A model for the evolution of traffic jams in multi-lane

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Abstract

In [7], Berthelin, Degond, Delitala and Rascle introduced a traffic flow model describing the formation and the dynamics of traffic jams. This model consists of a Pressureless Gas Dynamics system under a maximal constraint on the density and is derived through a singular limit of the Aw-Rascle model. In the present paper we propose an improvement of this model by allowing the road to be multi-lane piecewise. The idea is to use the maximal constraint to model the number of lanes. We also add in the model a parameter α which model the various speed limitations according to the number of lanes. We present the dynamical behaviour of clusters (traffic jams) and by approximation with such solutions, we obtain an existence result of weak solutions for any initial data.

Key words: Traffic flow models, Constrained Pressureless Gas Dynamics, Multi-lane, Weak solutions, Traffic jams

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Contents

1 Introduction

Classical models of traffic are splitted into three main categories: particle models (or "car-following" models) [15, 3], kinetic models [23, 24, 20, 18], and fluid dynamical models [19, 21, 22, 2, 26, 11, 16]. Obviously, these models are related; for example in [1], a fluid model is derived from a particle model. See also [17]. Here, we are interested in the third approach, which describes the evolution of macroscopic variables (like density, velocity, flow) in space and time. Let us recall briefly the history of such models.

The simplest fluid models of traffic are based on the single conservation law

$$\partial_t n + \partial_x f(n) = 0,$$

where $n = n(t, x)$ is the density of vehicles and $f(n)$ the associated flow. This model only assumes the conservation of the number of cars. Such models are called "first order" models, and the first one is due to Lighthill and Whitham [19] and Richards [25].

If we take the flux $f(n) = nu$ with $u = u(t, x)$ the velocity of the cars, we add a second equation of equilibrium related to the conservation of momentum. This approach starts with the Payne-Whitham model [21, 22].

But the analogy fluid-vehicles is not really convincing: in fact, in the paper [12], Daganzo shown the limits of this analogy, exhibiting absurdities which are implied by classical second-order models, for example, vehicles going backwards. To rehabilitate these models, Aw and Rascle proposed in [2] a new one which corrects the deficiencies pointed out by Daganzo. In particular, the density and velocity remain nonnegative.

The Aw-Rascle model is given by

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ (\partial_t + u\partial_x)(u + p(n)) = 0, \end{cases}$$

or in the conservative form

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(n(u + p(n))) + \partial_x(nu(u + p(n))) = 0, \end{cases}$$

where $p(n) \sim n^\gamma$ is the velocity offset, which bears analogies with the pressure in fluid dynamics.

In fact, this model can be derived from a microscopic "car-following" model, as it has been shown in [1]. But even the Aw-Rascle model exhibits some unphysical feature, namely the non-propagation of the upper bound of the density n, making a constraint such that $n \leq n^*$ impossible (where n^* stands for a maximal density of vehicles).

Some constraints models have been developed these last years in order to impose such bounds in hyperbolic models. See [9], [4], [6] for the first results of this topic and [5] for a numerical version of this kind of problem.

That is why recently, Berthelin, Degond, Delitala and Rascle [7] proposed a new second-order model, which aim is to allow to preserve the density constraint $n \leq n^*$ at any time. The main ideas are:

- modifying the Aw-Rascle model, changing the velocity offset into

  $$p(n) = \left(\frac{1}{n} - \frac{1}{n^*}\right)^{-\gamma}, \qquad n < n^*,$$

  thus $p(n)$ is increasing and tends to infinity when $n \to n^*$;
- rescaling this modified Aw-Rascle model (changing $p(n)$ into $\varepsilon p(n_\varepsilon)$) and taking the formal limit when $\varepsilon \to 0^+$.

This process leads to a limit system on (n, u) which corresponds to the *Pressureless Gas Dynamics system*:

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(nu) + \partial_x(nu^2) = 0, \end{cases}$$

in areas where $n < n^*$. But a new quantity appears, due to the singularity of the velocity offset in $n = n^*$. In fact, denoting by $\overline{p}(t,x)$ the formal limit of $\varepsilon p(n_\varepsilon)(t,x)$ when $\varepsilon \to 0^+$, we may have $\overline{p}$ non zero and finite at a point (t,x) such that $n(t,x) = n^*$. Thus, the function $\overline{p}$ turns out to be a Lagrangian multiplier of the constraint $n \leq n^*$. Finally, we obtain the *Constrained Pressureless Gas Dynamics* (designed as CPGD) model:

$$\begin{cases} \partial_t n + \partial_x(nu) = 0, \\ \partial_t(n(u+\overline{p})) + \partial_x(nu(u+\overline{p})) = 0, \\ 0 \le n \le n^*, \quad \overline{p} \ge 0, \quad (n^* - n)\overline{p} = 0. \end{cases}$$

The term $\overline{p}$ represents the speed capability which is not used if the road is blocked and that the cars in front imposes a speed smaller than that desired. We refer to [7] for more details on the derivation of the CPGD system in the case of the maximal density n^* being constant. The case where n^* depends on the velocity ($n^* = n^*(u)$) is more realistic (taking into account the fact that the maximal number of cars is smaller as the velocity is great, for safety reasons) and is treated in [8]. In [13], a numerical treatment of traffic jam is done.

In this paper, we propose another type of improvement based on the following idea: the idea is to use the maximal constraint to model the number of lanes. The constraint n^* will depend on the number of lanes in the portion of the road. Indeed, in a two-lane portion, n^* can be twice greater than it is in a one-lane portion of the road. This idea simplifies the model dramatically and we no longer need to consider as many equations of lanes which makes the modeling and use much simpler while reporting the same phenomenon.

The paper is organized as follows: in the next section, we make a modification of the CPGD system to model traffic jams in multi-lane. In section 3, we present the dynamics of jams. By approximation with such data, it is used in section 4 to prove the existence of weak solutions for any initial data.

2 The ML-CPGD model

We consider a piecewise constant maximal density of vehicles, given by

$$n^*(x) = \sum_{j=0}^{M} n_j^* \mathbb{1}_{]r_j, r_{j+1}[}(x)$$

where

$$n_j^* \in \{1, 2\}, \quad (r_j)_{1 \le j \le M} \text{ an increasing sequence of real numbers,}$$

$$r_0 = -\infty, \quad r_{M+1} = +\infty.$$

It means that we set on a road with one or two lanes, the road transitions (change of number of lanes) being at points $(r_j)_{1 \le j \le M}$. On a one-lane section, the maximal density is one (in view of simplification), whereas on a two-lane section, the maximal allowed density is two. It is the first improvement of our model: the constraint density changes with x to model the fact that there is one or two lanes. Evolution equations are given by the *Multi-lane Constrained Pressureless Gas Dynamics* system (designed by ML-CPGD), whose conservative form is

$$\partial_t n + \partial_x(nu) = 0\,, \tag{2.1}$$

$$\partial_t(n(u+p)I_\alpha) + \partial_x(nu(u+p)I_\alpha) = 0\,, \tag{2.2}$$

$$0 \le n \le n^*(x)\,, \quad u \ge 0\,, \quad p \ge 0\,, \quad (n^*(x) - n)p = 0\,, \tag{2.3}$$

where the function $I_\alpha = I_\alpha(x)$ is defined by

$$I_\alpha(x) = \begin{cases} 1 & \text{if} \quad n^*(x) = 1, \\ 1/\alpha & \text{if} \quad n^*(x) = 2. \end{cases}$$

The number $\alpha \geq 1$ stands for the rate between two-lane velocities and one-lane velocities. Thus a single car (we mean a car not into a jam) with speed u on a one-lane road will pass to the speed αu on a two-lane road. This represents the fact that on a two-lane section, the average velocity is higher than on a one-lane (on a highway, you drive faster than on a road even if you are alone). The preferred velocity depends on the road width according to α. It can also be understood as the speed limitation on the various kind of roads. This is the second improvement of our model. It only act on the second equation since it is the momentum quantity which has to be changed and not the conservation of the number of cars (first equation).

Of course, this model can be extended to case with three-lane, four-lane portion... In the case of three lanes, $n_j^* \in \{1, 2, 3\}$ and I_α is replaced by $I_{\alpha,\beta}(x) = \begin{cases} 1 & \text{if} \quad n^*(x) = 1, \\ 1/\alpha & \text{if} \quad n^*(x) = 2, \\ 1/\beta & \text{if} \quad n^*(x) = 3, \end{cases}$ with $\beta \geq \alpha \geq 1$, α being the rapport of speed between one and two lanes and β/α the rapport between three and two lanes.

3 Clusters dynamics

In this section, we present some particular solutions (n, u, p) of (2.1)-(2.3) which are clusters solutions. For these functions, $n = n(t, x)$ take as only values 0 and $n^*(x)$. In some sense, they are an extension of sticky particles of [10, 14] playing a crucial role in the proof of existence of solutions for constraint models. They have been introduced in [9] and used with various dynamics in [4, 6, 7, 8].

Let us consider the density $n(t, x)$, the flux $n(t, x)u(t, x)$ and the pressure $n(t, x)p(t, x)$ given respectively by

$$n(t,x) = n^*(x) \sum_{i=1}^{N} \mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{3.1}$$

$$n(t,x)u(t,x) = n^*(x) \sum_{i=1}^{N} u_i \mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{3.2}$$

$$n(t,x)p(t,x) = n^*(x) \sum_{i=1}^{N} p_i \mathbb{1}_{a_i(t)<x<b_i(t)}, \tag{3.3}$$

with

$$N \in \mathbb{N}^*, \qquad u_i \geq 0, \qquad p_i \geq 0,$$

as long as there is no collision and no change of $n^*(x)$. That is to say

$$a_1(t) < b_1(t) \leq a_2(t) < b_2(t) < \ldots \leq a_N(t) < b_N(t)$$

and the number of blocks N is constant until there is a shock or a change of width (thus we have $N = N(t)$).
This type of piecewise constant solution writes as a superposition of blocks with $(n, u, p) = (n^*, u_i, p_i)$ constant. Each block evolves according to the interactions with the other blocks and the changes of width.
We have to explain three dynamics:

- What happens when two blocks collide ? (how to describe a shock)
- What happens when the road narrows ($n^*(x)$ was 2 and becomes 1) ?
- What happens when the road widens ($n^*(x)$ was 1 and becomes 2) ?

First, let us present some technical properties that will be used in the various cases.

Lemma 3.1 *Let be $s, \sigma \in [0, +\infty[$, $\varphi \in \mathcal{D}(]0, +\infty[\times\mathbb{R})$, and $a, b \in C^1(]inf(s, \sigma), sup(s, \sigma)[)$. We set*

$$J(s, \sigma, a, b, u) := \int_s^\sigma \int_{a(t)}^{b(t)} (\partial_t \varphi(t, x) + u(t)\partial_x \varphi(t, x))dxdt.$$

Then we get

$$\begin{aligned} J(s, \sigma, a, b, u) &= \int_{a(\sigma)}^{b(\sigma)} \varphi(\sigma, x)\, dx - \int_{a(s)}^{b(s)} \varphi(s, x)\, dx \\ &\quad + \int_s^\sigma \varphi(t, b(t))\,(u(t) - b'(t))\, dt + \int_s^\sigma \varphi(t, a(t))\,(a'(t) - u(t))\, dt. \end{aligned} \tag{3.4}$$

Proof: We have

$$\frac{d}{dt}\left[\int_{a(t)}^{b(t)} \varphi(t, x)\, dx\right] = \int_{a(t)}^{b(t)} \partial_t \varphi(t, x)\, dx + \varphi(t, b(t))\, b'(t) - \varphi(t, a(t))\, a'(t),$$

thus

$$\begin{aligned} \int_s^\sigma \int_{a(t)}^{b(t)} \partial_t \varphi(t, x)\, dx\, dt = &\int_{a(\sigma)}^{b(\sigma)} \varphi(\sigma, x)\, dx - \int_{a(s)}^{b(s)} \varphi(s, x)\, dx \\ &- \int_s^\sigma \varphi(t, b(t))\, b'(t)\, dt + \int_s^\sigma \varphi(t, a(t))\, a'(t)\, dt. \end{aligned}$$

Moreover $\int_s^\sigma \int_{a(t)}^{b(t)} \partial_x \varphi(t, x)\, dx\, dt = \int_s^\sigma \varphi(t, b(t))\, dt - \int_s^\sigma \varphi(t, a(t))\, dt$ and the result follows. □

Remark 3.2 *We notice that*

$$J(\sigma, s, a, b, u) = -J(s, \sigma, a, b, u), \tag{3.5}$$

$$J(s, \sigma, b, a, u) = -J(s, \sigma, a, b, u). \tag{3.6}$$

If we have $a' = b' = u$, then

$$J(s, \sigma, a, b, u) = \int_{a(\sigma)}^{b(\sigma)} \varphi(\sigma, x)\, dx - \int_{a(s)}^{b(s)} \varphi(s, x)\, dx. \tag{3.7}$$

If we have $a' = u$ and c is constant, then

$$J(s,\sigma,a,c,u) = \int_{a(\sigma)}^{c} \varphi(\sigma,x)\,dx - \int_{a(s)}^{c} \varphi(s,x)\,dx + \int_{s}^{\sigma} \varphi(t,c)u(t)dt. \tag{3.8}$$

Lemma 3.3 *We have the following formulas:*
If $a' = b' = c' = u$, then

$$\begin{aligned} J(s,\sigma,a,b,u) + J(\sigma,\tau,a,c,u) = &-\int_{a(s)}^{b(s)} \varphi(s,x)\,dx \\ &+ \int_{c(\sigma)}^{b(\sigma)} \varphi(\sigma,x)\,dx + \int_{a(\tau)}^{c(\tau)} \varphi(\tau,x)\,dx. \end{aligned} \tag{3.9}$$

If $a' = b' = u$ and $c = b(\sigma) = a(\tau)$, then

$$J(s,\sigma,a,b,u) + J(\sigma,\tau,a,c,u) = -\int_{a(s)}^{b(s)} \varphi(s,x)dx + \int_{\sigma}^{\tau} u(t)\varphi(t,c)dt. \tag{3.10}$$

Proof: We have

$$\begin{aligned} & J(s,\sigma,a,b,u) + J(\sigma,\tau,a,c,u) \\ = \;& \int_{a(\sigma)}^{b(\sigma)} \varphi(\sigma,x)\,dx - \int_{a(s)}^{b(s)} \varphi(s,x)\,dx \\ & + \int_{s}^{\sigma} \varphi(t,b(t))\,(u(t) - b'(t))\,dt + \int_{s}^{\sigma} \varphi(t,a(t))\,(a'(t) - u(t))\,dt \\ & + \int_{a(\tau)}^{c(\tau)} \varphi(\tau,x)\,dx - \int_{a(\sigma)}^{c(\sigma)} \varphi(\sigma,x)\,dx \\ & + \int_{\sigma}^{\tau} \varphi(t,c(t))\,(u(t) - c'(t))\,dt + \int_{\sigma}^{\tau} \varphi(t,a(t))\,(a'(t) - u(t))\,dt \\ = \;& -\int_{a(s)}^{b(s)} \varphi(s,x)\,dx + \int_{\sigma}^{\tau} \varphi(t,c(t))\,(u(t) - c'(t))\,dt \\ & + \int_{c(\sigma)}^{b(\sigma)} \varphi(\sigma,x)\,dx + \int_{a(\tau)}^{c(\tau)} \varphi(\tau,x)\,dx \\ & + \int_{s}^{\sigma} \varphi(t,b(t))\,(u(t) - b'(t))\,dt + \int_{s}^{\tau} \varphi(t,a(t))\,(a'(t) - u(t))\,dt. \end{aligned}$$

Since $a' = b' = u$, the two last terms vanish and we have

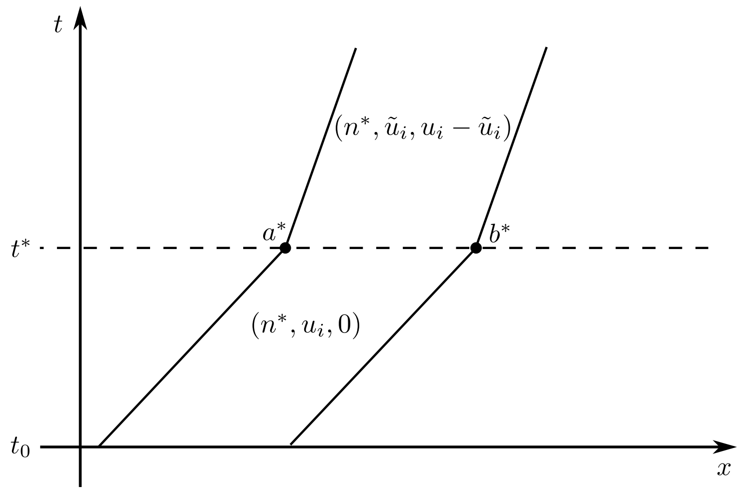
$$\begin{aligned} J(s,\sigma,a,b,u) + J(\sigma,\tau,a,c,u) \;=\; & -\int_{a(s)}^{b(s)} \varphi(s,x)\,dx \\ & + \int_{\sigma}^{\tau} \varphi(t,c(t))\,(u(t) - c'(t))\,dt \\ & + \int_{c(\sigma)}^{b(\sigma)} \varphi(\sigma,x)\,dx + \int_{a(\tau)}^{c(\tau)} \varphi(\tau,x)\,dx. \end{aligned}$$

The formulas (3.9) and (3.10) follow. □

3.1 About uniqueness of the dynamics

In order to work with the most realistic solution, it is necessary to impose a certain number of criteria on the dynamics in question. This discussion also improve the paper [7].

A single block for which $u + p$ stays constant is a solution, for example the function corresponding to the following figure:



Remark 3.4 *To understand the meaning of the dynamics, for every figure, the term (n, u, p) on a zone corresponds to the constant values of the functions on a block.*

In fact, in an open subset $\Omega \subset]0, +\infty[_t \times \mathbb{R}_x$, where n^* is constant, it is very easy to see that the dynamic displayed on figure satisfies (2.1)-(2.3), for any value of $0 \leq \tilde{u}_i \leq u_i$.

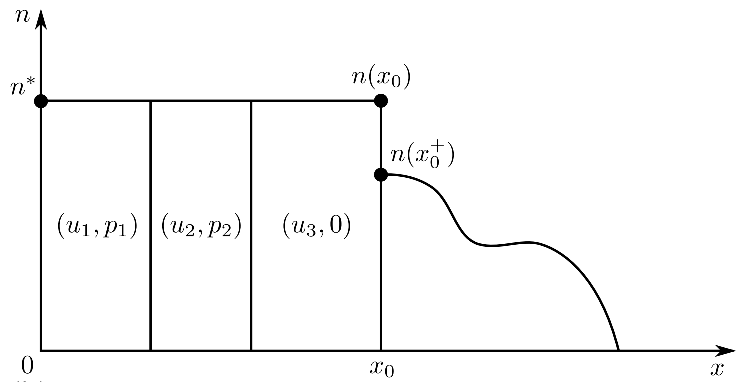
Now, remember that the term p represents the speed capability which is not used if the road is blocked and that the cars in front imposes a speed smaller than that desired. The term is 0 if the density is not $n^*(x)$ since in this case the car can go to its preferred velocity. Thus there is no reason for a single car to have a nonzero pressure term if there is no one before him. And the relation $(n^*(x) - n)p = 0$ do not impose $p = 0$ for the first car of the jam. This is why we assume that the blocks satisfy the additional constraint:

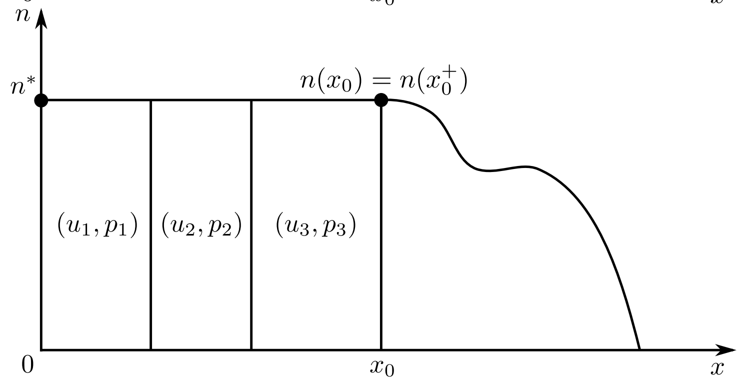
$$(n^*(x) - n(x^+))p = 0, \tag{3.11}$$

in zones where n^* is constant. In this property, we denote by $n(x^+)$ the limit, if it exists, of $n(y)$ when $y \to x$ with $y > x$. With this condition, the dynamics of the previous figure is a solution only if $\tilde{u}_i = u_i$.

The interpretation is the following : if the first car of the jam has the opportunity to use its preferred velocity, it uses it and p becomes zero. If not, p is not necessarily zero.

This is why for various blocks sticking one after the other, the constraint (3.11) on p gives the two situations of the above figures.

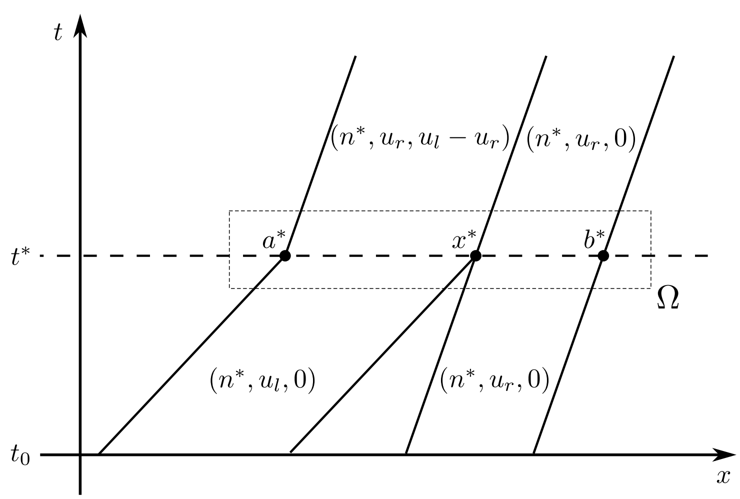




In fact, an other criteria than (3.11) related to the minimization of $p \geq 0$ can be used. It is clear than choosing $p = 0$ minimize the p term in the previous described situations. When the constraint (3.11) cannot be imposed, uniqueness criteria which is natural is the minimization of $p \geq 0$.
We now detail the various cases that can appear in the dynamic of clusters.

3.2 Collision between two blocks without change of width

In a zone where $n^*(x) = n^*$ is constant, we consider two blocks $(n^*, u_l, 0)$ and $(n^*, u_r, 0)$, with $u_l > u_r$. Thus, at a time $t^* > 0$, the left block reaches the right one, and collide with it. The dynamic is displayed in the following figure.



The density $n(t,x)$, the flux $n(t,x)u(t,x)$ and the function $p(t,x)$ are locally given respectively by

$$n(t,x) = \begin{cases} n^* \mathbb{1}_{a_l(t)<x<b_l(t)} + n^* \mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if} \quad t < t^*, \\ n^* \mathbb{1}_{\tilde{a}_l(t)<x<\tilde{b}_l(t)} + n^* \mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if} \quad t > t^*, \end{cases}$$

$$n(t,x)u(t,x) = \begin{cases} n^* u_l \mathbb{1}_{a_l(t)<x<b_l(t)} + n^* u_r \mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if} \quad t < t^*, \\ n^* u_r \mathbb{1}_{\tilde{a}_l(t)<x<\tilde{b}_l(t)} + n^* u_r \mathbb{1}_{a_r(t)<x<b_r(t)} & \text{if} \quad t > t^*, \end{cases}$$

and

$$n(t,x)p(t,x) = \begin{cases} 0 & \text{if} \quad t < t^*, \\ n^*(u_l - u_r) \mathbb{1}_{\tilde{a}_l(t)<x<\tilde{b}_l(t)} & \text{if} \quad t > t^*, \end{cases}$$

with the linear functions a_l, b_l, a_r, b_r, $\tilde{a}_l$, $\tilde{b}_l$ are given by

$$\frac{d}{dt} a_l(t) = \frac{d}{dt} b_l(t) = u_l, \quad a_l(t^*) = a^*, \quad b_l(t^*) = x^*,$$

$$\frac{d}{dt} a_r(t) = \frac{d}{dt} b_r(t) = u_r, \quad a_r(t^*) = x^*, \quad b_r(t^*) = b^*,$$

$$\frac{d}{dt} \tilde{a}_l(t) = \frac{d}{dt} \tilde{b}_l(t) = u_r, \quad \tilde{a}_l(t^*) = a^*, \quad \tilde{b}_l(t^*) = x^*,$$

and

$$u_l > u_r.$$

The left block obtains the velocity of the one being immediately on its right when they collide. We extend this when more than two blocks collide at a time t^*, by forming a new block with the velocity of the block on the right of the group.

Lemma 3.5 *The previous dynamic satisfies* (2.1)-(2.3).

Proof: Let Ω be an open neighborhood of the shock zone (displayed in the previous figure). Then, we have, for any continuous function S and any test function $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned} &\langle\ \partial_t(nS(u,p,I_\alpha)) + \partial_x(nuS(u,p,I_\alpha)), \varphi\ \rangle \\ &= \ -\int_0^{+\infty}\int_{\mathbb{R}} n(t,x)S(u(t,x),p(t,x),I_\alpha(x))(\partial_t\varphi + u\partial_x\varphi)dxdt \\ &= \ -n^*S(u_l,0,I_\alpha)J(0,t^*,a_l,b_l,u_l) \\ &\quad -n^*S(u_r,u_l-u_r,I_\alpha)J(t^*,\infty,\tilde{a}_l,\tilde{b}_l,u_r) \\ &\quad -n^*S(u_r,0,I_\alpha)J(0,\infty,a_r,b_r,u_r) \\ &= \ (-n^*S(u_l,0,I_\alpha) + n^*S(u_r,u_l-u_r,I_\alpha))\int_{a^*}^{x^*} \varphi(t^*,x)dx. \end{aligned}$$

For $S(u,p,I_\alpha)=1$, we get

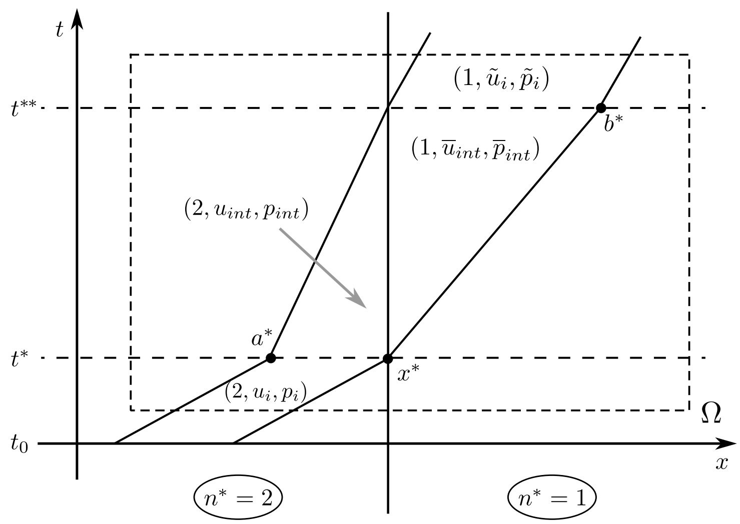
$$\langle\ \partial_t n+\partial_x(nu),\varphi\ \rangle=0.$$

For $S(u,p,I_\alpha)=(u+p)I_\alpha$, we get

$$\langle\ \partial_t(n(u+p)I_\alpha)+\partial_x(nu(u+p)I_\alpha),\varphi\ \rangle\ \ =\ \ 0.\ \square$$

3.3 Narrowing of the road without collision

Let us move to the situation where the road narrows ($n^*(x)$ was 2 and becomes 1). Here, we describe the evolution of a block which undergoes this narrowing. The speed will be divided by α.
The dynamic of the block is exhibited in the following figure.



The density $n(t,x)$, the flux $n(t,x)u(t,x)$ and the functional $p(t,x)$ are locally given respectively by

$$n(t,x)=\begin{cases}2\mathbb{1}_{a_i(t)<x<b_i(t)} & \text{if}\ \ t<t^*,\\ 2\mathbb{1}_{a_{int}(t)<x<x^*}+\mathbb{1}_{x^*<x<\bar{b}_{int}(t)} & \text{if}\ \ t^*<t<t^{**},\\ \mathbb{1}_{\tilde{a}_i(t)<x<\tilde{b}_i(t)} & \text{if}\ \ t>t^{**},\end{cases}$$

$$n(t,x)u(t,x)=\begin{cases}2u_i\mathbb{1}_{a_i(t)<x<b_i(t)} & \text{if}\ \ t<t^*,\\ 2u_{int}\mathbb{1}_{a_{int}(t)<x<x^*}+\overline{u}_{int}\mathbb{1}_{x^*<x<\bar{b}_{int}(t)} & \text{if}\ \ t^*<t<t^{**},\\ \tilde{u}_i\mathbb{1}_{\tilde{a}_i(t)<x<\tilde{b}_i(t)} & \text{if}\ \ t>t^{**},\end{cases}$$

and

$$n(t,x)p(t,x)=\begin{cases}2p_i\mathbb{1}_{a_i(t)<x<b_i(t)} & \text{if}\ \ t<t^*,\\ 2p_{int}\mathbb{1}_{a_{int}(t)<x<x^*}+\overline{p}_{int}\mathbb{1}_{x^*<x<\bar{b}_{int}(t)} & \text{if}\ \ t^*<t<t^{**},\\ \tilde{p}_i\mathbb{1}_{\tilde{a}_i(t)<x<\tilde{b}_i(t)} & \text{if}\ \ t>t^{**},\end{cases}$$

with

$$\frac{d}{dt}a_i(t) = \frac{d}{dt}b_i(t) = u_i, \quad a_i(t^*) = a^*, \quad b_i(t^*) = x^*,$$
$$\frac{d}{dt}a_{int}(t) = u_{int}, \quad a_{int}(t^*) = a^*, \quad a_{int}(t^{**}) = x^*,$$
$$\frac{d}{dt}\overline{b}_{int}(t) = \overline{u}_{int}, \quad \overline{b}_{int}(t^*) = x^*, \quad \overline{b}_{int}(t^{**}) = b^*,$$
$$\frac{d}{dt}\tilde{a}_i(t) = \frac{d}{dt}\tilde{b}_i(t) = \tilde{u}_i, \quad \tilde{a}_i(t^{**}) = x^*, \quad \tilde{b}_i(t^{**}) = b^*.$$

Lemma 3.6 *The previous dynamic satisfies* (2.1)-(2.3) *if and only if*

$$p_{int} = u_i + p_i - u_{int}, \quad (\overline{u}_{int}, \overline{p}_{int}) = \left(2u_{int}, \frac{u_i + p_i}{\alpha} - 2u_{int}\right), \quad \tilde{p}_i = \frac{u_i + p_i}{\alpha} - \tilde{u}_i,$$

with

$$0 \le u_{int} \le \frac{u_i + p_i}{2\alpha}, \quad 0 \le \tilde{u}_i \le \frac{u_i + p_i}{\alpha}.$$

Proof: We have, for any continuous function S and any function $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}
&\langle\ \partial_t(nS(u,p,I_\alpha) + \partial_x(nuS(u,p,I_\alpha)), \varphi\ \rangle \\
&= -\int_0^{+\infty}\int_{\mathbb{R}} n(t,x)S(u(t,x),p(t,x),I_\alpha(x))(\partial_t\varphi + u\partial_x\varphi)dxdt \\
&= -2S(u_i,p_i,1/\alpha)J(0,t^*,a_i,b_i,u_i) \\
&\quad -2S(u_{int},p_{int},1/\alpha)J(t^*,t^{**},a_{int},x^*,u_{int}) \\
&\quad -S(\overline{u}_{int},\overline{p}_{int},1)J(t^*,t^{**},x^*,\overline{b}_{int},\overline{u}_{int}) - S(\tilde{u}_i,\tilde{p}_i,1)J(t^{**},\infty,\tilde{a}_i,\tilde{b}_i,\tilde{u}_i) \\
&= -2S(u_i,p_i,1/\alpha)\int_{a^*}^{x^*}\varphi(t^*,x)dx \\
&\quad -2S(u_{int},p_{int},1/\alpha)\left(-\int_{a^*}^{x^*}\varphi(t^*,x)dx + u_{int}\int_{t^*}^{t^{**}}\varphi(t,x^*)dt\right) \\
&\quad -S(\overline{u}_{int},\overline{p}_{int},1)\left(\int_{x^*}^{b^*}\varphi(t^{**},x)dx - \overline{u}_{int}\int_{t^*}^{t^{**}}\varphi(t,x^*)dt\right) \\
&\quad +S(\tilde{u}_i,\tilde{p}_i,1)\int_{x^*}^{b^*}\varphi(t^{**},x)dx,
\end{aligned}$$

thus we have

$$\begin{aligned}
&\partial_t(nS(u,p,I_\alpha) + \partial_x(nuS(u,p,I_\alpha)) \\
&= 2\left(S(u_{int},p_{int},1/\alpha) - S(u_i,p_i,1/\alpha)\right)\mathbb{1}_{[a^*,x^*]}(x)\delta(t-t^*) \\
&\quad + \left(S(\tilde{u}_i,\tilde{p}_i,1) - S(\overline{u}_{int},\overline{p}_{int},1)\right)\mathbb{1}_{[x^*,b^*]}(x)\delta(t-t^{**}) \\
&\quad + \left(\overline{u}_{int}S(\overline{u}_{int},\overline{p}_{int},1) - 2u_{int}S(u_{int},p_{int},1/\alpha)\right)\mathbb{1}_{[t^*,t^{**}]}(t)\delta(x-x^*).
\end{aligned}$$

For $S(u,p,I_\alpha)=1$, we get

$$\partial_t n + \partial_x(nu) = (\overline{u}_{int} - 2u_{int})\, \mathbb{1}_{[t^*,t^{**}]}(t)\delta(x-x^*).$$

For $S(u,p,I_\alpha)=(u+p)I_\alpha$, we get

$$\begin{aligned}
&\partial_t(n(u+p)I_\alpha) + \partial_x(nu(u+p)I_\alpha)\\
&\quad= \tfrac{2}{\alpha}\,(u_{int}+p_{int}-u_i-p_i)\,\mathbb{1}_{[a^*,x^*]}(x)\delta(t-t^*)\\
&\qquad + (\tilde{u}_i+\tilde{p}_i-\overline{u}_{int}-\overline{p}_{int})\,\mathbb{1}_{[x^*,b^*]}(x)\delta(t-t^{**})\\
&\qquad + \left(\overline{u}_{int}(\overline{u}_{int}+\overline{p}_{int}) - 2u_{int}\left(\tfrac{u_{int}+p_{int}}{\alpha}\right)\right)\mathbb{1}_{[t^*,t^{**}]}(t)\delta(x-x^*).
\end{aligned}$$

Therefore, (n,u,p) is a solution of (2.1)-(2.2) if and only if

$$\left\{\begin{array}{lcl}
\overline{u}_{int} & = & 2u_{int}\\
u_{int}+p_{int} & = & u_i+p_i\\
\tilde{u}_i+\tilde{p}_i & = & \overline{u}_{int}+\overline{p}_{int}\\
\overline{u}_{int}+\overline{p}_{int} & = & \frac{1}{\alpha}(u_{int}+p_{int})
\end{array}\right. \iff \left\{\begin{array}{lcl}
\overline{u}_{int} & = & 2u_{int}\\
u_{int}+p_{int} & = & u_i+p_i\\
\overline{u}_{int}+\overline{p}_{int} & = & \frac{1}{\alpha}(u_i+p_i)\\
\tilde{u}_i+\tilde{p}_i & = & \frac{1}{\alpha}(u_i+p_i)
\end{array}\right. .$$

Since $u_{int}, p_{int}, \overline{u}_{int}$ and $\overline{p}_{int}$ are nonnegative, it concludes the proof of lemma. □

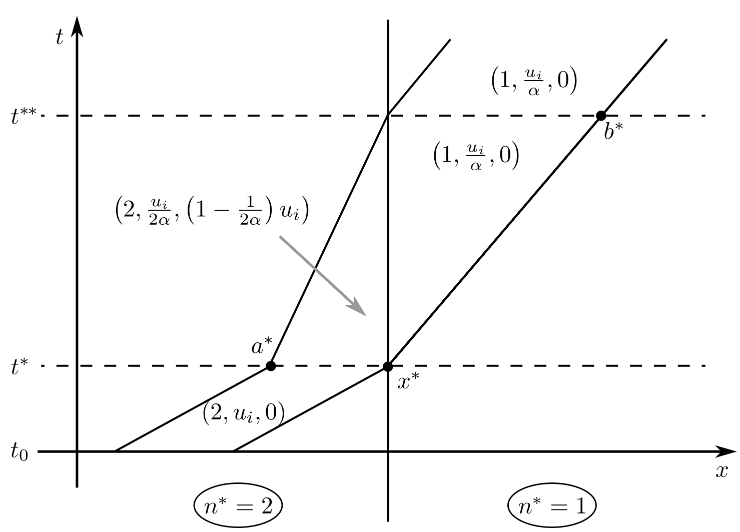
Now, we can find the dynamics governing a single block $(n^*, u_i, 0)$ which undergoes a narrowing of the road: according to subsection 3.1 and additional constraint (3.11), we have

$$p_i = 0, \qquad \tilde{p}_i = 0, \qquad \overline{p}_{int} = 0,$$

in the relations of lemma 3.6, which leads to

$$p_{int} = u_i - u_{int}, \quad \overline{u}_{int} = 2u_{int}, \quad u_{int} = \frac{u_i}{2\alpha}, \quad \tilde{u}_i = \frac{u_i}{\alpha}.$$

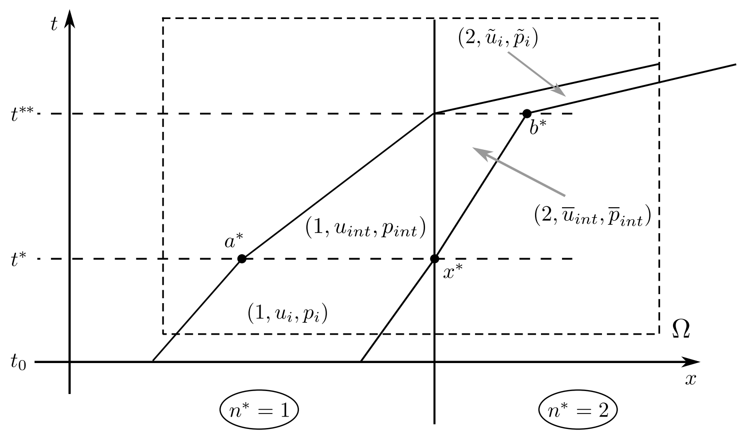
Finally, the only dynamics compatible with (3.11) for a narrowing is:



3.4 Enlargement of the road without collision

Now we explain what happens for a block when the road widens ($n^*(x)$ was 1 and becomes 2).

In fact, the block (which comes with $n = n^* = 1$) becomes a block with $n = n^* = 2$, but its speed will be multiplied by the parameter α. The dynamic is exhibited hereafter:



The density $n(t,x)$, the flux $n(t,x)u(t,x)$ and the functional $p(t,x)$ are locally given respectively by

$$n(t,x) = \begin{cases} \mathbb{1}_{a_i(t)<x<b_i(t)} & \text{if } \ t < t^*, \\ \mathbb{1}_{a_{int}(t)<x<x^*} + 2\mathbb{1}_{x^*<x<\overline{b}_{int}(t)} & \text{if } \ t^* < t < t^{**}, \\ 2\mathbb{1}_{\tilde{a}_i(t)<x<\tilde{b}_i(t)} & \text{if } \ t > t^{**}, \end{cases}$$

$$n(t,x)u(t,x) = \begin{cases} u_i\mathbb{1}_{a_i(t)<x<b_i(t)} & \text{if } \ t < t^*, \\ u_{int}\mathbb{1}_{a_{int}(t)<x<x^*} + 2\overline{u}_{int}\mathbb{1}_{x^*<x<\overline{b}_{int}(t)} & \text{if } \ t^* < t < t^{**}, \\ 2\tilde{u}_i\mathbb{1}_{\tilde{a}_i(t)<x<\tilde{b}_i(t)} & \text{if } \ t > t^{**}, \end{cases}$$

and

$$n(t,x)p(t,x) = \begin{cases} p_i\mathbb{1}_{a_i(t)<x<b_i(t)} & \text{if } \ t < t^*, \\ p_{int}\mathbb{1}_{a_{int}(t)<x<x^*} + 2\overline{p}_{int}\mathbb{1}_{x^*<x<\overline{b}_{int}(t)} & \text{if } \ t^* < t < t^{**}, \\ 2\tilde{p}_i\mathbb{1}_{\tilde{a}_i(t)<x<\tilde{b}_i(t)} & \text{if } \ t > t^{**}, \end{cases}$$

with

$$\frac{d}{dt}a_i(t) = \frac{d}{dt}b_i(t) = u_i, \quad a_i(t^*) = a^*, \quad b_i(t^*) = x^*,$$
$$\frac{d}{dt}a_{int}(t) = u_{int}, \quad a_{int}(t^*) = a^*, \quad a_{int}(t^{**}) = x^*,$$
$$\frac{d}{dt}\overline{b}_{int}(t) = \overline{u}_{int}, \quad \overline{b}_{int}(t^*) = x^*, \quad \overline{b}_{int}(t^{**}) = b^*,$$
$$\frac{d}{dt}\tilde{a}_i(t) = \frac{d}{dt}\tilde{b}_i(t) = \tilde{u}_i, \quad \tilde{a}_i(t^{**}) = x^*, \quad \tilde{b}_i(t^{**}) = b^*.$$

Lemma 3.7 *The previous dynamic satisfies* (2.1)-(2.3) *if and only if*

$$p_{int} = u_i + p_i - u_{int}, \quad (\overline{u}_{int}, \overline{p}_{int}) = \left(\frac{u_{int}}{2}, \alpha(u_i + p_i) - \frac{u_{int}}{2}\right), \quad \tilde{p}_i = \alpha(u_i + p_i) - \tilde{u}_i,$$

with

$$0 \leq u_{int} \leq u_i + p_i, \quad 0 \leq \tilde{u}_i \leq \alpha(u_i + p_i).$$

Proof: We have, for any continuous function S and any function $\varphi \in \mathcal{D}(\Omega)$,

$$\begin{aligned}
&\langle\ \partial_t(nS(u,p,I_\alpha) + \partial_x(nuS(u,p,I_\alpha)), \varphi\ \rangle \\
&\quad = -\int_0^{+\infty}\int_{\mathbb{R}} n(t,x)S(u(t,x),p(t,x),I_\alpha(x))(\partial_t\varphi + u\partial_x\varphi)dxdt \\
&\quad = -S(u_i,p_i,1)J(0,t^*,a_i,b_i,u_i) - S(u_{int},p_{int},1)J(t^*,t^{**},a_{int},x^*,u_{int}) \\
&\qquad -2S(\overline{u}_{int},\overline{p}_{int},1/\alpha)J(t^*,t^{**},x^*,\overline{b}_{int},\overline{u}_{int}) \\
&\qquad +2S(\tilde{u}_i,\tilde{p}_i,1/\alpha)J(t^{**},\infty,\tilde{a}_i,\tilde{b}_i,\tilde{u}_i) \\
&\quad = -S(u_i,p_i,1)\int_{a^*}^{x^*}\varphi(t^*,x)dx \\
&\qquad -S(u_{int},p_{int},1)\left(-\int_{a^*}^{x^*}\varphi(t^*,x)dx + u_{int}\int_{t^*}^{t^{**}}\varphi(t,x^*)dt\right) \\
&\qquad -2S(\overline{u}_{int},\overline{p}_{int},1/\alpha)\left(\int_{x^*}^{b^*}\varphi(t^{**},x)dx - \overline{u}_{int}\int_{t^*}^{t^{**}}\varphi(t,x^*)dt\right) \\
&\qquad +2S(\tilde{u}_i,\tilde{p}_i,1/\alpha)\int_{x^*}^{b^*}\varphi(t^{**},x)dx,
\end{aligned}$$

thus we have, in $\mathcal{D}'(\Omega)$,

$$\begin{aligned}
&\partial_t(nS(u,p,I_\alpha) + \partial_x(nuS(u,p,I_\alpha)) \\
&\quad = (S(u_{int},p_{int},1) - S(u_i,p_i,1))\,\mathbb{1}_{[a^*,x^*]}(x)\delta(t-t^*) \\
&\qquad +2\,(S(\tilde{u}_i,\tilde{p}_i,1/\alpha) - S(\overline{u}_{int},\overline{p}_{int},1/\alpha))\,\mathbb{1}_{[x^*,b^*]}(x)\delta(t-t^{**}) \\
&\qquad +(2\overline{u}_{int}S(\overline{u}_{int},\overline{p}_{int},1/\alpha) - u_{int}S(u_{int},p_{int},1))\,\mathbb{1}_{[t^*,t^{**}]}(t)\delta(x-x^*).
\end{aligned}$$

For $S(u,p,I_\alpha) = 1$, we get

$$\partial_t n + \partial_x(nu) = (2\overline{u}_{int} - u_{int})\,\mathbb{1}_{[t^*,t^{**}]}(t)\delta(x-x^*).$$

For $S(u,p,I_\alpha) = (u+p)I_\alpha$, we get

$$\begin{aligned}
&\partial_t(n(u+p)I_\alpha) + \partial_x(nu(u+p)I_\alpha) \\
&\quad = (u_{int} + p_{int} - u_i - p_i)\,\mathbb{1}_{[a^*,x^*]}(x)\delta(t-t^*) \\
&\qquad +\tfrac{2}{\alpha}\,(\tilde{u}_i + \tilde{p}_i - \overline{u}_{int} - \overline{p}_{int})\,\mathbb{1}_{[x^*,b^*]}(x)\delta(t-t^{**}) \\
&\qquad +\left(2\overline{u}_{int}\left(\tfrac{\overline{u}_{int}+\overline{p}_{int}}{\alpha}\right) - u_{int}(u_{int}+p_{int})\right)\mathbb{1}_{[t^*,t^{**}]}(t)\delta(x-x^*).
\end{aligned}$$

Therefore, such a function (n, u, p) is a solution of (2.1)-(2.2) if and only if

$$\begin{cases} 2\overline{u}_{int} & = & u_{int} \\ u_{int} + p_{int} & = & u_i + p_i \\ \tilde{u}_i + \tilde{p}_i & = & \overline{u}_{int} + \overline{p}_{int} \\ \overline{u}_{int} + \overline{p}_{int} & = & \alpha(u_{int} + p_{int}) \end{cases} \iff \begin{cases} 2\overline{u}_{int} & = & u_{int} \\ u_{int} + p_{int} & = & u_i + p_i \\ \overline{u}_{int} + \overline{p}_{int} & = & \alpha(u_i + p_i) \\ \tilde{u}_i + \tilde{p}_i & = & \alpha(u_i + p_i) \end{cases},$$

and we conclude as in lemma 3.6. □

Now, if a single block $(n^*, u_i, 0)$ undergoes a enlargement of the road, we have

$$p_i = 0, \qquad \tilde{p}_i = 0$$

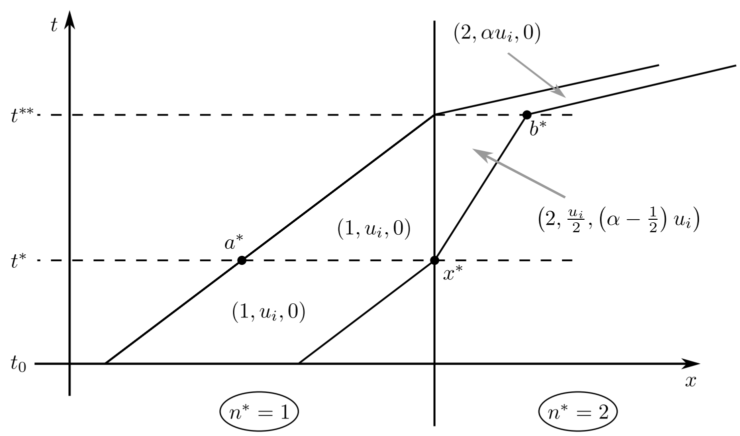
in the relations of lemma 3.7, which leads to

$$p_{int} = u_i - u_{int}, \quad (\overline{u}_{int}, \overline{p}_{int}) = \left(\frac{u_{int}}{2}, \alpha u_i - \frac{u_{int}}{2}\right), \quad \tilde{u}_i = \alpha u_i,$$

with

$$0 \leq u_{int} \leq u_i.$$

In this case, we can't impose $\overline{p}_{int} = 0$ since it would imply $u_{int} = 2\alpha u_i$ and then $p_{int} < 0$ which is impossible. Then, we use the second criteria of section 3.1 which is the minimization of p_{int} in this case. Here $p_{int} = 0$ is possible, that is the choice $u_{int} = u_i$, and the dynamics for a enlargement is the following:



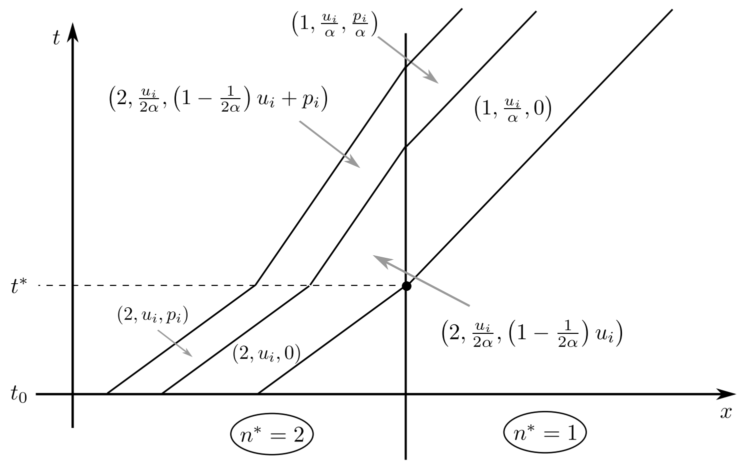
This situation is the only case where (3.11) can't be additionally asked. The physical explanation is that the increasing of the speed of the car going from u_i to αu_i is not instantaneous and has to be in two steps. Thus, in the intermediate state, the car is not yet at its preferred velocity and there is still a p term.

Remark 3.8 *We notice that the dynamics for enlargement is exactly the reverse process of the narrowing.*

3.5 Compatibility of the dynamics

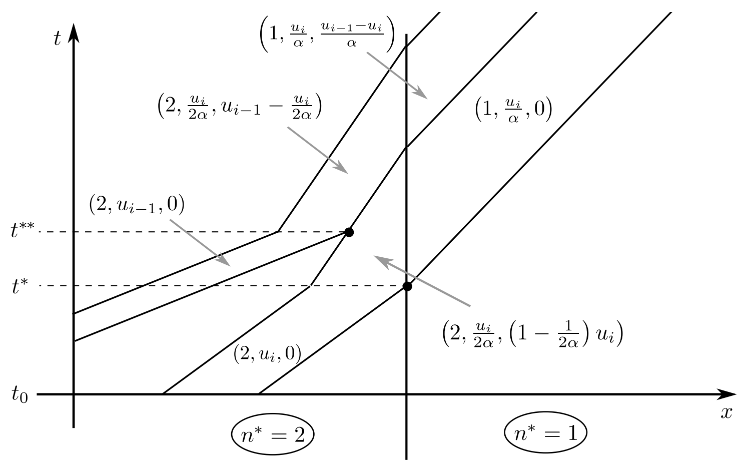
Since the previous dynamics are not instantaneous, they can interact before they are completed. In this subsection, we present the various compatibilities between these dynamics. Note that it is not just a superposition of various cases. In order to simplify the presentation, we only show figures that describe the various interactions.

3.5.1 A train of blocks undergoes an narrowing

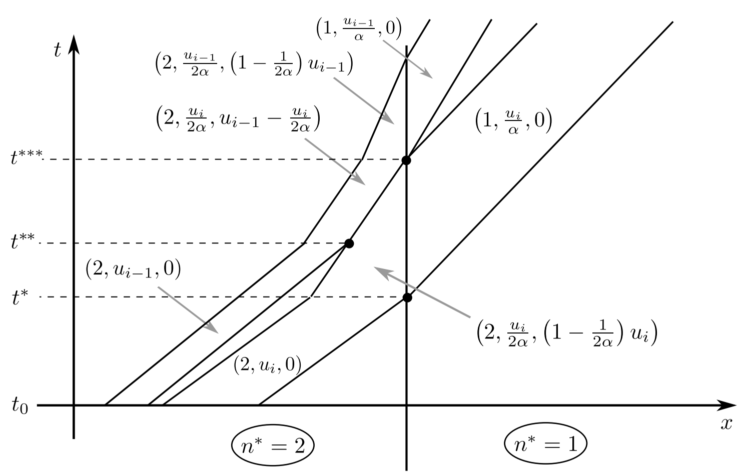


3.5.2 Two blocks collide just before the road narrows

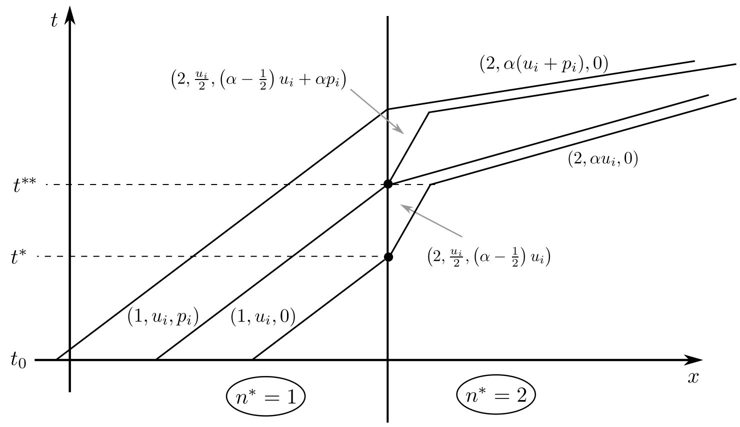
Case with $u_{i-1} > u_i$:



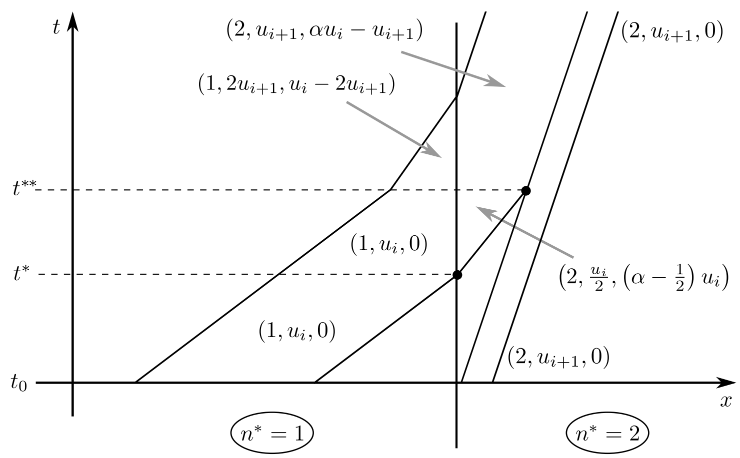
Case with $\frac{u_i}{2\alpha} < u_{i-1} \leq u_i$:



3.5.3 A train of blocks undergoes an enlargement



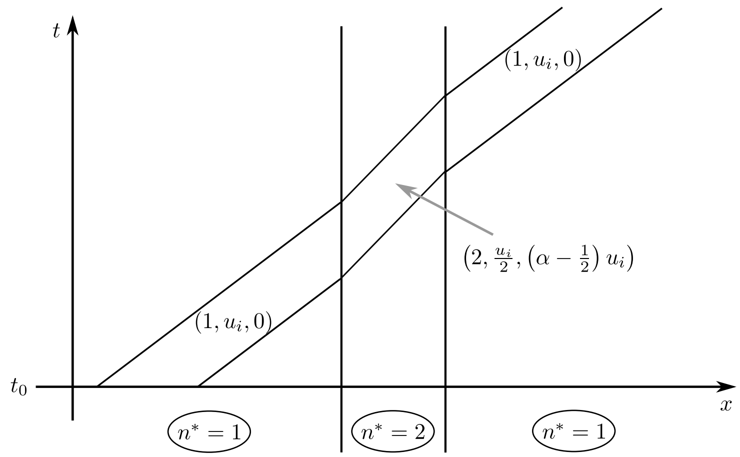
3.5.4 Two blocks collide just after the road widens



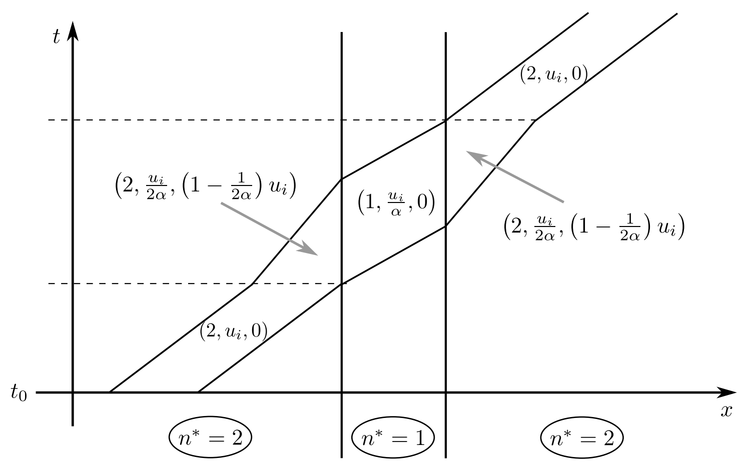
Here, we have $\frac{u_i}{2} > u_{i+1}$, thus

$$\alpha u_i - u_{i+1} \geq u_i - 2u_{i+1} > 0.$$

3.5.5 The road follows $1 \to 2 \to 1$ faster than the block



3.5.6 The road follows $2 \to 1 \to 2$ faster than the block



3.6 Block solutions and bounds

Using the above sections, we are able to state some results on the dynamics of blocks.

Remark 3.9 *The velocity u is assumed to be extended linearly in the vacuum (areas such that $n = 0$) between two successive blocks. Moreover, we assume that u is constant at $\pm\infty$. But concerning p, the constraint $(n^* - n)p = 0$ implies that $p = 0$ in the vacuum, and at $\pm\infty$. Thus, the computations of total variation in x of u and p are different.*

The previous computations show the following results:

Theorem 3.10 *With the various above dynamics, the quantities $n(t, x)$, $u(t, x)$ and $p(t, x)$ defined by (3.1)-(3.3) and Remark 3.9 are solutions to* (2.1), (2.2), (2.3).

We can also establish some bounds on these solutions:

Proposition 3.11 *We still denote by $n(t,x)$, $u(t,x)$ and $p(t,x)$ the functions of (3.1)-(3.3) and Remark 3.9. These functions satisfy the maximum principle*

$$0 \leq u(t,x) \leq 2\alpha \left(esssup_y u^0(y) + esssup_y p^0(y)\right), \tag{3.12}$$

$$0 \leq p(t,x) \leq 2\alpha \left(esssup_y u^0(y) + esssup_y p^0(y)\right). \tag{3.13}$$

If we assume furthermore that the initial data in the blocks u_i^0 and p_i^0 are BV functions, then we have, for all $t \in [0,T]$,

$$TV_K(u(t,.)) \leq 4\alpha M \left(TV_{\tilde{K}}(u^0) + TV_{\tilde{K}}(p^0) + \|u^0\|_{L^\infty}\right), \tag{3.14}$$

$$TV_K(p(t,.)) \leq 4\alpha M \left(TV_{\tilde{K}}(u^0) + TV_{\tilde{K}}(p^0) + \|u^0\|_{L^\infty}\right), \tag{3.15}$$

for any compact $K = [a,b]$ and with

$$\tilde{K} = [a - t\,(esssup_y u^0), b - t\,(essinf_y u^0)],$$

where TV_K (resp. $TV_{\tilde{K}}$) denotes the total variation on the set K (resp. $\tilde{K}$), and M is the number of road transitions (supposed to be finite).

Proof: We treat some examples which represent the critical cases. In these cases, we compute the total variation on $\mathbb{R}$ to simplify the presentation.

- Case of collisions without change of width:

  We obtain the bounds corresponding to the classical CPGD model (like in [8]).

  We assume the following dynamics: at time $t = 0$, there are N blocks (denoted by $B_1, \ldots, B_N$) with velocities $u_1^0 > u_2^0 > \cdots > u_N^0$ (which is the case with the most collisions) and pressures $p_1^0, \ldots, p_N^0 \geq 0$, thus

$$TV(u^0) = \sum_{i=1}^{N-1} |u_i^0 - u_{i+1}^0|, \qquad TV(p^0) = 2\sum_{i=1}^{N} p_i^0.$$

  Let $t > 0$ such that in the time interval $[0,t]$, the j first blocks $B_1, \ldots, B_j$ collide successively at $t_1 < \cdots < t_{j-1} \leq t$ for instance (i.e. B_i collide with B_{i+1} at the time t_i for all $1 \leq i \leq j-1$) and then the $q-j+1$ following blocks $B_j, \ldots, B_q$ collide at the same time t_j, with $t_{j-1} < t_j \leq t$. At the time t, the last $N-q+1$ blocks $B_q, \ldots, B_N$ have not collided yet.

  We have the relations

$$\forall k \in \{1, \ldots, j\}, \quad \forall i \in \{1, \ldots, N\}, \qquad p_i^{t_k} + u_i^{t_k} = p_i^0 + u_i^0,$$
$$\forall i \in \{1, \ldots, q\}, \qquad u_i^{t_j} = u_q^0, \qquad p_i^{t_j} = u_i^0 + p_i^0 - u_q^0,$$
$$\forall i \in \{q+1, \ldots, N\}, \qquad u_i^{t_j} = u_i^0, \qquad p_i^{t_j} = p_i^0.$$

  Then we get

$$
\begin{aligned}
TV(u(t,.)) &= |u_1^{t_j} - u_2^{t_j}| + \cdots + |u_{j-1}^{t_j} - u_j^{t_j}| \\
&+ \quad |u_j^{t_j} - u_{j+1}^{t_j}| + \cdots + \quad |u_{q-1}^{t_j} - u_q^{t_j}| \\
&+ \quad |u_q^{t_j} - u_{q+1}^{t_j}| + \cdots + |u_{N-1}^{t_j} - u_N^{t_j}| \\
&= \quad |u_q^0 - u_q^0| + \cdots + |u_q^0 - u_q^0| \\
&+ \quad |u_q^0 - u_{q+1}^0| + \cdots + |u_{N-1}^0 - u_N^0| \\
&\leq TV(u^0).
\end{aligned}
$$

Now, for p, we have

$$
\begin{aligned}
TV(p(t,.)) &= p_1^{t_j} + |p_1^{t_j} - p_2^{t_j}| + \cdots + |p_{j-1}^{t_j} - p_j^{t_j}| \\
&+ \quad |p_j^{t_j} - p_{j+1}^{t_j}| + \cdots + |p_{q-1}^{t_j} - p_q^{t_j}| \\
&+ \quad p_q^{t_j} \quad + \quad 2\left(p_{q+1}^{t_j} + \cdots + p_N^{t_j}\right) \\
&= u_1^0 + p_1^0 - u_q^0 + |u_1^0 + p_1^0 - u_2^0 - p_2^0| \\
&+ \cdots + |u_{j-1}^0 + p_{j-1}^0 - u_j^0 - p_j^0| \\
&+ \quad |u_j^0 + p_j^0 - u_{j+1}^0 - p_{j+1}^0| \\
&+ \cdots + |u_{q-1}^0 + p_{q-1}^0 - u_q^0 - p_q^0| \\
&+ \quad p_q^0 \quad + \quad 2\left(p_{q+1}^0 + \cdots + p_N^0\right) \\
&\leq u_1^0 - u_q^0 + |u_1^0 - u_2^0| + \cdots + |u_{q-1}^0 - u_q^0| \\
&+ \quad 2\left(p_1^0 + p_2^0 + \cdots + p_N^0\right) \\
&\leq 2TV(u^0) + TV(p^0).
\end{aligned}
$$

- Case of enlargement of the road without collision:

  We assume the following dynamics: at time $t = 0$, we consider two blocks $B_1 = (u_1^0, p_1^0)$ and $B_2 = (u_2^0, p_2^0)$, in a section of road where $n^* = 1$.

  We have
  $$TV(u^0) = |u_2^0 - u_1^0|, \qquad TV(p^0) = 2(p_2^0 + p_1^0).$$

  At time $t_1 > 0$, the block B_1 reach the two-lane section, and undergoes the change of width during the time interval $[t_1, t_2]$. Then, later in the interval $[t_3, t_4]$ (with $t_3 > t_2$) the block B_2 enter in the two-lane section.

For all $t \in]t_1, t_2[$, we have (with the notations of section 3.4)

$$\begin{aligned} TV(u(t,.)) &= |u_2^0 - u_{1,int}| + |u_{1,int} - \overline{u_{1,int}}| \\ &= |u_2^0 - u_{1,int}| + \frac{u_{1,int}}{2} \\ &\leq |u_2^0 - u_1^0| + |u_1^0 - u_{1,int}| + \frac{u_{1,int}}{2}. \end{aligned}$$

Since $u_{1,int} \leq u_1^0$ we obtain

$$TV(u(t,.)) \leq |u_2^0 - u_1^0| + u_1^0 - \frac{u_{1,int}}{2} \leq TV(u^0) + \|u^0\|_{L^\infty}.$$

Moreover

$$TV(p(t,.)) = 2p_2^0 + p_{1,int} + |p_{1,int} - \overline{p_{1,int}}| + \overline{p_{1,int}},$$

but

$$\overline{p_{1,int}} - p_{1,int} = (\alpha - 1)(u_1^0 + p_1^0) + \frac{u_{1,int}}{2} \geq 0,$$

thus

$$TV(p(t,.)) = 2(p_2^0 + \overline{p_{1,int}}) = 2(p_2^0 + \alpha(u_1^0 + p_1^0) - \frac{u_{1,int}}{2}),$$

and we deduce

$$TV(p(t,.)) \leq 2\alpha(p_1^0 + p_2^0) + 2\alpha u_1^0 \leq \alpha TV(p^0) + 2\alpha\|u^0\|_{L^\infty}.$$

For all $t \in]t_2, t_3[$, we have

$$\begin{aligned} TV(u(t,.)) = |u_2^0 - \alpha u_1^0| &\leq |u_2^0 - u_1^0| + |(1-\alpha)u_1^0| \\ &\leq TV(u^0) + (\alpha - 1)\|u^0\|_{L^\infty}, \end{aligned}$$

and

$$TV(p(t,.)) = 2p_2^0 + 2\alpha p_1^0 \leq \alpha TV(p^0).$$

For all $t \in]t_3, t_4[$, we have

$$\begin{aligned} TV(u(t,.)) &= |u_{2,int} - \overline{u_{2,int}}| + |\overline{u_{2,int}} - \alpha u_1^0| \\ &= \frac{u_{2,int}}{2} + |\frac{u_{2,int}}{2} - \alpha u_1^0| \\ &\leq \frac{u_{2,int}}{2} + |\frac{u_{2,int}}{2} - \alpha u_2^0| + |\alpha u_2^0 - \alpha u_1^0|. \end{aligned}$$

Since $u_{2,int} \leq u_2^0$, we obtain

$$
\begin{aligned}
TV(u(t,.)) &\leq \frac{u_{2,int}}{2} - \frac{u_{2,int}}{2} + \alpha u_2^0 + |\alpha u_2^0 - \alpha u_1^0| \\
&\leq \alpha TV(u^0) + \alpha \|u^0\|_{L^\infty}.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
TV(p(t,.)) &= p_{2,int} + |p_{2,int} - \overline{p_{2,int}}| + \overline{p_{2,int}} + 2\alpha p_1^0 \\
&= 2(\overline{p_{2,int}} + \alpha p_1^0) \\
&= 2(\alpha(u_2^0 + p_2^0) - \frac{u_{2,int}}{2} + \alpha p_1^0),
\end{aligned}
$$

and we deduce

$$
TV(p(t,.)) \leq 2\alpha(p_1^0 + p_2^0) + 2\alpha u_2^0 \leq \alpha TV(p^0) + 2\alpha \|u^0\|_{L^\infty}.
$$

At least, for $t > t_4$, we have

$$
TV(u(t,.)) = |\alpha u_2^0 - \alpha u_1^0| = \alpha TV(u^0),
$$

and

$$
TV(p(t,.)) = 2\alpha p_2^0 + 2\alpha p_1^0 = \alpha TV(p^0).
$$

Finally, the bound is

$$
TV(u(t,.)) \leq \alpha(TV(u^0) + \|u^0\|_{L^\infty}),
$$

$$
TV(p(t,.)) \leq \alpha(TV(u^0) + 2\|u^0\|_{L^\infty}),
$$

for all $t > 0$.

In the general case (if we follow N blocks along the time), we shall obtain the same bound because only one block at a time undergoes every change $n^* = 1 \to 2$.

But it is possible that many blocks undergo this enlargement together at different places. That is why the general estimate is the following:

$$
\begin{aligned}
TV(u(t,.)) &\leq \alpha(TV(u^0) + M\|u^0\|_{L^\infty}), \\
TV(p(t,.)) &\leq \alpha(TV(p^0) + 2M\|u^0\|_{L^\infty}),
\end{aligned}
$$

where M is the number of lane transitions.

- Case of narrowing of the road without collision:

  The computations are similar to the previous case. With the notations of section 3.3, we have:

For all $t \in]t_1, t_2[$,

$$TV(u(t,.)) = |u_2^0 - u_{1,int}| + |u_{1,int} - \overline{u_{1,int}}|$$
$$= |u_2^0 - u_{1,int}| + u_{1,int}$$
$$\leq |u_2^0 - u_1^0| + |u_1^0 - u_{1,int}| + u_{1,int}.$$

Since $u_{1,int} \leq u_1^0$ we obtain

$$TV(u(t,.)) \leq |u_2^0 - u_1^0| + u_1^0 \leq TV(u^0) + \|u^0\|_{L^\infty}.$$

Moreover,

$$TV(p(t,.)) = 2p_2^0 + p_{1,int} + |p_{1,int} - \overline{p_{1,int}}| + \overline{p_{1,int}},$$

but this time, $\overline{p_{1,int}} \leq p_{1,int}$, thus

$$TV(p(t,.)) = 2(p_2^0 + p_{1,int}) = 2(p_2^0 + u_1^0 + p_1^0 - u_{1,int}),$$

and we deduce

$$TV(p(t,.)) \leq 2(p_1^0 + p_2^0) + 2u_1^0 \leq TV(p^0) + 2\|u^0\|_{L^\infty}.$$

For all $t \in]t_2, t_3[$, we have

$$TV(u(t,.)) = |u_2^0 - \frac{1}{\alpha}u_1^0| \leq |u_2^0 - u_1^0| + |(1 - \frac{1}{\alpha})u_1^0|$$
$$\leq TV(u^0) + (1 - \frac{1}{\alpha})\|u^0\|_{L^\infty},$$

and

$$TV(p(t,.)) = 2p_2^0 + \frac{2}{\alpha}p_1^0 \leq TV(p^0).$$

For all $t \in]t_3, t_4[$, we have

$$TV(u(t,.)) = |u_{2,int} - \overline{u_{2,int}}| + |\overline{u_{2,int}} - \frac{1}{\alpha}u_1^0|$$
$$= u_{2,int} + |2u_{2,int} - \frac{1}{\alpha}u_1^0|$$
$$\leq u_{2,int} + |2u_{2,int} - \frac{1}{\alpha}u_2^0| + |\frac{1}{\alpha}u_2^0 - \frac{1}{\alpha}u_1^0|.$$

Since $u_{2,int} \leq u_2^0$, we obtain

$$
\begin{aligned}
TV(u(t,.)) &\leq u_{2,int} + 2u_2^0 - 2u_{2,int} + (2-\frac{1}{\alpha})u_2^0 + \frac{1}{\alpha}|u_2^0 - u_1^0| \\
&\leq 4u_2^0 + \frac{1}{\alpha}TV(u^0) \\
&\leq 4\|u^0\|_{L^\infty} + \frac{1}{\alpha}TV(u^0).
\end{aligned}
$$

Moreover

$$
\begin{aligned}
TV(p(t,.)) &= p_{2,int} + |p_{2,int} - \overline{p_{2,int}}| + \overline{p_{2,int}} + \frac{2}{\alpha}p_1^0 \\
&= 2(p_{2,int} + \frac{1}{\alpha}p_1^0) \\
&= 2(u_2^0 + p_2^0 - u_{2,int} + \frac{1}{\alpha}p_1^0),
\end{aligned}
$$

and we deduce

$$
TV(p(t,.)) \leq 2(p_1^0 + p_2^0) + 2u_2^0 \leq TV(p^0) + 2\|u^0\|_{L^\infty}.
$$

At least, for $t > t_4$, we have

$$
TV(u(t,.)) = |\frac{1}{\alpha}u_2^0 - \frac{1}{\alpha}u_1^0| = \frac{1}{\alpha}TV(u^0),
$$

and

$$
TV(p(t,.)) = \frac{2}{\alpha}p_2^0 + \frac{2}{\alpha}p_1^0 = \frac{1}{\alpha}TV(p^0).
$$

Finally, the bound is

$$
TV(u(t,.)) \leq TV(u^0) + 4\|u^0\|_{L^\infty},
$$

$$
TV(p(t,.)) \leq TV(u^0) + 2\|u^0\|_{L^\infty},
$$

for all $t > 0$.

Now $\|u^0\|_{L^\infty}$ can appear on every lane transition and the estimate is then

$$
\begin{aligned}
TV(u(t,.)) &\leq TV(u^0) + 4M\|u^0\|_{L^\infty}, \\
TV(p(t,.)) &\leq TV(p^0) + 2M\|u^0\|_{L^\infty},
\end{aligned}
$$

where M is the number of lane transitions.

- The general situation is a superposition of these cases and it gives the Proposition. □

4 Existence of weak solutions

In this section, we prove the existence of weak solutions using previous clusters dynamics, an approximation lemma of the initial data by these sticky blocks and a compactness result.

4.1 Approximation of the initial data by sticky blocks

We first start by proving the following approximation lemma of initial data.

Lemma 4.1 *Let $n^0 \in L^1(\mathbb{R})$, $u^0, p^0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ such that $0 \leq n^0 \leq n^*(x)$, $0 \leq u^0$, $0 \leq p^0$ and $(n^*(x) - n^0)p^0 = 0$. Then, there exists a sequence of block initial data $(n_k^0, u_k^0, p_k^0)_{k\geq 1}$ such that for all $k \in \mathbb{N}^*$,*

$$\int_{\mathbb{R}} n_k^0(x)dx \leq \int_{\mathbb{R}} n^0(x)dx, \tag{4.1}$$

$$essinf\ u^0 \leq u_k^0 \leq esssup\ u^0, \quad essinf\ p^0 \leq p_k^0 \leq esssup\ p^0, \tag{4.2}$$

$$TV(u_k^0) \leq TV(u^0), \quad TV(p_k^0) \leq TV(p^0), \tag{4.3}$$

and for which the convergences $n_k^0 \rightharpoonup n^0$, $n_k^0 u_k^0 \rightharpoonup n^0 u^0$ and $n_k^0 p_k^0 \rightharpoonup n^0 p^0$ hold in the distribution sense.

Proof: The proof is widely inspired from the ones in [4] and [8], but here, n^* is piecewise constant, and constant at $\pm\infty$.
Up to a negligible set, we can write

$$\mathbb{R} = \bigsqcup_{j\in\mathbb{Z}} I_j,$$

where $I_j =]a_j, a_{j+1}[$ is a bounded interval, $n^*(x) = n_j^*$ for $x \in I_j$, and $n_j^* \in \{1, 2\}$ (the assumption n^* constant at $\pm\infty$ implies that the sequence $(n_j^*)_{j\in\mathbb{Z}}$ is stationary).
For all $k \in \mathbb{N}^*$, we can divide (up to a negligible set) each interval I_j like this:

$$I_j = \bigsqcup_{i=0}^{k-1}]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[, \qquad a_{j,i}^{(k)} = a_j + \frac{i}{k}(a_{j+1} - a_j), \qquad i = 0, \ldots, k.$$

For $j \in \mathbb{Z}$, $k \in \mathbb{N}^*$, and $0 \leq i \leq k-1$, we set

$$m_{j,i}^{(k)} = \frac{1}{n_j^*} \int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)dx.$$

Since $0 \leq n^0 \leq n^*$, we have $0 \leq m_{j,i}^{(k)} \leq \frac{meas(I_j)}{k}$. thus

$$]a_{j,i}^{(k)}, a_{j,i}^{(k)} + m_{j,i}^{(k)}[\subset]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[.$$

We set

$$n_k^0(x) = \sum_{j=-k}^{k} \sum_{i=0}^{k-1} n_j^* \mathbb{1}_{]a_{j,i}^{(k)}, a_{j,i}^{(k)}+m_{j,i}^{(k)}[}(x).$$

Obviously n_k^0 satisfies (4.1).
Moreover, we can notice that

$$n_k^0 \equiv 0 \ a.e. \text{ on }]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[\quad \Longleftrightarrow \quad n^0 \equiv 0 \ a.e. \text{ on }]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[,$$

and

$$n_k^0 \equiv n_j^* \ a.e. \text{ on }]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[\quad \Longleftrightarrow \quad n^0 \equiv n_j^* \ a.e. \text{ on }]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[.$$

We also define

$$n_k^0(x) u_k^0(x) = \sum_{j=-k}^{k} \sum_{i=0}^{k-1} n_j^* u_{j,i}^{(k)} \mathbb{1}_{]a_{j,i}^{(k)}, a_{j,i}^{(k)}+m_{j,i}^{(k)}[}(x),$$

where $u_{j,i}^{(k)} = \underset{]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[}{essinf} \ u^0$, which makes sense because $u^0 \in BV(\mathbb{R})$. We have

$$a.e. \ x \in]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[, \qquad n_k^0(x) \neq 0 \quad \Longrightarrow \quad u_k^0(x) = u_{j,i}^{(k)}.$$

We extend u_k^0 linearly in the vacuum (areas where $n_k^0 = 0$) and at infinity, as in Remark 3.9.
Thus, areas where $n_k^0 = 0$ have no influence on the total variation and we have

$$\begin{aligned}
TV(u_k^0) \ &= \ |u_{-k,0}^{(k)} - u_{-k,1}^{(k)}| + \cdots + |u_{-k,k-2}^{(k)} - u_{-k,k-1}^{(k)}| \\
&+ \ |u_{-k,k-1}^{(k)} - u_{-k+1,0}^{(k)}| \\
&+ \ |u_{-k+1,0}^{(k)} - u_{-k+1,1}^{(k)}| + \cdots + |u_{k-1,k-1}^{(k)} - u_{k,0}^{(k)}| \\
&+ \ |u_{k,0}^{(k)} - u_{k,1}^{(k)}| + \cdots + |u_{k,k-2}^{(k)} - u_{k,k-1}^{(k)}| \\
&\leq \ TV_{[a_{-k}, a_{k+1}]}(u^0),
\end{aligned}$$

which shows that u_k^0 satisfies (4.3). We also have (4.2).
For any test function $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}
\int_{\mathbb{R}} n_k^0(x) \varphi(x) dx &= \sum_{|j| \leq k} \sum_{i=0}^{k-1} n_j^* \int_{a_{j,i}^{(k)}}^{a_{j,i}^{(k)}+m_{j,i}^{(k)}} \varphi(x) dx \\
&= \sum_{|j| \leq k} \sum_{i=0}^{k-1} n_j^* \left(m_{j,i}^{(k)} \varphi(a_{j,i}^{(k)}) + \frac{{m_{j,i}^{(k)}}^2}{2} \varphi'(\xi_{j,i}^{(k)}) \right)
\end{aligned}$$

with $a_{j,i}^{(k)} < \xi_{j,i}^{(k)} < a_{j,i}^{(k)} + m_{j,i}^{(k)}$ (if $m_{j,i}^{(k)} \neq 0$). Thus, we can rewrite

$$\int_{\mathbb{R}} n_k^0(x)\varphi(x)dx \;=\; \sum_{|j|\le k}\sum_{i=0}^{k-1}\left(\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)\varphi(a_{j,i}^{(k)})dx + \frac{n_j^* {m_{j,i}^{(k)}}^2}{2}\varphi'(\xi_{j,i}^{(k)})\right).$$

Let $j_0 \in \mathbb{N}^*$ such that $supp(\varphi) \subset \bigsqcup_{|j|\le j_0} I_{j_0}$ (it is possible because $\inf_{j\in\mathbb{Z}}(meas(I_j)) > 0$). Then we have, for all $k \ge j_0$,

$$\int_{\mathbb{R}} n_k^0(x)\varphi(x)dx \;=\; \sum_{|j|\le j_0}\sum_{i=0}^{k-1}\left(\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)\varphi(a_{j,i}^{(k)})dx + \frac{n_j^* {m_{j,i}^{(k)}}^2}{2}\varphi'(\xi_{j,i}^{(k)})\right).$$

We also have

$$\int_{\mathbb{R}} n^0(x)\varphi(x)dx \;=\; \sum_{|j|\le j_0}\sum_{i=0}^{k-1}\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)\varphi(x)dx.$$

Thus

$$\begin{aligned}
&\left|\int_{\mathbb{R}} n^0(x)\varphi(x)dx - \int_{\mathbb{R}} n_k^0(x)\varphi(x)dx\right| \\
\le\;& \sum_{|j|\le j_0}\sum_{i=0}^{k-1}\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)|\varphi(x)-\varphi(a_{j,i}^{(k)})|dx + \|\varphi'\|_\infty \sum_{|j|\le j_0}\sum_{i=0}^{k-1}\frac{n_j^*{m_{j,i}^{(k)}}^2}{2} \\
\le\;& \|\varphi'\|_\infty \sum_{|j|\le j_0}\sum_{i=0}^{k-1} n_j^* \int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} (x-a_{j,i}^{(k)})dx + \|\varphi'\|_\infty \sum_{|j|\le j_0}\sum_{i=0}^{k-1}{m_{j,i}^{(k)}}^2 \\
\le\;& 2\|\varphi'\|_\infty \sum_{|j|\le j_0}\sum_{i=0}^{k-1}\left(\frac{meas(I_j)}{k}\right)^2 \\
\le\;& C(\varphi, j_0)\times\frac{1}{k}.
\end{aligned}$$

Moreover, we have similarly

$$\begin{aligned}
&\int_{\mathbb{R}} n_k^0(x)u_k^0(x)\varphi(x)dx \\
=\;& \sum_{|j|\le j_0}\sum_{i=0}^{k-1}\left(\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)u_{j,i}^{(k)}\varphi(a_{j,i}^{(k)})dx + \frac{n_j^*{m_{j,i}^{(k)}}^2}{2}u_{j,i}^{(k)}\varphi'(\xi_{j,i}^{(k)})\right)
\end{aligned}$$

and

$$\int_{\mathbb{R}} n^0(x)u^0(x)\varphi(x)dx \;=\; \sum_{|j|\le j_0}\sum_{i=0}^{k-1}\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)u^0(x)\varphi(x)dx\;.$$

Thus

$$\left|\int_{\mathbb{R}} n^0(x)u^0(x)\varphi(x)dx - \int_{\mathbb{R}} n_k^0(x)u_k^0(x)\varphi(x)dx\right|$$

$$\leq \sum_{|j|\leq j_0}\sum_{i=0}^{k-1}\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)u_{j,i}^{(k)}|\varphi(x)-\varphi(a_{j,i}^{(k)})|dx$$

$$+ \quad \|\varphi'\|_\infty \sum_{|j|\leq j_0}\sum_{i=0}^{k-1} u_{j,i}^{(k)}\frac{n_j^* {m_{j,i}^{(k)}}^2}{2}$$

$$+ \quad \sum_{|j|\leq j_0}\sum_{i=0}^{k-1}\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} n^0(x)|u^0(x)-u_{j,i}^{(k)}||\varphi(x)|dx$$

$$\leq \quad C(\varphi, j_0)\times\|u_0\|_\infty \times \frac{1}{k} \quad + \quad 2\|\varphi\|_\infty \sum_{|j|\leq j_0}\sum_{i=0}^{k-1}\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} |u^0(x)-u_{j,i}^{(k)}|dx.$$

Therefore we just need to show that the last term vanishes when $k\to\infty$. This is raised because

$$\sum_{|j|\leq j_0}\sum_{i=0}^{k-1}\left(\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}} |u^0(x)-u_{j,i}^{(k)}|dx\right)$$

$$\leq \quad \sum_{|j|\leq j_0}\sum_{i=0}^{k-1}\int_{a_{j,i}^{(k)}}^{a_{j,i+1}^{(k)}}\left|\sup_{]a_{j,i}^{(k)},a_{j,i+1}^{(k)}[} u^0 - \inf_{]a_{j,i}^{(k)},a_{j,i+1}^{(k)}[} u^0\right| dx$$

$$\leq \quad \sum_{|j|\leq j_0}\frac{meas(I_j)}{k}\left(\sum_{i=0}^{k-1} TV_{]a_{j,i}^{(k)},a_{j,i+1}^{(k)}[}\left(u^0\right)\right)$$

$$\leq \quad \sum_{|j|\leq j_0}\frac{meas(I_j)}{k}\; TV_{I_j}\left(u^0\right)$$

$$\leq \quad TV(u^0)\times C(j_0)\times\frac{1}{k}.$$

We established that $< n_k^0,\varphi > \to < n^0,\varphi >$ and $< n_k^0 u_k^0,\varphi > \to < n^0u^0,\varphi >$. Finally, we define p_k^0 the same way as u_k^0:

$$n_k^0(x)p_k^0(x) = \sum_{j=-k}^{k}\sum_{i=0}^{k-1} n_j^* p_{j,i}^{(k)} \mathbb{1}_{]a_{j,i}^{(k)},a_{j,i}^{(k)}+m_{j,i}^{(k)}[}(x),$$

where $p_{j,i}^{(k)} = \underset{]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[}{essinf}\ p^0$. But in the vacuum (areas where $n_k^0 = 0$) we set $p_k^0 = 0$. Thus, we have $p_k^0 \equiv p_{j,i}^{(k)}$ on each interval $]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[$. In fact, there are two cases:

- If $n^0 \equiv n_j^*$ a.e. on $]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[$, then $n_k^0 \equiv n_j^*$ and $p_k^0 \equiv p_{j,i}^{(k)}$.
- Else, it exists a non negligible subset $\omega \subset]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[$ where $n^0 < n_j^*$, and $p^0 \equiv 0$ a.e. on ω, which implies $p_{j,i}^{(k)} = 0$, and $p_k^0 \equiv 0 = p_{j,i}^{(k)}$ a.e. on $]a_{j,i}^{(k)}, a_{j,i+1}^{(k)}[$.

We easily deduce that p_k^0 satisfies properties (4.2) and (4.3).

For the convergence $< n_k^0 p_k^0, \varphi > \to < n^0 p^0, \varphi >$, the proof is exactly the same as $n_k^0 u_k^0$. □

Remark 4.2 *The sequence* (n_k^0, u_k^0, p_k^0) *satisfies the constraint:*

$$(n^*(x) - n_k^0) p_k^0 = 0, \qquad \forall k \geq 1,$$

because $n_k^0(x) \in \{0, n_j^*\}$ *for all* $x \in \mathbb{R}$.

4.2 Existence result

Let us recall the ML-CPGD system:

$$\partial_t n + \partial_x (nu) = 0\,, \tag{4.4}$$
$$\partial_t (n(u+p) I_\alpha) + \partial_x (nu(u+p) I_\alpha) = 0\,, \tag{4.5}$$
$$0 \leq n \leq n^*(x)\,, \quad u \geq 0\,, \quad p \geq 0\,, \quad (n^*(x) - n)p = 0\,. \tag{4.6}$$

We prove now the existence of weak solutions. The idea is first to approximate the initial data in the distributional sense by sticky blocks. These special initial data give a sequence of solutions. Then we perform a compactness argument on this sequence of solutions. Finally, we prove that the obtained limit is a solution for the wanted initial data. The regularity of the solutions are

$$n \in L^\infty(]0, +\infty[_t, L^\infty(\mathbb{R}_x) \cap L^1(\mathbb{R}_x)), \tag{4.7}$$

$$u, p \in L^\infty(]0, +\infty[_t, L^\infty(\mathbb{R}_x)). \tag{4.8}$$

Theorem 4.3 *Let* (n^0, u^0, p^0) *be some initial data such that*

$$n^0 \in L^1(\mathbb{R}), \qquad u^0, p^0 \in L^\infty(\mathbb{R}) \cap BV(\mathbb{R}),$$

with $0 \leq u^0$, $0 \leq p^0$, $0 \leq n^0 \leq n^*(x)$ *and* $(n^*(x) - n^0)p^0 = 0$. *Then there exists* (n, u, p) *with regularities* (4.7), (4.8), *solution to the system* (4.4)−(4.6), *with initial data* (n^0, u^0, p^0). *The obtained solution also satisfies*

$$0 \leq u(t,x) \leq 2\alpha \left(esssup_y u^0(y) + esssup_y p^0(y) \right), \tag{4.9}$$

$$0 \leq p(t,x) \leq 2\alpha \left(esssup_y u^0(y) + esssup_y p^0(y) \right). \tag{4.10}$$

Proof: Let n_k^0, u_k^0, p_k^0 ($k \in \mathbb{N}^*$) be the block initial data associated respectively to n^0, u^0, p^0 provided by Lemma 4.1. For all k, the results of section 3 allow us to get (n_k, u_k, p_k) solutions of $(4.4) - (4.6)$ with initial data (n_k^0, u_k^0, p_k^0), with regularities (4.7), (4.8), and which satisfy the bounds

$$0 \le u_k(t,x) \le 2\alpha \left(esssup_y u_k^0(y) + esssup_y p_k^0(y) \right), \tag{4.11}$$

$$0 \le p_k(t,x) \le 2\alpha \left(esssup_y u_k^0(y) + esssup_y p_k^0(y) \right), \tag{4.12}$$

$$TV_K(u_k(t,.)) \le 4\alpha M \left(TV_{\tilde{K}}(u_k^0) + TV_{\tilde{K}}(p_k^0) + \|u_k^0\|_{L^\infty} \right), \tag{4.13}$$

$$TV_K(p_k(t,.)) \le 4\alpha M \left(TV_{\tilde{K}}(u_k^0) + TV_{\tilde{K}}(p_k^0) + \|u_k^0\|_{L^\infty} \right). \tag{4.14}$$

Since (n_k) is bounded in L^∞, then there exists a subsequence such that

$$n_k \rightharpoonup n \text{ in } L^\infty_{w*}(]0,+\infty[\times\mathbb{R}). \tag{4.15}$$

Thanks to (4.11), (4.12) and the bounds on u_k^0, p_k^0 provided by Lemma 4.1, the sequence (u_k) and (p_k) are bounded in $L^\infty(]0,+\infty[\times\mathbb{R})$, then, up to subsequences, we have

$$u_k \rightharpoonup u \text{ in } L^\infty_{w*}(]0,+\infty[\times\mathbb{R}), \tag{4.16}$$

$$p_k \rightharpoonup p \text{ in } L^\infty_{w*}(]0,+\infty[\times\mathbb{R}). \tag{4.17}$$

Next step is now to prove the passage to the limit in the equation.
First, for the sequence $(n_k)_{k\ge 1}$, we can obtain more compactness using the following lemma and the estimate:
$\forall T > 0, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}_x), \quad \forall t,s \in [0,T], \quad \forall k \in \mathbb{N}^*,$

$$\begin{aligned} &\left| \int_{\mathbb{R}} (n_k(t,x) - n_k(s,x))\varphi(x)dx \right| \\ &\qquad \le \quad n^* \sup_{k\ge 1} \|u_k^0\|_{L^\infty(\mathbb{R}_x)} \left(\int_{\mathbb{R}} |\partial_x \varphi| dx \right) |t-s|, \end{aligned} \tag{4.18}$$

which can be obtained by integrating (4.4).

Lemma 4.4 *Let $(n_k)_{k\in\mathbb{N}^*}$ be a bounded sequence in $L^\infty(]0,T[\times\mathbb{R})$ which satisfies: for all $\varphi \in \mathcal{D}(\mathbb{R}_x)$, the sequence $\left(\int_{\mathbb{R}} n_k(t,x)\varphi(x)dx\right)_k$ is uniformly Lipschitz continuous on $[0,T]$, i.e.*

$$\exists C_\varphi > 0, \quad \forall k \in \mathbb{N}^*, \quad \forall s,t \in [0,T],$$

$$\left| \int_{\mathbb{R}} (n_k(t,x) - n_k(s,x))\varphi(x)dx \right| \le C_\varphi |t-s|.$$

Then, up to a subsequence, it exists $n \in L^\infty(]0,T[\times\mathbb{R})$ such that $n_k \to n$ in $C([0,T], L^\infty_{w*}(\mathbb{R}_x))$, i.e.

$$\forall \Gamma \in L^1(\mathbb{R}_x), \quad \sup_{t\in[0,T]} \left| \int_{\mathbb{R}} (n_k(t,x) - n(t,x))\Gamma(x)dx \right| \underset{k}{\longrightarrow} 0.$$

<u>Proof</u>: Let $(\varphi_m)_{m\geq 1}$ be a countable set dense in $\mathcal{D}(\mathbb{R}_x)$ for the L^1-norm, which exists because of the separability of $L^1(\mathbb{R}_x)$. We denote

$$g_{k,m}(t) := \int_{\mathbb{R}} n_k(t,x)\varphi_m(x)dx.$$

The sequence $(g_{k,1})_{k\geq 1}$ is bounded and equicontinuous in $C([0,T],\mathbb{R})$, thus, the Ascoli Theorem entails that it exists an extraction $\sigma_1(k)$ such that

$$g_{\sigma_1(k),1} \underset{k}{\longrightarrow} l_1 \quad \text{in} \quad C([0,T],\mathbb{R}).$$

The same applies to $(g_{\sigma_1(k),2})_{k\geq 1}$, thus it exists an extraction σ_2 such that

$$g_{\sigma_1(\sigma_2(k)),2} \underset{k}{\longrightarrow} l_2 \quad \text{in} \quad C([0,T],\mathbb{R}).$$

A simple recursion shows that we can build a sequence of extractions σ_m such that

$$g_{\sigma_1(\sigma_2(\ldots\sigma_m(k))\ldots)),m} \underset{k}{\longrightarrow} l_m \quad \text{in} \quad C([0,T],\mathbb{R}).$$

Therefore, setting $\sigma(k) := \sigma_1 \circ \cdots \circ \sigma_k(k)$, we have (by diagonal extraction)

$$\forall m \geq 1, \qquad g_{\sigma(k),m} \underset{k}{\longrightarrow} l_m \quad \text{in} \quad C([0,T],\mathbb{R}). \tag{4.19}$$

Now, we can identify the limit l_m because since $(n_{\sigma(k)})_k$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$, there exists a subsequence (still denoted by the same way) such that $n_{\sigma(k)} \rightharpoonup n$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$. Thus, we have, for all $m \geq 1$, and for all $\psi \in \mathcal{D}(]0,T[_t)$,

$$\int_0^T \int_{\mathbb{R}} n_{\sigma(k)}(t,x)\psi(t)\varphi_m(x)dxdt \underset{k}{\to} \int_0^T \int_{\mathbb{R}} n(t,x)\psi(t)\varphi_m(x)dxdt,$$

which rewrites

$$\int_0^T g_{\sigma(k),m}(t)\psi(t)dt \underset{k}{\to} \int_0^T \left(\int_{\mathbb{R}} n(t,x)\varphi_m(x)dx\right)\psi(t)dt.$$

Moreover, (4.19) easily implies that

$$\int_0^T g_{\sigma(k),m}(t)\psi(t)dt \underset{k}{\to} \int_0^T l_m(t)\psi(t)dt,$$

thus $l_m(t) = \int_{\mathbb{R}} n(t,x)\varphi_m(x)dx$, a.e. $t \in [0,T]$, from which we can deduce

$$\forall m \geq 1, \qquad \sup_{t\in[0,T]} \left|\int_{\mathbb{R}} (n_{\sigma(k)}(t,x) - n(t,x))\varphi_m(x)dx\right| \underset{k}{\longrightarrow} 0.$$

Finally, this convergence stays available for all $\varphi \in \mathcal{D}(\mathbb{R}_x)$, because of the inequality

$$\sup_{t\in[0,T]} \left|\int_{\mathbb{R}} (n_{\sigma(k)} - n)(t,x)\varphi(x)dx\right|$$

$$\leq \sup_{t\in[0,T]} \left|\int_{\mathbb{R}} (n_{\sigma(k)} - n)(t,x)\varphi_m(x)dx\right| + C\|\varphi - \varphi_m\|_{L^1(\mathbb{R})},$$

where

$$C := \sup_{k\geq 1}(\|n_k\|_{L^\infty(]0,T[\times\mathbb{R})}) + \|n\|_{L^\infty(]0,T[\times\mathbb{R})} < +\infty.$$

We conclude that it is also true for $\Gamma \in L^1(\mathbb{R}_x)$ by density, using the same inequality. □

Following of the proof of Theorem 4.3: According to (4.6) and (4.18), the lemma 4.4 applies to the sequence $(n_k)_{k\geq 1}$, and thus

$$n_k \to n \text{ in } C([0,T], L^\infty_{w*}(\mathbb{R}_x)), \text{ for all } T > 0. \tag{4.20}$$

As the same, we obtain (integrating (4.5)) an estimate similar to (4.18) for the sequence $(n_k(u_k + p_k)I_\alpha)_{k\geq 1}$, thus it exists $q \in L^\infty(]0,+\infty[\times\mathbb{R})$ such that

$$n_k(u_k + p_k)I_\alpha \to q \text{ in } C([0,T], L^\infty_{w*}(\mathbb{R}_x)), \text{ for all } T > 0. \tag{4.21}$$

Now, the key point of the proof is passing to the limit in the products and is treated by the following technical lemma:

Lemma 4.5 *Let us assume that $(\gamma_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R})$ that tends to γ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$, and satisfies for any $\Gamma \in \mathcal{D}(\mathbb{R}_x)$,*

$$\int_{\mathbb{R}} (\gamma_k - \gamma)(t,x)\Gamma(x)dx \underset{k}{\longrightarrow} 0, \tag{4.22}$$

either i) a.e. $t \in]0,T[$ or ii) in $L^1(]0,T[_t)$.
Let us also assume that $(\omega_k)_{k\in\mathbb{N}}$ is a bounded sequence in $L^\infty(]0,T[\times\mathbb{R})$ that tends to ω in $L^\infty_{w*}(]0,T[\times\mathbb{R})$, and such that for all compact interval $K = [a,b]$, there exists $C > 0$ such that the total variation (in x) of ω_k and ω over K satisfies

$$\forall k \in \mathbb{N}, \quad TV_K(\omega_k(t,.)) \leq C, \qquad TV_K(\omega(t,.)) \leq C. \tag{4.23}$$

Then, $\gamma_k\omega_k \rightharpoonup \gamma\omega$ in $L^\infty_{w*}(]0,T[\times\mathbb{R})$, as $k \to +\infty$.

Remark 4.6 *This is a result of compensated compactness, which uses the compactness in x for $(\omega_k)_k$ given by (4.23) and the weak compactness in t for $(\gamma_k)_k$ given by (4.22) to pass to the weak limit in the product $\gamma_k\omega_k$.*

Proof: We can refer to [4] for a complete proof, even in the case where

$$\forall k \in \mathbb{N}, \quad TV_K(\omega_k(t,.)) \leq C(1 + \frac{1}{t}), \qquad TV_K(\omega(t,.)) \leq C(1 + \frac{1}{t}),$$

which is more general. □

End of the proof of Theorem 4.3: The convergence (4.20) allows to apply Lemma 4.5 with $\gamma_k = n_k$. Moreover, thanks to (4.13) and the BV bounds on u_k^0 provided by Lemma 4.1, we can set $\omega_k = u_k$ in Lemma 4.5 (in fact, the sequence $u_k(t,.)$ is uniformly bounded in BV with respect to t, and also $u(t,.)$ thanks to the lower semi-continuity to the BV norm). Thus, we have

$$n_k u_k \rightharpoonup nu \text{ in } L^\infty_{w*}(]0,+\infty[\times\mathbb{R}). \tag{4.24}$$

The same applies to the sequences $(\gamma_k, \omega_k) = (n_k, p_k)$ and $(\gamma_k, \omega_k) = (n_k(u_k + p_k)I_\alpha, u_k)$: we have

$$n_k p_k \rightharpoonup np \text{ in } L^\infty_{w*}(]0, +\infty[\times\mathbb{R}), \tag{4.25}$$

$$n_k(u_k + p_k)I_\alpha u_k \rightharpoonup qu \text{ in } L^\infty_{w*}(]0, +\infty[\times\mathbb{R}). \tag{4.26}$$

Furthermore, we easily have

$$n_k(u_k + p_k)I_\alpha \rightharpoonup n(u + p)I_\alpha \text{ in } L^\infty_{w*}(]0, +\infty[\times\mathbb{R}),$$

thus $q = n(u + p)I_\alpha$, and

$$n_k u_k(u_k + p_k)I_\alpha \rightharpoonup nu(u + p)I_\alpha \text{ in } L^\infty_{w*}(]0, +\infty[\times\mathbb{R}). \tag{4.27}$$

We deduce that (n, u, p) satisfies (4.4), (4.5) in $\mathcal{D}'(]0, +\infty[\times\mathbb{R})$, and the constraints (4.6).
The last step is to show that (n^0, p^0, u^0) is really the initial data of the problem, according to the weak formulation:

$$\forall\varphi \in C_c^\infty([0, +\infty[_t\times\mathbb{R}_x),$$

$$\int_0^\infty\int_{\mathbb{R}} (n\partial_t\varphi + nu\partial_x\varphi)(t, x)dxdt + \int_{\mathbb{R}} n^0(x)\varphi(0, x)dx = 0,$$

$$\int_0^\infty\int_{\mathbb{R}} (n(u + p)I_\alpha\partial_t\varphi + nu(u + p)I_\alpha\partial_x\varphi)(t, x)dxdt$$

$$+ \int_{\mathbb{R}} n^0(x)(u^0(x) + p^0(x))I_\alpha(x)\varphi(0, x)dx = 0.$$

It comes easily, because we have, for all $k \geq 1$:

$$\forall\varphi \in C_c^\infty([0, +\infty[_t\times\mathbb{R}_x),$$

$$\int_0^\infty\int_{\mathbb{R}} (n_k\partial_t\varphi + n_k u_k\partial_x\varphi)(t, x)dxdt + \int_{\mathbb{R}} n_k^0(x)\varphi(0, x)dx = 0,$$

$$\int_0^\infty\int_{\mathbb{R}} (n_k(u_k + p_k)I_\alpha\partial_t\varphi + n_k u_k(u_k + p_k)I_\alpha\partial_x\varphi)(t, x)dxdt$$

$$+ \int_{\mathbb{R}} n_k^0(x)(u_k^0(x) + p_k^0(x))I_\alpha(x)\varphi(0, x)dx = 0,$$

and we can pass to the limit when $k \to +\infty$ because of the convergences $n_k^0 \rightharpoonup n^0$, $n_k^0 u_k^0 \rightharpoonup n^0 u^0$ and $n_k^0 p_k^0 \rightharpoonup n^0 p^0$ in $\mathcal{D}'(\mathbb{R})$, and the convergences (4.24), (4.26) and (4.27) in $L^\infty_{w*}(]0, +\infty[\times\mathbb{R})$. $\square$

4.3 Compactness result

To finalize the paper, we set a compactness result which is contained into the proof of the previous existence Theorem.

Theorem 4.7 *Let us consider a sequence of solutions* (n_k, u_k, p_k) *with regularity* (4.7), (4.8)*, satisfying* $(4.4) - (4.6)$*, and the following bounds:*

$$\forall k \in \mathbb{N}, \quad a.e.\ (t,x) \in]0,+\infty[\times\mathbb{R}, \qquad 0 \le u_k(t,x) \le C_\alpha,$$

$$\forall k \in \mathbb{N}, \quad a.e.\ (t,x) \in]0,+\infty[\times\mathbb{R}, \qquad 0 \le p_k(t,x) \le C_\alpha,$$

$$\forall K = [a,b] \subset \mathbb{R}, \quad \forall k \in \mathbb{N}, \quad a.e.\ t \in]0,+\infty[, \qquad TV_K(u_k(t,.)) \le C_{\alpha,M,K},$$

$$\forall K = [a,b] \subset \mathbb{R}, \quad \forall k \in \mathbb{N}, \quad a.e.\ t \in]0,+\infty[, \qquad TV_K(p_k(t,.)) \le C_{\alpha,M,K},$$

with C_α *(resp.* $C_{\alpha,M,K}$*) some positive constant depending only on* α *(resp.* α*,* M *and* K*).*
Then, up to a subsequence, $(n_k, u_k, p_k) \rightharpoonup (n,u,p)$ *in* $L^\infty_{w*}(]0,+\infty[\times\mathbb{R})$*, where* (n,u,p) *is a solution to the system* $(4.4) - (4.6)$*. This solution* (n,u,p) *also satisfies*

$$a.e.\ (t,x) \in]0,+\infty[\times\mathbb{R}, \qquad 0 \le u(t,x) \le C_\alpha,$$

$$a.e.\ (t,x) \in]0,+\infty[\times\mathbb{R}, \qquad 0 \le p(t,x) \le C_\alpha.$$

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