explain the project from which these manipulations come, although the reader is assured that they are not without purpose.

We suppose given an element θ in π_r^S for $r = 2^{f-1} - 1$, such that $2^f\theta = 0$, $e_R(\theta) = 2^{-f}$ and $\theta\theta = 0$. For example, there is such an element if $f = 5$. We assume $f \geqq 4$, so that $r \equiv -1 \bmod 8$.

By [18, p.30, Theorem 3.6] there are elements ρ, φ in the 2-component of π_{2r-1} (where $2r - 1 = 2^f - 1$) such that

$$\rho \in \{\theta, \theta, 2^f\}$$

and

$$2\rho + 2^f\varphi \in \{\theta, 2^f, \theta\}.$$

EXAMPLE 12.20. *In the last equation the element φ cannot be zero; indeed we have*

$$e_R(\varphi) = 2^{-f-1} \qquad \mathit{mod}\ 2^{-f}.$$

Proof. By Theorem 11.1 we have

$$e_R\{\theta, 2^f, \theta\} = -2^{-f}.$$

By Corollary 11.7 we have

$$\begin{aligned} e_R\{\theta, \theta, 2^f\} &= -2^f . \tfrac{1}{2}(1 + 2^{f-1})2^{-f}2^{-f} && \bmod \tfrac{1}{2} \\ &= -(2^{-f-1} + \tfrac{1}{4}) && \bmod \tfrac{1}{2}. \end{aligned}$$

Thus

$$e_R(2\rho) = -2^{-f} + \tfrac{1}{2} \qquad \bmod 1.$$

Hence

$$e_R(\varphi) = 2^{-f-1} \qquad \bmod 2^{-f}.$$

This completes the proof.

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