A second order traffic-flow model with constraint on the velocity for the Modeling of Traffic Jams

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Joint work with

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Summary

1. Traffic models: overview on fluid models

Fluid models (1)

- Conservation of car density

$$\partial_t n + \partial_x q = 0$$

- What expression for the flux q ?

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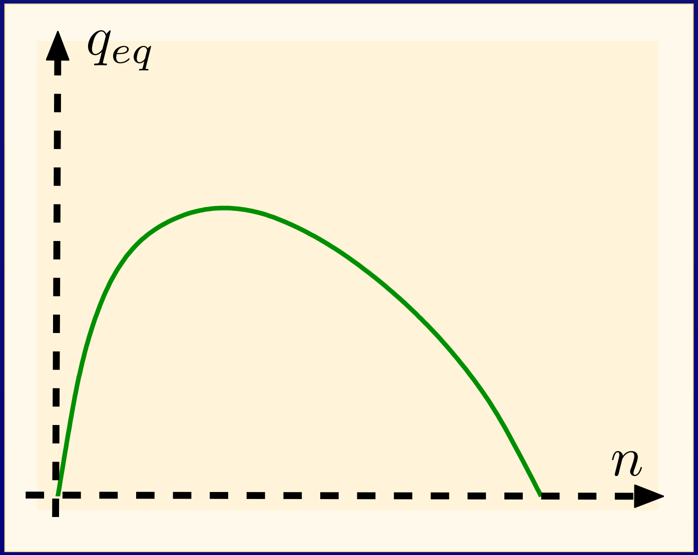
- First order models:

$$q = q_{eq}(n)$$

[Lighthill, Witham (1955)], ...



Fluid models (2)

- Second order models: $q = nu$ and gas dynamics-like eq. for u:

$$\partial_t nu + \partial_x (nu^2 + p) = -\frac{nu - q_{eq}(n)}{\tau}$$

  - [Payne (1971)], ...

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  - [Payne (1971)], ...

- [Daganzo (1995)]: Inacceptable properties (e.g. Vehicles going backwards)
  - Fluid $\Rightarrow$ sound propagation is isotropic in a comoving frame
  - Traffic: information propagates backwards

The Aw-Rascle model (1)

- Modified 2nd order model (see also [Zhang (2002)])
- Preferred velocity w is a Lagrangian quantity:

$$\dot{w} := (\partial_t + u\partial_x)w = 0$$

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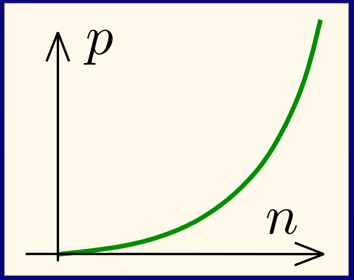
- The actual velocity u offsets the preferred velocity w by a quantity $p(n)$ which increases with n

$$w = u + p(n)\,, \quad p \nearrow \text{ as } n \nearrow$$

- Typically $p(n) = n^\gamma$, $\gamma > 0$



AR model (2)

$$\partial_t n + \partial_x(nu) = 0$$
$$(\partial_t + u\partial_x)(u + p(n)) = 0$$

- Second eq. equivalent to

$$(\partial_t + (u - np'(n))\partial_x)u = 0$$

- Two characteristic velocities:
  - $\lambda_1 = u - np'(n)$ (assoc.w. u, GNL)
  - $\lambda_2 = u$ (assoc.w. $w = u + p(n)$, LD)

Properties of AR model

- Invariant regions: (u, w) - rectangles
  - If
    $$a < u_0 < b \quad \text{and} \quad c < w_0 < d$$
    then for all times
    $$a < u(t) < b \quad \text{and} \quad c < w(t) < d$$
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- AR model in Lagrangian coordinates = continuous version of Follow-the-Leader model [Aw, Klar, Materne, Rascle (2002)]

No invariant region for n !

- Problem: there is no invariant region for n
  - $n > 0$ BUT:
  - n can exceed the upper limit n^* (if any) even if initially $n < n^*$)
- Modified AR model (M-AR):
  - AR model which guarantees the constraint

$$n < n^*$$

at all times

Rescaled Modified AR model (RM-AR)

- Perturbed AR system

$$\partial_t n^\varepsilon + \partial_x (n^\varepsilon u^\varepsilon) = 0$$
$$(\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(n^\varepsilon)) = 0$$

- with modified velocity offset:

$$p(n) = \frac{1}{\left(\frac{1}{n} - \frac{1}{n^*}\right)^\gamma}$$

Constrained Pressureless Gas Dynamics (CPGD)

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$$
\begin{aligned}
&\partial_t n + \partial_x (nu) = 0 \\
&(\partial_t + u\partial_x)(u + \bar{p}) = 0 \\
&\bar{p}(n^* - n) = 0 \\
&\bar{p} \geq 0\,, \quad 0 \leq n \leq n^*
\end{aligned}
$$

- see e.g. [Brenier, ...], [B. and Bouchut] for gaseous corks in pipes

About the constraint

- We want to improve CPGD model with

$$n^* = n^*(u)$$

since it is well known that in practice, the distribution of vehicles on a highway, depends on their velocity

2. Rescaled Modified Aw-Rascle

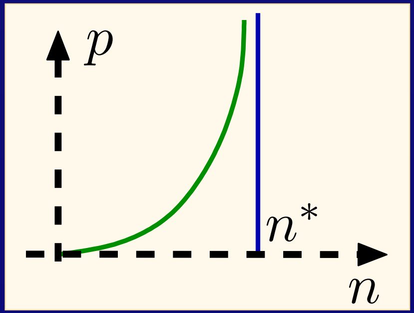
From the Modified AR model $(M - AR^*)$

- Modify $p(n)$ s.t.

$$p(n,u) \longrightarrow \infty \quad \text{as} \quad n \longrightarrow n^*(u)$$

- For instance

$$p(n,u) = \frac{1}{\left(\frac{1}{n} - \frac{1}{n^*(u)}\right)^{\gamma}}$$



Density constraint

- $M - AR^*$ has the same properties as the standard AR model
  - Hyperbolicity
  - Invariant regions
- One linearly degenerate eigenvalue
- Under assumptions on $n^*(u)$, the other eigenvalue is genuinely non linear
- Satisfies the density constraint

$$n < n^*(u)$$

at all times

Assumptions on $n^*(u)$

- $n^*(u)$ is twice continuously differentiable
- $n^*(u)$ is strictly decreasing
- $n^*(u)$ is concave

- The second assumption is natural since the minimum distance between drivers is an increasing function of the velocity

A singular situation

- In practice: two traffic regimes:
  - Uncongested traffic ($n < n^*(u)$): driver goes its preferred velocity
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  - Uncongested traffic ($n < n^*(u)$): driver goes its preferred velocity
  - Congested traffic ($n \sim n^*(u)$): velocity is determined by the traffic conditions.
- in the $M - AR^*$ model:
  - $p(n,u)$ very small as long as n not close to $n^*(u)$
  - $p(n,u)$ large (and possibly ∞) only when $n \underset{\sim}{\leq} n^*(u)$

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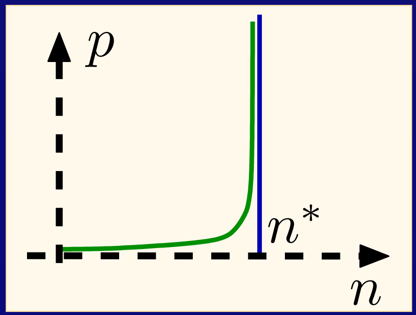

- Modeled by the rescaling:

$$p(n, u) = \varepsilon \tilde{p}(n, u)$$

Rescaled Modified AR^* model ($RM - AR^*$)

- Perturbed AR^* system

$$\partial_t n^\varepsilon + \partial_x (n^\varepsilon u^\varepsilon) = 0$$
$$(\partial_t + u^\varepsilon \partial_x)(u^\varepsilon + \varepsilon p(n^\varepsilon, u^\varepsilon)) = 0$$

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- Question: what happens in the limit

$$\varepsilon \longrightarrow 0$$

3. Limit $\varepsilon \to 0$: The Second Order Model with Constraint

$\varepsilon \longrightarrow 0$: Case I (uncongested)

- Suppose $n^\varepsilon \longrightarrow n < n^*(u)$ (uncongested case)
  - Then $\varepsilon p(n^\varepsilon, u^\varepsilon) \longrightarrow 0$ in $(RM - AR^*)$ model:

$$\partial_t n^\varepsilon + \partial_x (n^\varepsilon u^\varepsilon) = 0$$
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- Limit system = Pressureless Gas Dynamics

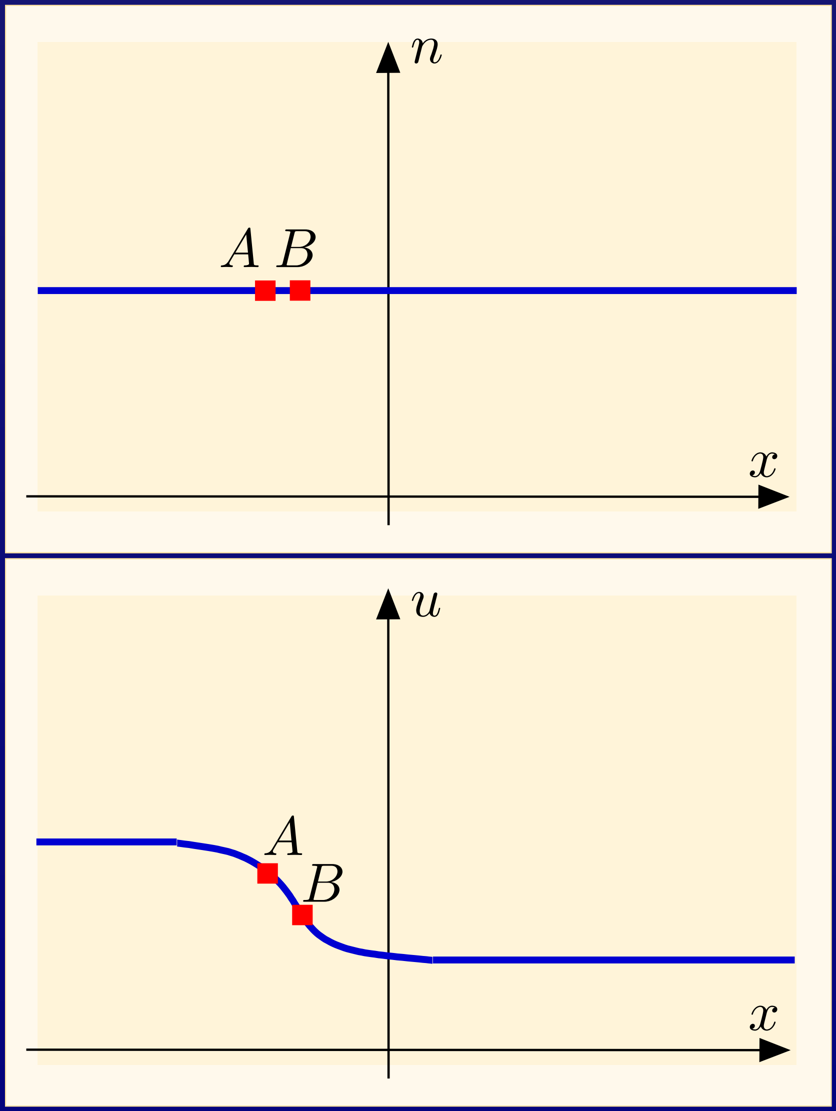
$$\partial_t n + \partial_x (nu) = 0$$
$$(\partial_t + u\partial_x) u = 0$$

-
  - Mass conservation
  - Burger's eq. for the velocity

Pressureless Gas Dyn. properties

- Not strictly hyperbolic
  - 2 identical eigenvalues u
  - But not diagonalizable: Jacobian $= \begin{pmatrix} u & n \\ 0 & u \end{pmatrix}$
- Weak instability:
  - linearized solution increase like $O(t)$
- Generates mass concentrations

Pressureless Gas Dyn. concentrations



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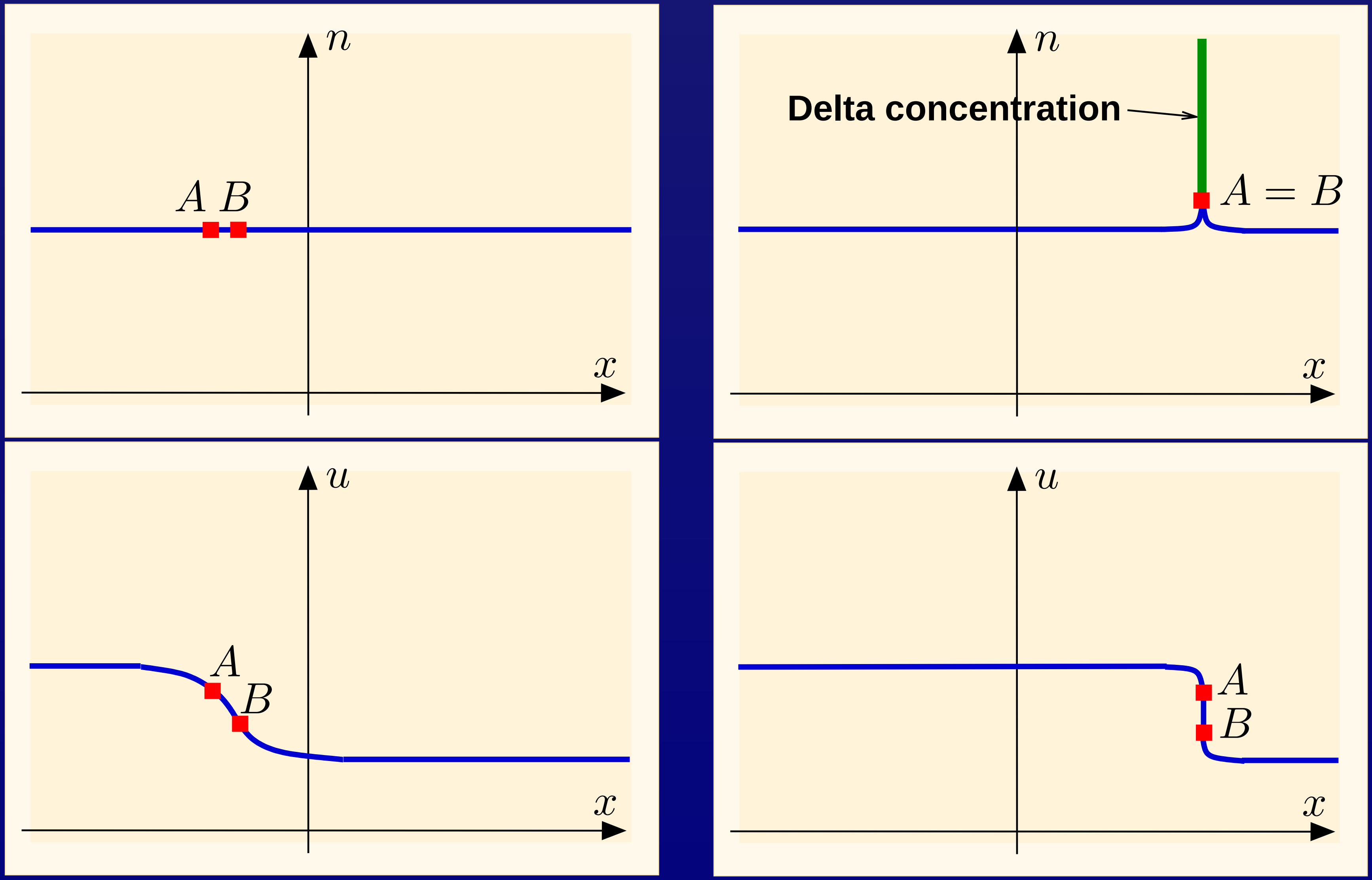

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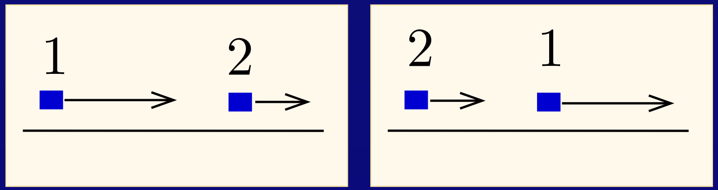

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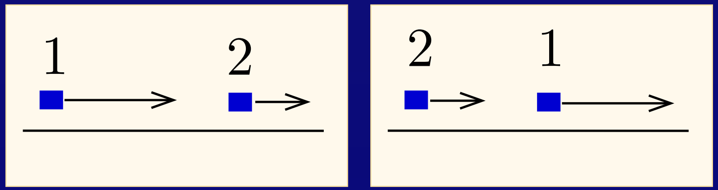
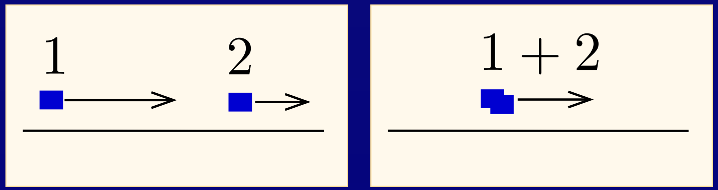


  - Sticky particles (Zeldowitch, E, ...)



- see e.g. [Bouchut (94)], [Grenier (95)], [Rykov, Sinai (96)], [Brenier, Grenier (98)], ...

Here: no concentrations

- Density constraint: no concentration formation
  - No need to define a particle dynamics
- Instead: formation of ’clusters’ (traffic jams)
  - Cluster dynamics follows from the asymptotic limit

$\varepsilon \to 0$: Case II (congested)

- Suppose $n^\varepsilon \to n^*$ (then $p(n^\varepsilon, u^\varepsilon) \to \infty$)
  - Suppose $\varepsilon p(n^\varepsilon, u^\varepsilon) \to \bar{p} < \infty$
- Then $\varepsilon \to 0$ in $(RM - AR^*)$ model:

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- Gives

$$\partial_t n + \partial_x(nu) = 0$$
$$(\partial_t + u\partial_x)(u + \bar{p}) = 0$$
$$n = n^*(u)$$

- $\bar{p}$ unknown: Lagrange multiplier

$\varepsilon \to 0$: Case II (congested)

- Formaly, it is

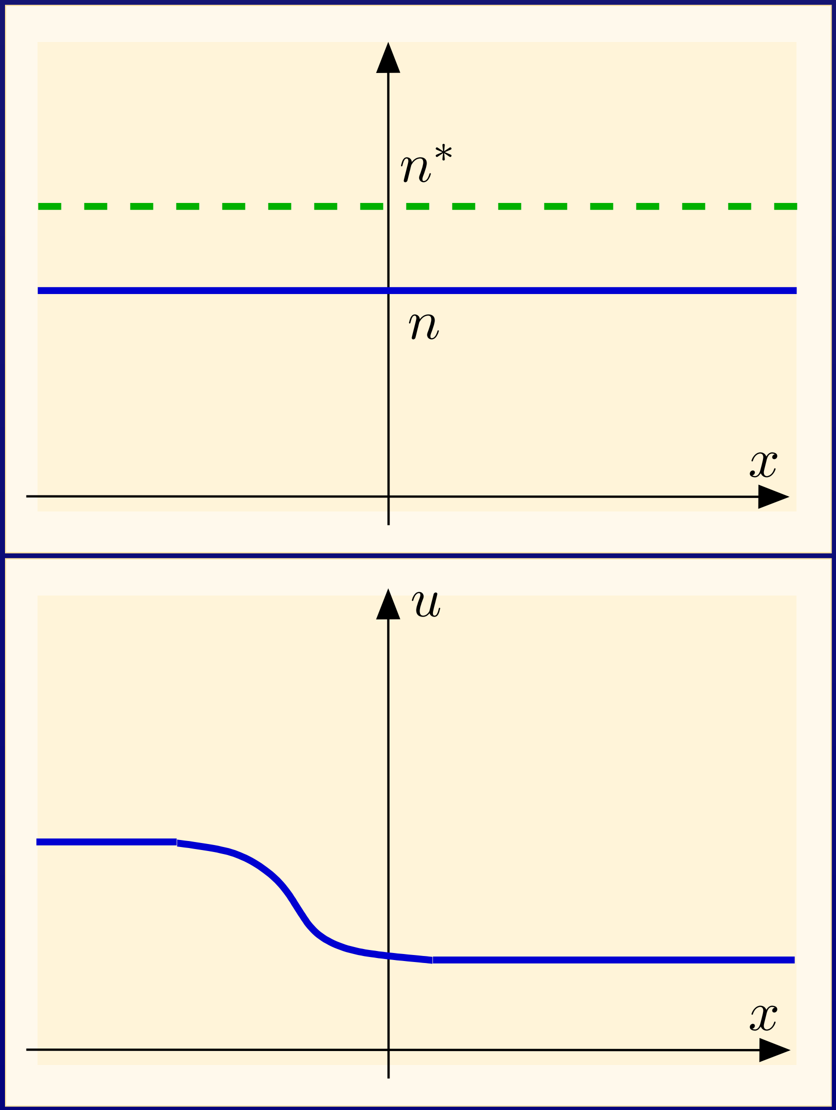
$$\partial_t n^*(u) + \partial_x (n^*(u) u) = 0,$$

- Let $n \mapsto u^*(n)$ the inverse functional of $u \mapsto n^*(u)$, it rewrites

$$\partial_t n + \partial_x (n u^*(n)) = 0,$$

- Therefore the second order model "relaxes" to the Lighthill, Witham first order model with the flux $q(n) = n u^*(n)$ when the maximal density constraint is saturated

Cluster formation



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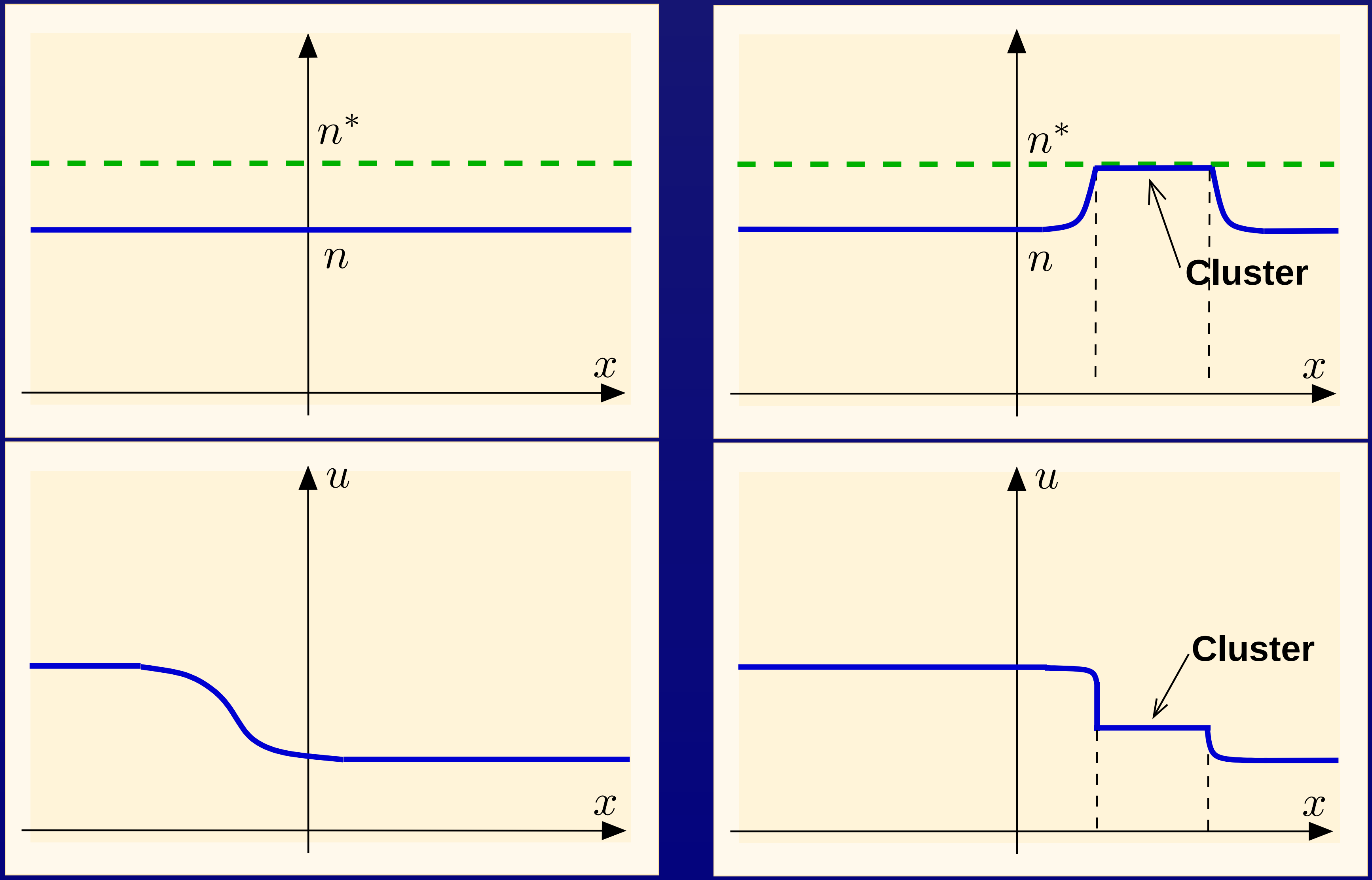

Unified formulation

➠ Constrained Pressureless Gas Dynamics (CPGD)

$$\partial_t n + \partial_x(nu) = 0$$
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$$\bar{p}(n^*(u) - n) = 0$$
$$\bar{p} \geq 0\,, \quad 0 \leq n \leq n^*(u)$$

4. Second Order Model with Constraint: additional laws

SOMC model

- SOMC formulation ill-posed
  lack of information for defining a unique solution

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  lack of information for defining a unique solution

- To be defined
  - Cluster dynamics
  - Value of $\bar{p}$ inside clusters
  - What if clusters meet ?

Characteristic velocities

- If $n^\varepsilon \to n^*(u)$ with $\varepsilon p(n^\varepsilon, u^\varepsilon) \to \bar{p} < \infty$, then the Characteristic velocities:
  - $\lambda_1^\varepsilon \to u + \frac{n^*(u)}{(n^*)'(u)}$
  - $\lambda_2^\varepsilon = u^\varepsilon \to u$
- A velocity variation in front of the cluster propagates with a finite speed
- In the case $n^* =$ constant, any variation of the velocity of the leading car instantaneously propagates to the whole cluster since $\lambda_1^\varepsilon \to -\infty$

How to complete SOMC formulation

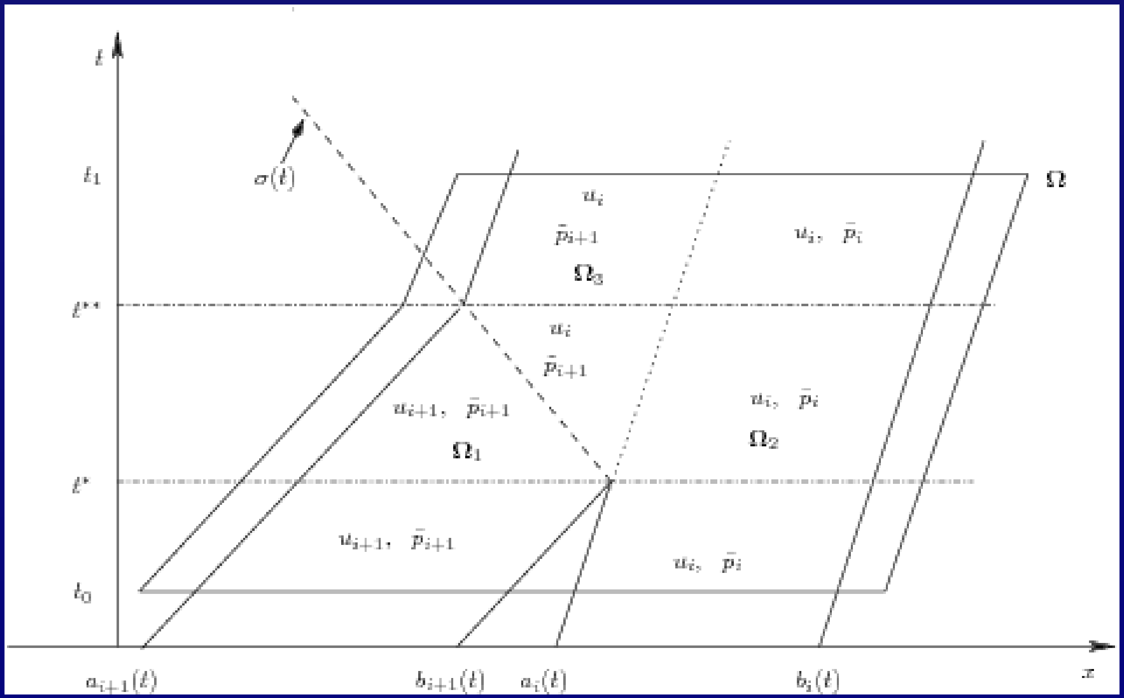
- Limit $(RM - AR^*) \to$ (SOMC) is formal
  - Gives no information about cluster dynamics beyond what has been noticed above

How to complete SOMC formulation

- Limit $(RM - AR^*) \longrightarrow$ (SOMC) is formal
  - Gives no information about cluster dynamics beyond what has been noticed above
- But Riemann problem solutions of $(RM - AR^*)$ are explicit
  - Limit $\varepsilon \longrightarrow 0$ in these solutions give information about cluster dynamics

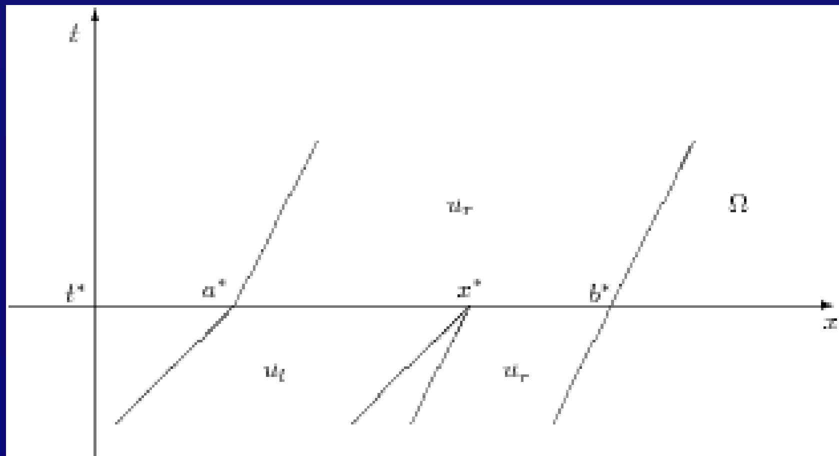
Cluster dynamics (from Riemann pbm)

- When two clusters meet, a shock wave appears at the front of the cluster behind and propagates upstream with a finite speed

- 

Cluster dynamics (difference with constant case)

- When two clusters meet, they merge
  - The resulting cluster takes instantaneously the velocity of the front cluster (the slowest one)

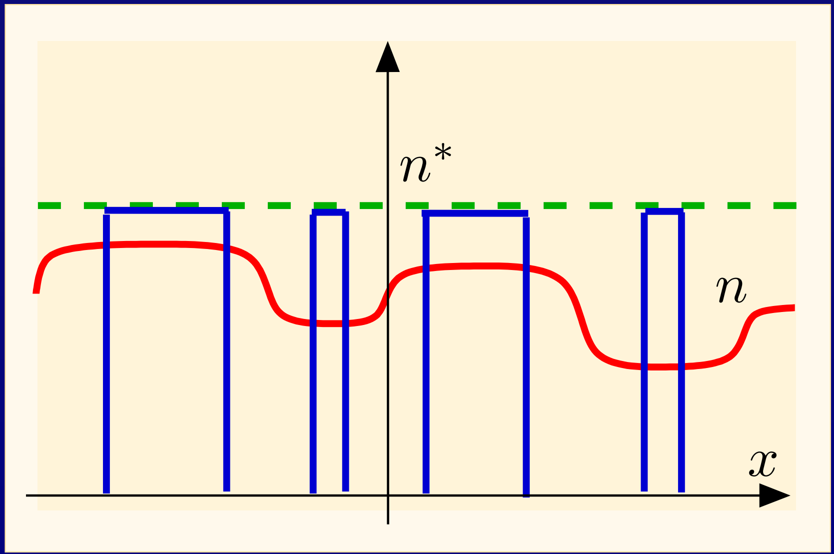
→ 

5. Existence theorem for SOMC

Cluster approximate solution

- Idea (follows from [B. and Bouchut (2002, 2003)]),
  - Approximate (in $\mathcal{D}'$) the solution by clusters

$$\begin{pmatrix} n(x,t) \\ (nu)(x,t) \end{pmatrix} \approx \sum_1^N \begin{pmatrix} n^*(u_i) \\ n^*(u_i)u_i(t) \end{pmatrix} \chi_{a_i(t) \le x \le b_i(t)}$$



Properties of cluster dynamics

- Is a weak solution of (SOMC)
- Satisfies L^∞ and BV bounds:

$$\operatorname*{essinf}_y u^0(y) \le u(x,t) \le \operatorname*{esssup}_y u^0(y),$$

$$0 \le \bar{p}(x,t) \le \operatorname*{esssup}_y u^0(y) + \operatorname*{esssup}_y \bar{p}^0(y)$$

$$TV_K(u(.,t)) \le TV_{\tilde{K}}(u^0),$$

$$TV_K(\bar{p}(.,t)) \le TV_{\tilde{K}}(\bar{p}^0) + 2TV_{\tilde{K}}(u^0),$$

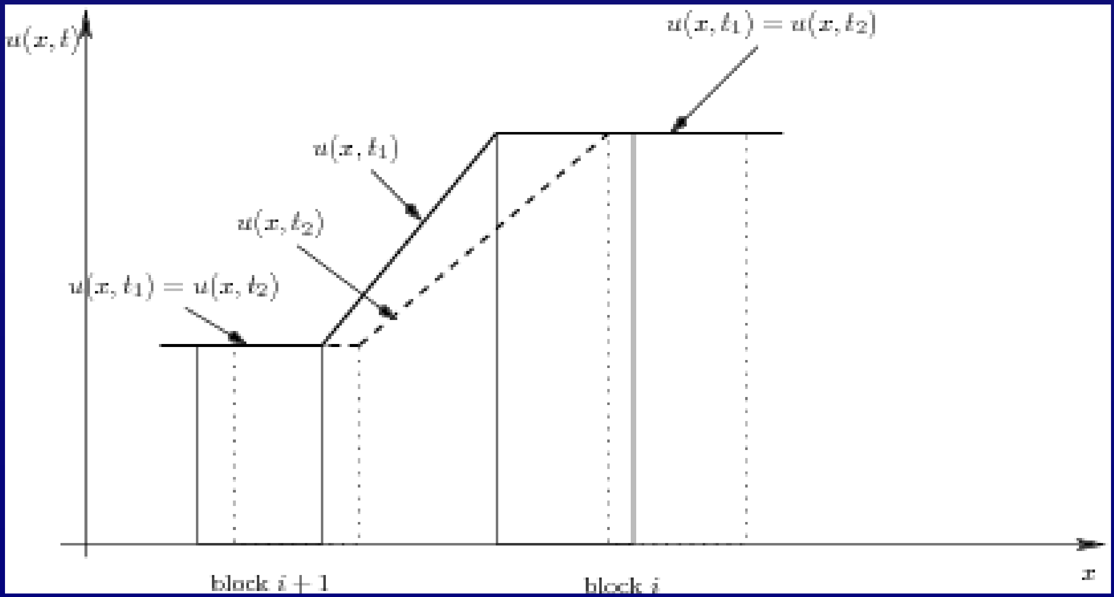
for any compact $K = [a,b]$ and with

$\tilde{K} = \left[a - t \operatorname{esssup} \left|u^0\right|, b - t \operatorname{essinf} \left|u^0\right|\right]$

Properties of cluster dynamics (2)

- We have equicontinuity in time:

$$\int_{\mathbb{R}} |u(x,t_2) - u(x,t_1)|\, dx \le \|u\|_\infty \, |t_2 - t_1| \, TV(u^0)$$

- 

Properties of cluster dynamics (2)

- We have equicontinuity in time:

$$\int_{\mathbb{R}} |u_k(x,t_2) - u_k(x,t_1)|\,dx \le \|u_k\|_\infty |t_2 - t_1|\, TV(u^0)$$

- With furthermore BV bound on u_k, a Cantor diagonal process argument implies

$$u_k \underset{k\to\infty}{\longrightarrow} u \text{ in } L^1(\mathbb{R}\times[0,T]).$$

Proof of existence

- Step 1: approximate initial condition $(n_0, u_0, \bar{p}_0)$ by a converging sequence of clusters (n_0^k, u_0^k, p_0^k)
  - Defines a sequence of cluster sol. $(n^k, u^k, \bar{p}^k)$ satisfying the above a priori bounds

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- Step 3: Prove the convergence of the products $n^k u^k$, $n^k \bar{p}^k$, $n^*(u_k)\bar{p}_k$, ... in $L^\infty_{w*}((0,\infty)\times\mathbb{R})$ and obtain a solution of (SOMC).

Existence result

- Suppose
  - $n_0 \in L^1 \cap L^\infty$,
  - $u_0 \in L^\infty \cap BV, \qquad 0 \le n_0 \le n^*(u_0)$
  - $\bar{p}^0 \in L^\infty \cap BV$ in cluster form

Existence result

- Suppose
  - $n_0 \in L^1 \cap L^\infty$,
  - $u_0 \in L^\infty \cap BV, \qquad 0 \le n_0 \le n^*(u_0)$
  - $\bar{p}^0 \in L^\infty \cap BV$ in cluster form
- $\exists n \in L^\infty_t(L^\infty_x \cap L^1_x), u, \bar{p} \in L^\infty_{x,t}$
  - a solution of SOMC
  - satisfying L^∞ and BV bounds

6. Conclusion

Summary

- Modified Aw-Rascle model
  - Density constraint
  - Rescaled for small difference between preferred velocity and actual velocity in uncongested situations

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- Modified Aw-Rascle model
  - Density constraint
  - Rescaled for small difference between preferred velocity and actual velocity in uncongested situations
- Limit model
  - Constrained Pressureless Gas Dynamics
  - Describes well cluster formation and dynamics
  - Existence theorem

Perspectives

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  - Convergence proof $(RM - AR^*) \longrightarrow$ (SOMC)
  - About unicity of the solution ?
  - Lagrangian formulation and scheme

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- SOMC:
  - Convergence proof $(RM - AR^*) \longrightarrow$ (SOMC)
  - About unicity of the solution ?
  - Lagrangian formulation and scheme
- More elaborate model
  - Multi-lane
  - Multi-class
  - etc.