Stochastic isentropic Euler equations

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Abstract

We study the stochastically forced system of isentropic Euler equations of gas dynamics with a γ-law for the pressure. We show the existence of martingale weak entropy solutions; we also discuss the existence and characterization of invariant measures in the concluding section.

Keywords: Stochastic partial differential equations, isentropic Euler equations, entropy solutions.

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Contents

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1 Introduction

In this paper, we study the stochastically forced system of isentropic Euler equations of gas dynamics with a γ-law for the pressure.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis, let $\mathbb{T}$ be the one-dimensional torus, let $T > 0$ and set $Q_T := \mathbb{T} \times (0,T)$. We study the system

$$d\rho + \partial_x(\rho u)dt = 0, \qquad\qquad \text{in } Q_T, \qquad (1.1a)$$

$$d(\rho u) + \partial_x(\rho u^2 + p(\rho))dt = \Phi(\rho, u)dW(t), \qquad\qquad \text{in } Q_T, \qquad (1.1b)$$

$$\rho = \rho_0, \quad \rho u = \rho_0 u_0, \qquad\qquad \text{in } \mathbb{T} \times \{0\}, \qquad (1.1c)$$

where p follows the γ-law

$$p(\rho) = \kappa\rho^\gamma, \quad \kappa = \frac{\theta^2}{\gamma}, \quad \theta = \frac{\gamma - 1}{2}, \qquad (1.2)$$

for $\gamma > 1$, W is a cylindrical Wiener process and $\Phi(0,u) = 0$. Therefore the noise affects the momentum equation only and vanishes in vacuum regions. Our aim is to prove the existence of solutions to (1.1) for general initial data (including vacuum), *cf.* Theorem 2.1 below.

There are to our knowledge no existing results on stochastically forced systems of first-order conservation laws, with the exception of the papers by Kim, [Kim11], and Audusse, Boyaval, Goutal, Jodeau, Ung, [ABG+15]. In [Kim11] the problematic is the possibility of global existence of *regular* solutions to symmetric hyperbolic systems under suitable assumptions on the structure of the stochastic forcing term. In [ABG+15] is derived a shallow water system with a stochastic

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Exner equation as a model for the dynamics of sedimentary river beds. On second-order stochastic systems, and specifically on the stochastic compressible Navier-Stokes equation[1], different results have been obtained recently, see the papers by Breit, Feireisl, Hofmanová, Maslowski, Novotny, Smith, [FMN13, BH14, BFH15, Smi15] (see also the older work by Tornare and Fujita, [TFY97]).

The *incompressible* Euler equations with stochastic forcing terms have been studied in particular by Bessaih, Flandoli,[Bes99, BF99, Bes00, Bes08], Capiński, Cutland, [CC99], Brzeźniak, Peszat, [BP01], Cruzeiro, Flandoli, Malliavin, [CFM07], Brzeźniak, Flandoli, Maurelli, [BFM14], Glatt-Holtz and Vicol, [GHV14], Cruzeiro and Torrecilla, [CT15]. We refer in particular to [GHV14] for results in space dimension 3.

In the deterministic case, and in in space dimension 1, the existence of weak entropy solutions to the isentropic Euler system has been proved by Lions, Perthame, Souganidis in [LPS96]. Let us mention also the anterior papers by Di Perna [DiP83a], Ding, Chen, Luo [DCL85], Chen [Che86], Lions, Perthame, Tadmor [LPT94]. The uniqueness of weak entropy solutions is still an open question.

For *scalar* non-linear hyperbolic equations with a stochastic forcing term, the theory has recently known a lot of developments. Well-posedness has been proved in different contexts and under different hypotheses and also with different techniques: by Lax-Oleinik formula (E, Khanin, Mazel, Sinai [EKMS00]), Kruzhkov doubling of variables for entropy solutions (Kim [Kim03], Feng, Nualart [FN08], Vallet, Wittbold [VW09], Chen, Ding, Karlsen [CDK12], Bauzet, Vallet, Wittbold [BVW12]), kinetic formulation (Debussche, Vovelle [DV10, DV13]). Resolution in L^1 has been given in [DV14]. Let us also mention the works of Hofmanová in this fields (extension to second-order scalar degenerate equations, convergence of the BGK approximation [Hof13, DHV15, Hof15a]) and the recent works by Hofmanová, Gess, Lions, Perthame, Souganidis [LPS13b, LPS13a, LPS14, GP15, GP14, Hof15b] on scalar conservation laws with quasilinear stochastic terms.

We will show existence of martingale solutions to (1.1), see Theorem 2.1 below. The procedure is standard: we prove the convergence of (subsequence of) solutions to the parabolic approximation to (1.1). For this purpose we have to adapt the concentration compactness technique (*cf.* [DiP83a, LPS96]) of the deterministic case to the stochastic case. Such an extension has already been done for scalar conservation laws by Feng and Nualart [FN08] and what we do is quite similar. The mode of convergence for which there is compactness, if we restrict ourselves to the sample variable ω, is the convergence in law. That is why we obtain martingale solutions. There is a usual trick, the Gyöngy-Krylov characterization of convergence in probability, that allows to recover pathwise solutions once pathwise uniqueness of solutions is known (*cf.* [GK96]). However for the stochastic problem (1.1) (as it is already the case for the deterministic one), no such results of uniqueness are known.

A large part of our analysis is devoted to the proof of existence of solutions to the parabolic approximation. What is challenging and more difficult than in the deterministic framework for the stochastic parabolic problem is the issue of positivity of the density. We solve this problem by using a regularizing effect of parabolic equations with drifts and a bound given by the entropy, quite in the spirit of Mellet, Vasseur, [MV09], *cf.* Theorem A.1. Then, the proof of convergence of the parabolic approximation (3.1) to Problem (1.1) is adapted from the proof in the deterministic case to circumvent two additional difficulties:

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[1] which, to be exact, is first-order in the density and second-order in the velocity

1. there is a lack of compactness with respect to ω; one has to pass to the limit in some stochastic integrals,

2. there are no "uniform in ε" L^∞ bounds on solutions (here ε is the regularization parameter in the parabolic problem (3.1)).

Problem 1. is solved by use of convergence in law and martingale formulations, Problem 2. is solved by using higher moment estimates (see (3.10) and (3.11)-(3.12)). We will give more details about the main problematic of the paper in Section 2.4, after our framework has been introduced more precisely. Note that Problem 2. also occurs in the resolution of the isentropic Euler system for flows in non-trivial geometry, as treated by Le Floch, Westdickenberg, [LW07].

2 Notations and main result

2.1 Stochastic forcing

Our hypotheses on the stochastic forcing term $\Phi(\rho, u)W(t)$ are the following ones. We assume that $W = \sum_{k\geq 1} \beta_k e_k$ where the β_k are independent Brownian processes and $(e_k)_{k\geq 1}$ is a complete orthonormal system in a Hilbert space $\mathfrak{U}$. For each $\rho \geq 0, u \in \mathbb{R}$, $\Phi(\rho, u)\colon \mathfrak{U} \to L^2(\mathbb{T})$ is defined by

$$\Phi(\rho, u)e_k = \sigma_k(\cdot, \rho, u) = \rho\sigma_k^*(\cdot, \rho, u), \tag{2.1}$$

where $\sigma_k^*(\cdot, \rho, u)$ is a 1-periodic continuous function on $\mathbb{R}$. More precisely, we assume $\sigma_k^* \in C(\mathbb{T}_x \times \mathbb{R}_+ \times \mathbb{R})$ and the bound

$$\mathbf{G}(x, \rho, u) := \left(\sum_{k\geq 1} |\sigma_k(x, \rho, u)|^2\right)^{1/2} \leq A_0\rho\left[1 + u^2 + \rho^{2\theta}\right]^{1/2}, \tag{2.2}$$

for all $x \in \mathbb{T}$, $\rho \geq 0$, $u \in \mathbb{R}$, where A_0 is some non-negative constant. Depending on the statement, we will sometimes also make the following localization hypothesis: for $\varkappa > 0$, denote by $z = u - \rho^\theta$, $w = u + \rho^\theta$ the Riemann invariants for (1.1) and by $\Lambda_\varkappa$ the domain

$$\Lambda_\varkappa = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R}; -\varkappa \leq z \leq w \leq \varkappa\}. \tag{2.3}$$

We will establish some of our results (more precisely: the resolution of the approximate parabolic Problem (3.1)) under the hypothesis that there exists $\varkappa > 0$ such that

$$\mathrm{supp}(\mathbf{G}) \subset \mathbb{T}_x \times \Lambda_\varkappa. \tag{2.4}$$

We define the auxiliary space $\mathfrak{U}_0 \subset \mathfrak{U}$ by

$$\mathfrak{U}_0 = \left\{v = \sum_{k\geq 1} \alpha_k e_k; \sum_{k\geq 1} \frac{\alpha_k^2}{k^2} < \infty\right\}, \tag{2.5}$$

and the norm

$$\|v\|_{\mathfrak{U}_0}^2 = \sum_{k\geq 1} \frac{\alpha_k^2}{k^2}, \qquad v = \sum_{k\geq 1} \alpha_k e_k.$$

The embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is then an Hilbert-Schmidt operator. Moreover, trajectories of W are $\mathbb{P}$-a.s. in $C([0, T]; \mathfrak{U}_0)$ (see Da Prato, Zabczyk [DPZ92]). We use the path space $C([0, T]; \mathfrak{U}_0)$ to recover the cylindrical Wiener process W in certain limiting arguments, *cf.* Section 3.2.7 for example.

2.2 Notations

We denote by

$$\mathbf{U} = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{pmatrix}, \quad q = \rho u, \tag{2.6}$$

the 2-dimensional unknown and flux of the conservative part of the problem. We also set

$$\psi_k(\mathbf{U}) = \begin{pmatrix} 0 \\ \sigma_k(\mathbf{U}) \end{pmatrix}, \quad \boldsymbol{\Psi}(\mathbf{U}) = \begin{pmatrix} 0 \\ \Phi(\mathbf{U}) \end{pmatrix}.$$

With the notations above, (1.1) can be more concisely rewritten as the following stochastic first-order system

$$d\mathbf{U} + \partial_x \mathbf{F}(\mathbf{U})dt = \boldsymbol{\Psi}(\mathbf{U})dW(t). \tag{2.7}$$

If E is a space of real-valued functions on $\mathbb{T}$, we will denote $\mathbf{U}(t) \in E$ instead of $\mathbf{U}(t) \in E \times E$ when this occurs. Similarly, we will denote $\mathbf{U} \in E$ instead of $\mathbf{U} \in E \times E$ if E is a space of real-valued functions on $\mathbb{T} \times [0, T]$ (see the statement of Definition 2.1 as an example).

We denote by $\mathcal{P}_T$ the predictable σ-algebra on $\Omega \times [0, T]$ generated by $(\mathcal{F}_t)$.

We will also use the following notation in various estimates below:

$$A = \mathcal{O}(1)B,$$

where $A, B \in \mathbb{R}_+$, with the meaning $A \leq CB$ for a constant $C \geq 0$. In general, the dependence of C over the data and parameters at stake will be given in detail, see for instance Theorem 3.1 below. We use the notation

$$A \lesssim B$$

with the same meaning $A \leq CB$, but when the constant $C \geq 0$ depends only on γ and nothing else, C being bounded for γ in a compact subset of $[1, +\infty)$. In this last case, C depends sometimes even not on γ and is simply a numerical constant (see Appendix B for instance).

2.3 Entropy Solution

In relation with the kinetic formulation for (1.1) in [LPT94], there is a family of entropy functionals

$$\eta(\mathbf{U}) = \int_{\mathbb{R}} g(\xi)\chi(\rho, \xi - u)d\xi, \quad \text{with } q = \rho u, \tag{2.8}$$

for (1.1), where

$$\chi(\mathbf{U}) = c_\lambda(\rho^{2\theta} - u^2)^\lambda_+, \quad \lambda = \frac{3 - \gamma}{2(\gamma - 1)}, \quad c_\lambda = \left(\int_{-1}^1 (1 - z^2)^\lambda_+ \, dz \right)^{-1},$$

$s^\lambda_+ := s^\lambda \mathbf{1}_{s>0}$. Indeed, if $g \in C^2(\mathbb{R})$ is a convex function, then η is of class C^2 on the set

$$\mathcal{U} := \left\{ \mathbf{U} = \begin{pmatrix} \rho \\ q \end{pmatrix} \in \mathbb{R}^2; \rho > 0 \right\}$$

and η is a convex function of the argument $\mathbf{U}$. Formally, by the Itō Formula, solutions to (1.1) satisfy

$$d\mathbb{E}\eta(\mathbf{U}) + \partial_x \mathbb{E}H(\mathbf{U})dt = \frac{1}{2}\mathbb{E}\partial_{qq}^2\eta(\mathbf{U})\mathbf{G}^2(\mathbf{U})dt, \tag{2.9}$$

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where the entropy flux H is given by

$$H(\mathbf{U}) = \int_{\mathbb{R}} g(\xi)[\theta\xi + (1-\theta)u]\chi(\rho, \xi - u)d\xi, \quad \text{with } q = \rho u. \tag{2.10}$$

Note that, by a change of variable, we also have

$$\eta(\mathbf{U}) = \rho c_\lambda \int_{-1}^1 g\left(u + z\rho^\theta\right)(1 - z^2)_+^\lambda dz \tag{2.11}$$

and

$$H(\mathbf{U}) = \rho c_\lambda \int_{-1}^1 g\left(u + z\rho^\theta\right)\left(u + z\theta\rho^\theta\right)(1 - z^2)_+^\lambda dz. \tag{2.12}$$

In particular, for $g(\xi) = 1$ we obtain the density $\eta_0(\mathbf{U}) = \rho$. To $g(\xi) = \xi$ corresponds the impulsion $\eta(\mathbf{U}) = q$ and to $g(\xi) = \frac{1}{2}\xi^2$ corresponds the energy

$$\eta_E(\mathbf{U}) = \frac{1}{2}\rho u^2 + \frac{\kappa}{\gamma - 1}\rho^\gamma. \tag{2.13}$$

Note the form of the energy, in particular the fact that the hypothesis (2.2) on the noise gives a bound

$$\mathbf{G}^2(x, \mathbf{U}) = \sum_{k \geq 1} |\Phi(\rho, u)e_k(x)|^2 \leq \rho A_0^\sharp(\eta_0(\mathbf{U}) + \eta_E(\mathbf{U})), \tag{2.14}$$

for a constant $A_0^\sharp$ depending on A_0 and γ (recall that $\eta_0(\mathbf{U}) := \rho$). If (2.9) is satisfied with an inequality $\leq$, then formally (2.14) and the Gronwall Lemma give a bound on $\mathbb{E}\int_{\mathbb{T}}(\eta_0 + \eta_E)(\mathbf{U})(t)dx$ in terms of $\mathbb{E}\int_{\mathbb{T}}(\eta_0 + \eta_E)(\mathbf{U})(0)dx$. Indeed, we have $\partial_{qq}^2\eta_E(\mathbf{U}) = \frac{1}{\rho}$ and, therefore,

$$\mathbb{E}\partial_{qq}^2\eta_E(\mathbf{U})\mathbf{G}^2(\mathbf{U}) \leq A_0^\sharp\mathbb{E}(\eta_0(\mathbf{U}) + \eta_E(\mathbf{U})).$$

We will prove rigorously uniform bounds for approximate parabolic solutions in Section 3.2.2. The above formal computations are however sufficient for the moment to introduce the following definition.

Definition 2.1 (Entropy solution). Let $\rho_0, u_0 \in L^2(\mathbb{T})$ with $\rho_0 \geq 0$ a.e. and let $\mathbf{U}_0 = \begin{pmatrix} \rho_0 \\ \rho_0 u_0 \end{pmatrix}$ satisfy

$$\int_{\mathbb{T}} \rho_0(1 + u_0^2 + \rho_0^{2\theta})dx < +\infty.$$

A process $(\mathbf{U}(t))$ with values in $W^{-2,2}(\mathbb{T})$ is said to be a pathwise weak entropy solution to (1.1) with initial datum $\mathbf{U}_0$ if

1. almost surely, $\mathbf{U} \in C([0, T], W^{-2,2}(\mathbb{T}))$ and $(\mathbf{U}(t))$ is predictable,

2. almost surely, $\mathbf{U}$ is an integrable function on Q_T,

3. the bound

$$\mathbb{E} \operatorname*{ess\,sup}_{0 \leq t \leq T} \int_{\mathbb{T}} \eta(\mathbf{U}(x, t))dx < +\infty, \tag{2.15}$$

is satisfied for $\eta = \eta_E$, the energy defined in (2.13),

4. $\Phi(\mathbf{U})$ satisfies
$$\Phi(\mathbf{U}) \in L^2\big(\Omega \times [0,T], \mathcal{P}_T, d\mathbb{P} \times dt; L_2(\mathfrak{U}; L^2(\mathbb{T}))\big), \tag{2.16}$$
where $L_2(\mathfrak{U}; L^2(\mathbb{T}))$ is the space of Hilbert-Schmidt operators from $\mathfrak{U}$ into $L^2(\mathbb{T})$,

5. for any (η, H) given by (2.8)-(2.10), where $g \in C^2(\mathbb{R})$ is convex and subquadratic[2], for all $t \in (0, T]$, for all non-negative $\varphi \in C^1(\mathbb{T})$, and non-negative $\alpha \in C_c^1([0,t))$, the following entropy inequality is almost surely satisfied:

$$\int_0^t \big\langle \eta(\mathbf{U})(s), \varphi \big\rangle \alpha'(s) + \big\langle H(\mathbf{U})(s), \partial_x \varphi \big\rangle \alpha(s)\, ds$$
$$+ \frac{1}{2} \int_0^t \big\langle \mathbf{G}^2(x, \mathbf{U}) \partial_{qq}^2 \eta(\mathbf{U}), \varphi \big\rangle \alpha(s)\, ds + \big\langle \eta(\mathbf{U}_0), \varphi \big\rangle \alpha(0)$$
$$+ \sum_{k \geq 1} \int_0^t \big\langle \sigma_k(x, \mathbf{U}) \partial_q \eta(\mathbf{U}), \varphi \big\rangle \alpha(s)\, d\beta_k(s) \geq 0. \tag{2.17}$$

Remark 2.1. A pathwise weak entropy solution $\mathbf{U}$ is a priori a process $(\mathbf{U}(t))$ with values in $W^{-2,2}(\mathbb{T})$, a space of distributions. In item *2.* we require that, almost surely, $\mathbf{U}$ is an integrable function of (t, x): $\rho(x, t)$ and $q(x, t)$ are defined a.e. To give a sense to $\eta_E(\mathbf{U}(x, t))$, we need to know $u(x, t)$, or to be able to interpret the quotient $\frac{q(x,t)^2}{\rho(x,t)}$. We will prove the existence of a martingale weak entropy solution $\mathbf{U}$ to (1.1) (see Theorem (2.1)) satisfying $u = 0$ in the vacuum region $\rho = 0$ (see (5.37)). Note besides, to make an additional comment on Definition 2.1, that, with the choice $(\eta, H)(\mathbf{U}) = \pm(\rho, q)$, we infer from (2.17) the weak formulation of Equation (1.1).

Remark 2.2. By (2.16), the stochastic integral $t \mapsto \int_0^t \Phi(\mathbf{U})(s) dW(s)$ is a well defined process taking values in $L^2(\mathbb{T})$ (see [DPZ92] for the details of the construction). There is a little redundancy here in the definition of entropy solutions since, apart from the predictability, the integrability property (2.16) will follow from (2.2) and the bounds (2.15), *cf.* (2.14).

In Definition 2.1, the notion of solution considered is weak in space-time, strong with respect to ω. The following notion of solution is weak in (x, t, ω).

Definition 2.2 (Martingale solution). Let $\rho_0, u_0 \in L^2(\mathbb{T})$ with $\rho_0 \geq 0$ a.e. and let $\mathbf{U}_0 = \begin{pmatrix} \rho_0 \\ \rho_0 u_0 \end{pmatrix}$ satisfy
$$\int_{\mathbb{T}} \rho_0(1 + u_0^2 + \rho_0^{2\theta}) dx < +\infty.$$

A martingale weak entropy solution to (1.1) with initial datum $\mathbf{U}_0$ is a multiplet
$$(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t), \tilde{W}, \tilde{\mathbf{U}}),$$
where $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a probability space, with filtration $(\tilde{\mathcal{F}}_t)$ satisfying the usual conditions, $\tilde{W}$ a $(\tilde{\mathcal{F}}_t)$-cylindrical Wiener process, and $(\tilde{\mathbf{U}}(t))$ defines, according to Definition 2.1, a pathwise weak entropy solution to (1.1) with initial datum $\mathbf{U}_0$.

In summary, if after the substitution
$$\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W\big) \leftarrow \big(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}\big), \tag{2.18}$$

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[2]in the sense that g satisfies (5.1)

$\tilde{\mathbf{U}}$ is a pathwise weak entropy solution to (1.1), then we say that $\tilde{\mathbf{U}}$ (or, to be more rigorous, $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, (\tilde{\mathcal{F}}_t), \tilde{W}, \tilde{\mathbf{U}}))$ is a martingale weak entropy solution to (1.1). The substitution (2.18) leaves invariant the *law* of the resulting process $(\mathbf{U}(t))$. The fact is that we are in most cases interested only in the law of the process. An example is the discussion on the large time behaviour and invariant measures given in Section 6.

Theorem 2.1 (Main result). *Let $p \in \mathbb{N}$ satisfy $p \geq 4 + \frac{1}{2\theta}$. Assume that the structure and growth hypothesis (2.2) on the noise are satisfied. Let $\rho_0, u_0 \in L^2(\mathbb{T})$ with $\rho_0 \geq 0$ a.e. and let $\mathbf{U}_0 = \begin{pmatrix} \rho_0 \\ \rho_0 u_0 \end{pmatrix}$ satisfy*

$$\int_{\mathbb{T}} \rho_0(1 + u_0^{4p} + \rho_0^{4\theta p})dx < +\infty.$$

Then there exists a martingale solution to (1.1) with initial datum $\mathbf{U}_0$.

Remark 2.3. Our study and our results can be extended to the case where[3] $x \in \mathbb{R}$ instead of $x \in \mathbb{T}$, but is restricted to the space-dimension one. In space dimension above two, some of the regularity results for the heat equation which we use are not true any more. Furthermore, in space dimension above two, it is not possible to apply the div-curl lemma as done in Section 5). One may also wonder if it is possible to consider some different kind of noise, as a Lévy noise for example, in (1.1b). There is no theoretical obstruction to such an extension. The martingale method used in Section 3.2 and in Section 5.3.2 can be extended to this situation indeed, *cf.* [JS03, p. 155]. However, proving a version of Theorem 2.1 for Lévy noise requires a careful work and we do not claim the result to be true without a specific study.

2.4 Organization of the paper and main problematic

The paper is organized as follows. In Section 3, we prove the existence of strong solutions to the parabolic approximation of Problem (1.1), see Problem (3.1). The parabolic approximation to Problem (1.1) is a stochastic parabolic PDE with singularity at the state-value $\rho = 0$. To get existence of a solution to (3.1), we use a priori estimates: some are naturally furnished by the entropy balance equations, see Corollary 3.7, Corollary 3.11. These estimates are however of no use in the vacuum region $\{\rho = 0\}$ (observe that, indeed, a factor ρ is present in each of the estimates stated in Corollary 3.7, Corollary 3.11). For the isentropic Euler system, an estimate still of use in the vacuum region is an L^∞ estimate given by the invariance of some regions $\Lambda_\varkappa$ defined with the help of the Riemann invariants (see the definition of $\Lambda_\varkappa$ in (2.3)). In our stochastic setting, we can use such invariant regions provided the noise is compactly supported (but here the L^∞ estimates will be lost when $\varepsilon \to 0$). This is what we assume, see hypothesis (3.3). We need crucially this estimate "still of use in the vacuum" to prove the last a priori estimate necessary for the existence of a solution to the parabolic approximation (3.1), which is the positivity of the density, see Section 3.2.5. The positivity results is obtained by means of the regularizing effects of the heat equation. This is the subject of Appendix A.

All these a priori estimates are proved rigorously on an approximation of the solution to the parabolic approximation obtained by time splitting in Section 3.2. Once the existence of solutions to the parabolic approximation of Problem (1.1) has been proved, we want to take the limit on the regularizing parameter to obtain a martingale solution to (1.1). As in the deterministic case [DiP83a, DiP83b, LPS96], we use the concept of measure-valued solution (Young measure) to

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[3]Our discussion on the invariant measure in Section 6 makes sense in the periodic case only.

achieve this. In Section 4 we develop the tools on Young measure (in our stochastic framework) which are required. This is taken in part (but quite different) from Section 4.3 in [FN08]. We also use the probabilistic version of Murat's Lemma from [FN08, Appendix A], to identify the limiting Young measure. This is the content of Section 5, which requires two other fundamental tools: the analysis of the consequences of the div-curl lemma in [LPS96, Section I.5] and an identification result for densely defined martingales from [Hof13, Appendix A]. We obtain then the existence of a martingale solution to (1.1). In Section 6 we discuss the existence of invariant measures to (1.1). Besides, as explained above, we need at some point some bounds from below on solutions to (1-dimensional here) parabolic equations, which are developed in Appendix A. We also need some regularity results, with few variations, on the (1-dimensional) heat semi-group, and those are given in Appendix B.

Acknowledgements

3 Parabolic Approximation

For $\varepsilon > 0$, we consider the following second-order approximation to (1.1)

$$d\mathbf{U}_\varepsilon + \partial_x \mathbf{F}(\mathbf{U}_\varepsilon)dt = \varepsilon \partial_{xx}^2 \mathbf{U}_\varepsilon dt + \boldsymbol{\Psi}^\varepsilon(\mathbf{U}_\varepsilon)dW(t), \tag{3.1a}$$

$$\mathbf{U}_{\varepsilon|t=0} = \mathbf{U}_{\varepsilon 0}. \tag{3.1b}$$

Recall that $\mathbf{U}$ and $\mathbf{F}(\mathbf{U})$ are defined by

$$\mathbf{U} = \begin{pmatrix} \rho \\ q \end{pmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{pmatrix} q \\ \frac{q^2}{\rho} + p(\rho) \end{pmatrix}.$$

Problem (3.1) is a regularized version of Problem (1.1): this is a parabolic regularization of (1.1) and we will also assume more regularity than in (1.1) on the coefficients of the noise (see (3.5)). More precisely, as in (1.1) we assume that there is no noise in the evolution equation for ρ_ε: the first component of $\boldsymbol{\Psi}^\varepsilon(\mathbf{U}_\varepsilon)$ is 0. For each given $\mathbf{U}$, the second component is the map $\Phi^\varepsilon(\mathbf{U}) \colon \mathfrak{U} \to L^2(\mathbb{T})$ given by

$$[\Phi^\varepsilon(\rho, u)e_k](x) = \sigma_k^\varepsilon(x, \rho, u),$$

where σ_k^ε is a continuous function of its arguments. We assume (compare to (2.2))

$$\mathbf{G}^\varepsilon(x, \rho, u) := \left(\sum_{k \geq 1} |\sigma_k^\varepsilon(x, \rho, u)|^2 \right)^{1/2} \leq A_0 \rho \left[1 + u^2 + \rho^{2\theta} \right]^{1/2}, \tag{3.2}$$

for all $x \in \mathbb{T}$, $\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$. We will also assume that $\mathbf{G}^\varepsilon$ is supported in an invariant region: there exists $\varkappa_\varepsilon > 0$ such that

$$\operatorname{supp}(\mathbf{G}^\varepsilon) \subset \mathbb{T}_x \times \Lambda_{\varkappa_\varepsilon}, \tag{3.3}$$

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where the region $\Lambda_\varkappa$ is defined by (2.3). Note that this gives (3.2), but with a constant A_0 depending on $\varkappa_\varepsilon$: we have indeed

$$|\mathbf{G}^\varepsilon(x, \mathbf{U})| \leq M(\varkappa_\varepsilon), \tag{3.4}$$

for all $x \in \mathbb{T}$, $\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$. Note however that, in (3.2), A_0 is assumed independent on ε. Eventually, we will assume that the following Lipschitz condition is satisfied:

$$\sum_{k \geq 1} |\sigma_k^\varepsilon(x, \mathbf{U}_1) - \sigma_k^\varepsilon(x, \mathbf{U}_2)|^2 \leq C(\varepsilon, R)|\mathbf{U}_1 - \mathbf{U}_2|^2, \tag{3.5}$$

for all $x \in \mathbb{T}$, $\mathbf{U}_1, \mathbf{U}_2 \in \mathcal{U}_R$, where $C(\varepsilon, R)$ is a constant depending on ε and R. Here, for $R > 1$, $\mathcal{U}_R$ denotes the set of $\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$ such that

$$R^{-1} \leq \rho \leq R, \quad |q| \leq R. \tag{3.6}$$

We also denote by D_R be the set of functions $\mathbf{U} \in L^2(\mathbb{T})$ such that $\mathbf{U}(x) \in \mathcal{U}_R$ for a.e. $x \in \mathbb{T}$. Note that D_R is a closed subset of $L^2(\mathbb{T})$.

3.1 Pathwise solution to the parabolic problem

Definition 3.1 (Bounded solution to the parabolic approximation). Let $\mathbf{U}_0 \in L^\infty(\mathbb{T})$ satisfy $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$, where $c_0 > 0$. Let $T > 0$. Assume (3.2). A process $(\mathbf{U}(t))_{t \in [0,T]}$ with values in $(L^2(\mathbb{T}))^2$ is said to be a bounded solution to (3.1) if it is a predictable process such that

1. almost surely, $\mathbf{U} \in C([0, T]; L^2(\mathbb{T}))$,

2. $\mathbf{U} \in D_R$ with high probability, *i.e.* for all $\alpha > 0$, there exists $R > 0$ such that the probability of the event "for all $t \in [0, T]$, $\mathbf{U}(t) \in D_R$" is greater than $1 - \alpha$,

3. almost surely, for all $t \in [0, T]$, for all test function $\varphi \in C^2(\mathbb{T}; \mathbb{R}^2)$, the following equation is satisfied:

$$\langle \mathbf{U}(t), \varphi \rangle = \langle \mathbf{U}_0, \varphi \rangle + \int_0^t \langle \mathbf{F}(\mathbf{U}), \partial_x \varphi \rangle + \varepsilon \langle \mathbf{U}, \partial_{xx}^2 \varphi \rangle \, ds + \int_0^t \langle \mathbf{\Psi}^\varepsilon(\mathbf{U}) \, dW(s), \varphi \rangle. \tag{3.7}$$

Let us make some comments about item 2. in Definition 3.1. By continuity of $\mathbf{U}(t)$ with values in $L^2(\mathbb{T})$ and by continuity of the filtration $(\mathcal{F}_t)$, we can introduce the stopping time T_R defined by

$$T_R = \inf \{t \in [0, T]; \mathbf{U}(t) \notin D_R\} \tag{3.8}$$

(with the convention that $T_R = T$ if $\mathbf{U}(t) \in D_R$ for all $t \in [0, T]$), and then item 2. in Definition 3.1 is equivalent to

$$\lim_{R \to +\infty} \mathbb{P}(T_R < T) = 0. \tag{3.9}$$

We will prove the existence of pathwise solutions to the parabolic stochastic problem (3.1) satisfying uniform (or weighted) estimates with respect to ε. If η is an entropy function given by (2.8) with a convex function g of class C^2, we denote by

$$\Gamma_\eta(\mathbf{U}) = \int_{\mathbb{T}} \eta(\mathbf{U}(x)) dx,$$

the total entropy of a function $\mathbf{U} \colon \mathbb{T} \to \mathbb{R}^2$.

Theorem 3.1 (Existence of pathwise solution to (3.1))**.** *Let* $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ *satisfy* $\rho_{\varepsilon 0} \geq c_0$ *a.e. in* $\mathbb{T}$, *for a positive constant* c_0. *For* $m \in \mathbb{N}$, *let* η_m *denote the entropy associated to* $\xi \mapsto \xi^{2m}$ *by (2.8). Assume that hypotheses (3.2), (3.3), (3.5) are satisfied and that* $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$. *Then the problem (3.1) admits a unique bounded solution* $\mathbf{U}_\varepsilon$, *which has the following property:*

1. *it satisfies some moment estimates: for all* $m \in \mathbb{N}$,

$$
\mathbb{E} \sup_{t \in [0,T]} \int_{\mathbb{T}} \left(|u_\varepsilon|^{2m} + |\rho_\varepsilon|^{m(\gamma-1)} \right) \rho_\varepsilon dx = \mathcal{O}(1), \tag{3.10}
$$

   where $\mathcal{O}(1)$ *depends on* T, γ, *on the constant* A_0 *in (3.2), on* m *and on* $\mathbb{E}\Gamma_\eta(\mathbf{U}_{\varepsilon 0})$ *for* $\eta \in \{\eta_0, \eta_{2m}\}$,

2. *it satisfies the following gradient estimates: for all* $m \in \mathbb{N}$,

$$
\varepsilon\mathbb{E} \iint_{Q_T} \left(|u_\varepsilon|^{2m} + \rho_\varepsilon^{2m\theta} \right) \rho_\varepsilon^{\gamma-2} |\partial_x \rho_\varepsilon|^2 dxdt = \mathcal{O}(1), \tag{3.11}
$$

   and

$$
\varepsilon\mathbb{E} \iint_{Q_T} \left(|u_\varepsilon|^{2m} + \rho_\varepsilon^{2m\theta} \right) \rho_\varepsilon |\partial_x u_\varepsilon|^2 dxdt = \mathcal{O}(1), \tag{3.12}
$$

   where $\mathcal{O}(1)$ *depends on* T, γ, *on the constant* A_0 *in (3.2) and on the initial quantities* $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$ *for* $\eta \in \{\eta_0, \eta_{2m+2}\}$,

3. *the region* $\Lambda_{\varkappa_\varepsilon}$ *is an invariant region: a.s., for all* $t \in [0,T]$, $\mathbf{U}_\varepsilon(t) \in \Lambda_{\varkappa_\varepsilon}$.

Besides, $\mathbf{U}_\varepsilon$ *has the regularity* $L_\omega^2 C_t^\alpha W_x^{1,2}$ $(\alpha < 1/4)$ *and* $L_\omega^2 C_t^0 W_x^{2,2}$, *see (3.25)-(3.26), and* $\mathbf{U}_\varepsilon$ *satisfies the following entropy balance equation: for all entropy-entropy flux pair* (η, H) *where* η *is of the form (2.8) with a convex function* g *of class* C^2, *almost surely, for all* $t \in [0,T]$, *for all test function* $\varphi \in C^2(\mathbb{T})$,

$$
\langle \eta(\mathbf{U}_\varepsilon(t)), \varphi \rangle + \varepsilon \int_0^t \langle \eta''(\mathbf{U}_\varepsilon) \cdot (\partial_x \mathbf{U}_\varepsilon, \partial_x \mathbf{U}_\varepsilon), \varphi \rangle ds
$$

$$
= \langle \eta(\mathbf{U}_{\varepsilon 0}), \varphi \rangle + \int_0^t \left[\langle H(\mathbf{U}_\varepsilon), \partial_x \varphi \rangle + \varepsilon \langle \eta(\mathbf{U}_\varepsilon), \partial_x^2 \varphi \rangle \right] ds
$$

$$
+ \int_0^t \langle \eta'(\mathbf{U}_\varepsilon) \boldsymbol{\Psi}^\varepsilon(\mathbf{U}_\varepsilon) \, dW(s), \varphi \rangle
$$

$$
+ \frac{1}{2} \int_0^t \langle \mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)^2 \partial_{qq}^2 \eta(\mathbf{U}_\varepsilon), \varphi \rangle ds. \tag{3.13}
$$

To prove the existence of such pathwise solutions, we will prove first the existence of martingale solution and then use the Gyöngy-Krylov argument [GK96] to conclude (section 3.2.9). This means that we have to prove a result of pathwise uniqueness, which is given by the following theorem.

Theorem 3.2 (Uniqueness of bounded solution to (3.1))**.** *Let* $\mathbf{U}_{\varepsilon 0} \in L^\infty(\mathbb{T})$ *satisfy* $\rho_{\varepsilon 0} \geq c_0$ *a.e. in* $\mathbb{T}$, *for a positive constant* c_0. *Let* $T > 0$. *Assume that hypotheses (3.3), (3.5) are satisfied. Then, the problem (3.1) admits at most one bounded solution* $\mathbf{U}_\varepsilon$.

Proof. Let $S_\varepsilon(t) = S(\varepsilon^{-1}t)$, where $S(t)$ is the heat semi-group on $\mathbb{T}$. From the weak formulation (3.7) follows the mild formulation: almost surely, for all $t \in [0,T]$,

$$\mathbf{U}(t) = S_\varepsilon(t)\mathbf{U}_0 - \int_0^t \partial_x S_\varepsilon(t-s)\mathbf{F}(\mathbf{U}(s))ds + \int_0^t S_\varepsilon(t-s)\mathbf{\Psi}^\varepsilon(\mathbf{U}(s))\,dW(s), \qquad (3.14)$$

(see, *e.g.*, [Bal77] in the deterministic case and [GR00, Proposition 3.7] for a stochastic version of that result). Note that each member of (3.14) is almost surely in $C([0,T]; L^2(\mathbb{T}))$: this is the case of $\mathbf{U}$ by Definition 3.1; the term $S_\varepsilon(t)\mathbf{U}_0$ is deterministic and continuous in t with values in $L^2(\mathbb{T})$ by continuity of the semi-group $(S_\varepsilon(t))$. To prove the continuity of the two remaining terms in (3.14), let us set

$$\mathcal{T}_{\text{det}}\mathbf{U}(t) = \int_0^t \partial_x S_\varepsilon(t-s)\mathbf{F}(\mathbf{U}(s))ds,$$

$$\mathcal{T}_{\text{sto}}\mathbf{U}(t) = \int_0^t S_\varepsilon(t-s)\mathbf{\Psi}^\varepsilon(\mathbf{U}(s))\,dW(s).$$

Let $L(R)$ denote the Lipschitz constant of $\mathbf{F}$ on D_R. Let $\omega \in \Omega$ be such that $\mathbf{U}(x,t) \in D_R$ for a.e. $(x,t) \in Q_T$. Since $\mathbf{U}$ is a bounded solution, such a bound is satisfied for almost all ω, provided $R = R(\omega)$ is large enough (*cf.* (3.9)). By (B.4) with $j = 1$, $k = 1$, $p = 2$, we have, with $S(t)u = K_t * u$ (where K_t is defined in (B.3)),

$$\|\partial_x S_\varepsilon(t_2 - s)\mathbf{F}(\mathbf{U}(s)) - \partial_x S_\varepsilon(t_1 - s)\mathbf{F}(\mathbf{U}(s))\|_{L^2(\mathbb{T})}$$

$$\lesssim \|\partial_x S_\varepsilon(t_2 - s)\mathbf{F}(\mathbf{U}(s)) - \partial_x S_\varepsilon(t_1 - s)\mathbf{F}(\mathbf{U}(s))\|_{L^\infty(\mathbb{T})}$$

$$\leq \|\partial_x K_{\varepsilon(t_2-s)} - \partial_x K_{\varepsilon(t_1-s)}\|_{L^2(\mathbb{T})}\|\mathbf{F}(\mathbf{U}(s))\|_{L^\infty(\mathbb{T})}$$

$$\lesssim \varepsilon^{-7/4} \int_{t_1-s}^{t_2-s} t^{-7/4}dt \, \|\mathbf{F}\|_{L^\infty(D_R)}$$

$$\lesssim \varepsilon^{-7/4} \left[(t_2 - s)^{-3/4} - (t_1 - s)^{-3/4} \right] \|\mathbf{F}\|_{L^\infty(D_R)}.$$

Similarly, taking $j = 1$, $k = 0$, $p = 2$ in (B.4), we obtain

$$\left\| \int_{t_1}^{t_2} \partial_x S_\varepsilon(t-s)\mathbf{F}(\mathbf{U}(s))ds \right\|_{L^2(\mathbb{T})} \lesssim \varepsilon^{-1/2}(\sqrt{t_2} - \sqrt{t_1})\|\mathbf{F}\|_{L^\infty(D_R)}.$$

It follows that

$$\|\mathcal{T}_{\text{det}}\mathbf{U}(t_2) - \mathcal{T}_{\text{det}}\mathbf{U}(t_1)\|_{L^2(\mathbb{T})} \lesssim \varepsilon^{-7/4}\|\mathbf{F}\|_{L^\infty(D_R)}\delta_{\text{det}}(t_1, t_2), \qquad (3.15)$$

where

$$\delta_{\text{det}}(t_1, t_2) = \sqrt{t_2} - \sqrt{t_1} + \int_0^{t_1} \left[(t_2 - s)^{-3/4} - (t_1 - s)^{-3/4} \right] ds. \qquad (3.16)$$

We use the same kind of estimates to show the continuity of the stochastic term. Instead of fixed times t_1, t_2, let us consider some stopping times $T_1 \leq T_2$ satisfying $T_i \leq T$ a.s. for $i = 1, 2$. Recall (see Corollary 5.10 p.52 in [DD10] for example) that

$$\int_0^{T_i} S_\varepsilon(T_i - s)\mathbf{\Psi}^\varepsilon(\mathbf{U}(s))\,dW(s) = \int_0^T \mathbf{1}_{s \in [0, T_i]}S_\varepsilon(T_i - s)\mathbf{\Psi}^\varepsilon(\mathbf{U}(s))\,dW(s).$$

By Itō's Isometry and the bound (3.4), we have therefore

$$\mathbb{E} \left\| \mathcal{T}_{\text{sto}} \mathbf{U}(T_2) - \mathcal{T}_{\text{sto}} \mathbf{U}(T_1) \right\|_{L^2(\mathbb{T})}^2$$

$$= \mathbb{E} \int_{T_1}^{T_2} \| S_\varepsilon(T_2 - s) \mathbf{G}^\varepsilon(\mathbf{U}(s)) \|_{L^2(\mathbb{T})}^2 ds + \mathbb{E} \int_0^{T_1} \| \left[S_\varepsilon(T_2 - s) - S_\varepsilon(T_1 - s) \right] \mathbf{G}^\varepsilon(\mathbf{U}(s)) \|_{L^2(\mathbb{T})}^2 ds$$

$$\lesssim \mathbb{E}(T_2 - T_1) M(\varkappa_\varepsilon)^2 + \mathbb{E} \int_0^{T_1} \left| \varepsilon^{-5/4} \left[(T_2 - s)^{-1/4} - (T_1 - s)^{-1/4} \right] \right|^2 ds \, M(\varkappa_\varepsilon)^2$$

$$\lesssim \varepsilon^{-5/2} M(\varkappa_\varepsilon)^2 \mathbb{E} \delta_{\text{sto}}(T_1, T_2)^2, \tag{3.17}$$

where

$$\delta_{\text{sto}}(t_1, t_2)^2 = (t_2 - t_1) + \int_0^{t_1} \left[(t_2 - s)^{-1/4} - (t_1 - s)^{-1/4} \right]^2 ds. \tag{3.18}$$

Note that the estimate on $\mathcal{T}_{\text{det}} \mathbf{U}$ can also be adapted to the case where $t_i = T_i(\omega)$ for $T_1 \leq T_2$ some stopping times as above. In particular, we have

$$\mathbb{E} \left\| \mathcal{T}_{\text{det}} \mathbf{U}(T_2 \wedge T_R) - \mathcal{T}_{\text{det}} \mathbf{U}(T_1 \wedge T_R) \right\|_{L^2(\mathbb{T})}^2 \lesssim \varepsilon^{-7/2} \| \mathbf{F} \|_{L^\infty(D_R)}^2 \mathbb{E} \delta_{\text{det}}(T_1 \wedge T_R, T_2 \wedge T_R)^2, \tag{3.19}$$

where T_R is defined by (3.8).

Let σ be a stopping time such that $\sigma \leq T$ almost surely. If σ takes a finite number on values $\sigma_1, \dots, \sigma_n$, then by (3.14), almost surely on $\{\sigma = \sigma_k\}$, for all $t \in [0, \sigma_k]$, (3.14) is satisfied. Equivalently, we have: almost surely, for all $t \in [0, T]$,

$$\mathbf{U}(t \wedge \sigma) = S_\varepsilon(t \wedge \sigma) \mathbf{U}_0 - \int_0^{t \wedge \sigma} \partial_x S_\varepsilon(t \wedge \sigma - s) \mathbf{F}(\mathbf{U}(s)) ds$$

$$+ \int_0^{t \wedge \sigma} S_\varepsilon(t \wedge \sigma - s) \mathbf{\Psi}^\varepsilon(\mathbf{U}(s)) \, dW(s). \tag{3.20}$$

Let σ^n be a sequence of simple stopping times converging to σ in $L^1(\Omega)$ and such that $\sigma^n \geq \sigma$ for all n, e.g. $\sigma^n = 2^{-n} [2^n \sigma + 1]$, where $[t]$ is the integer part of t. If $\alpha > 0$, we have, by (3.19) and the Markov inequality, for $R > 0$,

$$\mathbb{P} \left[\| \mathcal{T}_{\text{det}} \mathbf{U}(\sigma^n) - \mathcal{T}_{\text{det}} \mathbf{U}(\sigma) \|_{L^2(\mathbb{T})} > \alpha \right] \lesssim \mathbb{P}(T_R < T) + \alpha^{-1} \varepsilon^{-7/4} \| \mathbf{F} \|_{L^\infty(D_R)} \mathbb{E} \delta_{\text{det}}(\sigma, \sigma^n).$$

Since $\mathbb{P}(T_R < T) \to 0$ when $R \to +\infty$, it follows that $\mathcal{T}_{\text{det}} \mathbf{U}(\sigma^n) \to \mathcal{T}_{\text{det}} \mathbf{U}(\sigma)$ in $L^2(\mathbb{T})$ in probability. Using (3.17), we can also pass to the limit in the stochastic term to show that (3.20) holds true when σ is a general stopping time.

Now we consider two bounded solutions $\mathbf{U}_1, \mathbf{U}_2$ to (3.1). Let $R > 1$ be such that $\mathbf{U}_{\varepsilon 0} \in D_R$, let

$$T_R^{1,2} = \inf \left\{ t \in [0, T]; \mathbf{U}^1(t) \text{ or } \mathbf{U}^2(t) \notin D_R \right\}.$$

By (B.6), we have: almost surely, for $0 \leq s \leq t \wedge T_R^{1,2}$,

$$\| \partial_x S_\varepsilon(t \wedge T_R^{1,2} - s) \left[\mathbf{F}(\mathbf{U}_1(s)) - \mathbf{F}(\mathbf{U}_2(s)) \right] \|_{L^2(\mathbb{T})}$$

$$\leq \varepsilon^{-1/2} (t \wedge T_R^{1,2} - s)^{-1/2} L(R) \sup_{s \in [0, t \wedge T_R^{1,2}]} \| \mathbf{U}_1(s) - \mathbf{U}_2(s) \|_{L^2(\mathbb{T})}.$$

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This gives

$$\mathbb{E}\left\|\mathcal{T}_{\det}\mathbf{U}_1(t \wedge T_R^{1,2}) - \mathcal{T}_{\det}\mathbf{U}_2(t \wedge T_R^{1,2})\right\|_{L^2(\mathbb{T})}^2$$
$$\leq 4\varepsilon^{-1}L(R)^2\, t\, \mathbb{E}\sup_{s \in [0,t]}\|\mathbf{U}_1(s \wedge T_R^{1,2}) - \mathbf{U}_2(s \wedge T_R^{1,2})\|_{L^2(\mathbb{T})}^2. \quad (3.21)$$

By Itō's Isometry and the bound (3.5), we have

$$\mathbb{E}\left\|\mathcal{T}_{\mathrm{sto}}\mathbf{U}_1(t \wedge T_R^{1,2}) - \mathcal{T}_{\mathrm{sto}}\mathbf{U}_2(t \wedge T_R^{1,2})\right\|_{L^2(\mathbb{T})}^2$$
$$\leq C(\varepsilon, R)\, t\, \mathbb{E}\sup_{s \in [0,t]}\|\mathbf{U}_1(s \wedge T_R^{1,2}) - \mathbf{U}_2(s \wedge T_R^{1,2})\|_{L^2(\mathbb{T})}^2. \quad (3.22)$$

It follows from (3.20), (3.21), (3.22) that

$$\mathbb{E}\sup_{s \in [0,t]}\|\mathbf{U}_1(s \wedge T_R^{1,2}) - \mathbf{U}_2(s \wedge T_R^{1,2})\|_{L^2(\mathbb{T})}^2$$
$$\leq \tilde{C}(\varepsilon, R)\, t\, \mathbb{E}\sup_{s \in [0,t]}\|\mathbf{U}_1(s \wedge T_R^{1,2}) - \mathbf{U}_2(s \wedge T_R^{1,2})\|_{L^2(\mathbb{T})}^2,$$

where $\tilde{C}(\varepsilon, R) = 4\varepsilon^{-1}L(R)^2 + C(\varepsilon, R)$. For $t < t_1 := 1/\tilde{C}(\varepsilon, R)$, we obtain: almost surely, $\mathbf{U}_1 = \mathbf{U}_2$ on the interval $[0, t_1 \wedge T_R^{1,2}]$. We then repeat the argument on the intervals $[kt_1, (k+1)t_1]$, $k = 1, \dots$ This is licit since the semi-group property shows that (3.20) holds true when starting from time t_1:

$$\mathbf{U}(t \wedge \sigma + t_1 \wedge \sigma) = S_\varepsilon(t \wedge \sigma)\mathbf{U}(t_1 \wedge \sigma) - \int_0^{t \wedge \sigma} \partial_x S_\varepsilon(t \wedge \sigma - s)\mathbf{F}(\mathbf{U}(s + t_1 \wedge \sigma))ds$$
$$+ \int_0^{t \wedge \sigma} S_\varepsilon((t \wedge \sigma)\mathbf{\Psi}^\varepsilon(\mathbf{U}(s + t_1 \wedge \sigma))\, dW(s).$$

This gives $\mathbf{U}_1 = \mathbf{U}_2$ a.s. on $[0, T_R^{1,2}]$. Since $T_R^{1,2} \to T$ almost surely as $R \to +\infty$, we conclude to $\mathbf{U}_1 = \mathbf{U}_2$ a.s. ∎

Remark 3.1. Assume $\mathbf{\Psi}^\varepsilon = 0$. In this deterministic case the stopping time T_R is deterministic: $\mathbb{P}(T_R = T) > 0$ implies $T_R = T$. For R large enough, and by the bound (3.21), we obtain the following estimate:

$$\sup_{t \in [0,T]}\|\mathbf{U}_1(t) - \mathbf{U}_2(t)\|_{L^2(\mathbb{T})} \leq C(T, R, \varepsilon)\|\mathbf{U}_1(0) - \mathbf{U}_2(0)\|_{L^2(\mathbb{T})},$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are two bounded solutions to Problem (3.1) and $C(T, R, \varepsilon)$ is a constant depending on T, R and ε.

In the following proposition, we use the fractional Sobolev space $W^{s,2}(\mathbb{T})$, defined in Appendix B.

Proposition 3.3 (Regularity of bounded solutions to (3.1))**.** *Let* $\mathbf{U}_{\varepsilon 0} \in W^{1,2}(\mathbb{T})$ *satisfy* $\rho_{\varepsilon 0} \geq c_0$ *a.e. in* $\mathbb{T}$*, for a positive constant* c_0*. Let* $T > 0$*. Assume that hypothesis (3.3) is satisfied. Let* $\mathbf{U}_\varepsilon$ *be a bounded solution to Problem (3.1). Then, for all* $\alpha \in [0, 1/4)$*,* $\mathbf{U}_\varepsilon(\cdot \wedge T_R)$ *has a modification whose trajectories are almost surely in* $C^\alpha([0, T]; L^2(\mathbb{T}))$ *and such that*

$$\mathbb{E}\|\mathbf{U}_\varepsilon(\cdot \wedge T_R)\|_{C^\alpha([0,T];L^2(\mathbb{T}))}^2 \leq C(R, \varepsilon, T, \alpha, \mathbf{U}_{\varepsilon 0}), \quad (3.23)$$

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where T_R is the exit time from D_R (see (3.8)) and $C(R, \varepsilon, T, \alpha)$ is a constant depending on R, T, ε, α and $\|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}$. Furthermore, for every $s \in [0, 1)$, $\mathbf{U}_\varepsilon$ satisfies the estimate

$$\sup_{t \in [0,T]} \mathbb{E}\|\mathbf{U}_\varepsilon(t \wedge T_R)\|_{W^{s,2}(\mathbb{T})}^2 \le C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0}) \tag{3.24}$$

where $C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0})$ is a constant depending on R, T, ε, s and $\|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}$.

If additionally $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ and the Lipschitz condition (3.5) is satisfied, then

$$\mathbb{E}\|\mathbf{U}_\varepsilon(t \wedge T_R)\|_{C^\alpha([0,T];W^{1,2}(\mathbb{T}))}^2 \le C(R, \varepsilon, T, \alpha, \mathbf{U}_{\varepsilon 0}), \tag{3.25}$$

for all $\alpha \in [0, 1/4)$, and

$$\sup_{t \in [0,T]} \mathbb{E}\|\mathbf{U}_\varepsilon(t \wedge T_R)\|_{W^{2,2}(\mathbb{T})}^2 \le C(R, \varepsilon, T, \mathbf{U}_{\varepsilon 0}), \tag{3.26}$$

where $C(R, \varepsilon, T, \mathbf{U}_{\varepsilon 0})$ is a constant depending on R, T, ε, on the constant $C(\varepsilon, R)$ in (3.5), and on $\|\mathbf{U}_{\varepsilon 0}\|_{W^{2,2}(\mathbb{T})}$.

Proof. **Step 1.** Note first that $\mathbf{U}_{\varepsilon 0} \in W^{1,2}(\mathbb{T})$ gives (see (B.9))

$$(x, t) \mapsto S_\varepsilon(t) \mathbf{U}_{\varepsilon 0}(x) \in C^{1/2}([0, T]; L^2(\mathbb{T})),$$

with

$$\|S_\varepsilon(t_2)\mathbf{U}_{\varepsilon 0} - S_\varepsilon(t_1)\mathbf{U}_{\varepsilon 0}\|_{L^2(\mathbb{T})} \lesssim \varepsilon^{-1/2}|t_2 - t_1|^{1/2}\|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}. \tag{3.27}$$

Next, to prove the Hölder regularity of $\mathbf{U}_\varepsilon$ in t, we use the estimates (3.17) and (3.19) established in the proof of Theorem 3.2. By (3.16) and (3.18), we have

$$\delta_{\det}(t_2, t_1) \le (t_2 - t_1)^{1/2} + \int_0^{+\infty}\left[(1+s)^{-3/4} - s^{-3/4}\right]ds\,(t_2 - t_1)^{1/4},$$

and

$$\delta_{\mathrm{sto}}(t_1, t_2)^2 \le (t_2 - t_1) + \int_0^{+\infty}\left[(1+s)^{-1/4} - s^{-1/4}\right]^2 ds\,(t_2 - t_1)^{1/2}.$$

It follows that

$$\mathbb{E}\|\mathbf{U}(t_2 \wedge T_R) - \mathbf{U}(t_1 \wedge T_R)\|_{L^2(\mathbb{T})}^2 \le C(R, \varepsilon, T, \mathbf{U}_{\varepsilon 0})\max\left(t_2 - t_1, (t_2 - t_1)^{1/2}\right), \tag{3.28}$$

for all $0 \le t_1 \le t_2 \le T$, where $C(R, \varepsilon, T, \mathbf{U}_{\varepsilon 0})$ is a constant depending on R, T, ε and $\|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}$. We can improve the bound (3.28) as follows: first, we deduce from (3.15) that, for all $k \ge 1$,

$$\mathbb{E}\left\|\mathcal{T}_{\det}\mathbf{U}(t_2 \wedge T_R) - \mathcal{T}_{\det}\mathbf{U}(t_1 \wedge T_R)\right\|_{L^2(\mathbb{T})}^{2k}$$
$$\lesssim \varepsilon^{-7k/2}\|\mathbf{F}\|_{L^\infty(D_R)}^{2k}\mathbb{E}\delta_{\det}(t_1 \wedge T_R, t_2 \wedge T_R)^{2k}$$
$$\le C(R, \varepsilon, T, k)\max\left((t_2 - t_1)^{k/2}, (t_2 - t_1)^k\right), \tag{3.29}$$

where $C(R, \varepsilon, T)$ is a constant depending on R, T, ε, k. By the Burkholder-Davis-Gundy inequality, we also have the following analogue to (3.17):

$$\mathbb{E}\left\|\mathcal{T}_{\mathrm{sto}}\mathbf{U}(T_2) - \mathcal{T}_{\mathrm{sto}}\mathbf{U}(T_1)\right\|_{L^2(\mathbb{T})}^{2k}$$

$$\lesssim \mathbb{E}\left[\int_{T_1}^{T_2} \|S_\varepsilon(T_2 - s)\mathbf{G}^\varepsilon(\mathbf{U}(s))\|_{L^2(\mathbb{T})}^2 ds\right]^k$$

$$+ \mathbb{E}\left[\int_0^{T_1} \|\left[S_\varepsilon(T_2 - s) - S_\varepsilon(T_1 - s)\right]\mathbf{G}^\varepsilon(\mathbf{U}(s))\|_{L^2(\mathbb{T})}^2 ds\right]^k$$

$$\lesssim \mathbb{E}(T_2 - T_1)^k M(\varkappa_\varepsilon)^{2k} + \mathbb{E}\left[\int_0^{T_1} \left|\varepsilon^{-5/4}\left[(T_2 - s)^{-1/4} - (T_1 - s)^{-1/4}\right]\right|^2 ds\right]^k M(\varkappa_\varepsilon)^{2k}$$

$$\leq C(R, \varepsilon, T, k) \max\left((T_2 - T_1)^{k/2}, (T_2 - T_1)^k\right), \tag{3.30}$$

where $C(R, \varepsilon, T, k)$ is a constant depending on R, T, ε, k. By (3.27), (3.29) and (3.30), we obtain

$$\mathbb{E}\|\mathbf{U}(t_2 \wedge T_R) - \mathbf{U}(t_1 \wedge T_R)\|_{L^2(\mathbb{T})}^{2k} \leq C(R, \varepsilon, T, k) \max\left((t_2 - t_1)^{k/2}, (t_2 - t_1)^k\right), \tag{3.31}$$

for all $0 \leq t_1 \leq t_2 \leq T$, where $C(R, \varepsilon, T, k, \mathbf{U}_{\varepsilon 0})$ is a constant depending on R, T, ε, k and $\|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}$. By the Kolmogorov's criterion, the existence of a modification with trajectories almost surely C^α and (3.23) follow from (3.31).

Step 2. The proof of the regularity in x of $\mathbf{U}_\varepsilon$ is also standard: by the contraction property, we have

$$\|S_\varepsilon(\cdot)\mathbf{U}_{\varepsilon 0}\|_{C([0,T];W^{1,2}(\mathbb{T}))} \leq \|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}. \tag{3.32}$$

Let $s \in (0, 1)$. To prove (3.24), we use the identity (B.13). By (B.8), we have

$$\|J^s \mathcal{T}_{\det}\mathbf{U}_\varepsilon(t \wedge T_R)\|_{L^2(\mathbb{T})} \leq C(R, \varepsilon, T, s), \tag{3.33}$$

$$\mathbb{E}\|J^s \mathcal{T}_{\mathrm{sto}}\mathbf{U}_\varepsilon(t \wedge T_R)\|_{L^2(\mathbb{T})}^2 \leq C(R, \varepsilon, T, s), \tag{3.34}$$

where $C(R, \varepsilon, T, s)$ is a constant depending on R, T, ε, s. Indeed, the left-hand side in (3.33) is bounded by

$$\int_0^t (t - r)^{-\frac{1+s}{2}} dr \, C(R, \varepsilon), \tag{3.35}$$

and the left-hand side in (3.34) is bounded by

$$\int_0^t (t - r)^{-s} dr \, C(R, \varepsilon), \tag{3.36}$$

where $C(R, \varepsilon)$ depends on R and ε. With (3.32), (3.33) and (3.34) give (3.24).

Step 3. Let us assume now that $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ and that the Lipschitz condition (3.5) is satisfied. By (B.12) and (3.24), we have

$$\sup_{t \in [0,T]} \mathbb{E}\|\mathbf{F}(\mathbf{U}_\varepsilon)(t \wedge T_R)\|_{W^{s,2}(\mathbb{T})}^2, \leq C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0}),$$

and

$$\sup_{t \in [0,T]} \mathbb{E}\|\mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)(t \wedge T_R)\|_{W^{s,2}(\mathbb{T})}^2 \leq C(R, \varepsilon, T, s, \mathbf{U}_\varepsilon)$$

where $C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0})$ is a constant depending on R, T, ε, s, $\|\mathbf{U}_{\varepsilon 0}\|_{W^{1,2}(\mathbb{T})}$ and also on $\mathbf{F}$ and on the constant $C(\varepsilon, R)$ in (3.5). Using the decompositions

$$J^{2s}\partial_x S_\varepsilon(t - r)\mathbf{F}(\mathbf{U}_\varepsilon) = J^s\partial_x S_\varepsilon(t - r)J^s\mathbf{F}(\mathbf{U}_\varepsilon),$$

and

$$J^{2s}\partial_x S_\varepsilon(t - r)\sigma_k(\mathbf{U}_\varepsilon) = J^s\partial_x S_\varepsilon(t - r)J^s\sigma_k(\mathbf{U}_\varepsilon),$$

we deduce as in (3.33)-(3.34) that, for all $s \in [\frac{1}{2}, 1)$, and for some constants $C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0})$ possibly varying from lines to lines,

$$\sup_{t\in[0,T]} \mathbb{E}\|J^{2s-1}J\mathcal{T}_{\det}\mathbf{U}_\varepsilon(t \wedge T_R)\|_{L^2(\mathbb{T})} \leq C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0})$$

and

$$\sup_{t\in[0,T]} \mathbb{E}\|J^{2s-1}J\mathcal{T}_{\mathrm{sto}}\mathbf{U}_\varepsilon(t \wedge T_R)\|_{L^2(\mathbb{T})} \leq C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0}).$$

This shows that

$$\sup_{t\in[0,T]} \mathbb{E}\|J\mathbf{U}_\varepsilon(t \wedge T_R)\|_{W^{2s-1,2}(\mathbb{T})} \leq C(R, \varepsilon, T, s, \mathbf{U}_{\varepsilon 0}).$$

In particular, almost surely,

$$\partial_x\mathbf{U}_\varepsilon(\cdot \wedge T_R) \in C([0, T]; W^{2s-1,2}(\mathbb{T})), \tag{3.37}$$

and $\partial_x\mathbf{U}_\varepsilon$ is solution to the fixed-point equation

$$\partial_x\mathbf{U}_\varepsilon = S_\varepsilon(\cdot)\partial_x\mathbf{U}_{\varepsilon 0} + \mathcal{T}_{\det}(D\mathbf{F}(\mathbf{U}_\varepsilon) \cdot \partial_x\mathbf{U}_\varepsilon) + \mathcal{T}_{\mathrm{sto}}(D\mathbf{F}(\mathbf{U}_\varepsilon) \cdot \partial_x\mathbf{U}_\varepsilon), \tag{3.38}$$

on $[0, T_R]$. By (3.38), we can estimate $J\partial_x\mathbf{U}_\varepsilon$. Indeed, (3.37) gives

$$\partial_x\mathbf{U}_\varepsilon(\cdot \wedge T_R) \in C([0, T] \times \mathbb{T}),$$

almost surely. Using (B.11), we obtain

$$\sup_{t\in[0,T]} \mathbb{E}\|\mathbf{U}_\varepsilon(t)\|^2_{W^{2,2}(\mathbb{T})} \leq C(R, \varepsilon, T, \mathbf{U}_{\varepsilon 0}), \tag{3.39}$$

and therefore (3.26). By (3.38) and (B.11) we obtain (3.25) by the same proof as (3.23). $\blacksquare$

Remark 3.2. By using (3.24) (*resp.* (3.39)) it is possible to improve (3.23) (*resp.* (3.25)) to the range $\alpha \in [0, 3/8)$. We will not need it anyhow.

3.2 Solution to the parabolic problem

3.2.1 Time splitting

To prove the existence of a solution to (3.1), we perform a splitting in time. Let $\tau > 0$. Set $t_k = k\tau$, $k \in \mathbb{N}$. We solve alternatively the deterministic, parabolic part of (3.1) on time intervals $[t_{2k}, t_{2k+1})$ and the stochastic part of (3.1) on time intervals $[t_{2k+1}, t_{2k+2})$, *i.e.*

- for $t_{2k} \leq t < t_{2k+1}$,

$$\partial_t\mathbf{U}^\tau + 2\partial_x\mathbf{F}(\mathbf{U}^\tau) = 2\varepsilon\partial^2_{xx}\mathbf{U}^\tau \qquad\qquad \text{in } Q_{t_{2k}, t_{2k+1}}, \tag{3.40a}$$

$$\mathbf{U}^\tau(t_{2k}) = \mathbf{U}^\tau(t_{2k}-) \qquad\qquad \text{in } \mathbb{T}, \tag{3.40b}$$

- for $t_{2k+1} \leq t < t_{2k+2}$,

$$d\mathbf{U}^\tau = \sqrt{2}\mathbf{\Psi}^{\varepsilon,\tau}(\mathbf{U}^\tau)dW(t) \qquad\qquad \text{in } Q_{t_{2k+1},t_{2k+2}}, \qquad (3.41a)$$

$$\mathbf{U}^\tau(t_{2k+1}) = \mathbf{U}^\tau(t_{2k+1}-) \qquad\qquad \text{in } \mathbb{T}. \qquad (3.41b)$$

Note that we took care to speed up the deterministic equation (3.40a) by a factor 2 and the stochastic equation (3.41a) by a factor $\sqrt{2}$. This rescaling procedure should yield a solution $\mathbf{U}^\tau$ consistent with the solution of (3.1) when $\tau \to 0$. In (3.41) we have also truncated (in the number of "modes") the coefficient $\mathbf{\Psi}^\varepsilon$ into a coefficient $\mathbf{\Psi}^{\varepsilon,\tau}$: we assume that, for a finite integer $K^\tau \geq 1$, for each $\rho \geq 0, u \in \mathbb{R}$, we have

$$[\Phi^{\varepsilon,\tau}(\rho,u)e_k](x) = \sigma_k^{\varepsilon,\tau}(x,\rho,u) := \zeta_{\alpha_\tau} * \sigma_k^\varepsilon(x,\rho,u)\mathbf{1}_{k \leq K^\tau}. \qquad (3.42)$$

Then $\mathbf{\Psi}^{\varepsilon,\tau}$ is defined as the vector with first component 0 and second component $\Phi^{\varepsilon,\tau}(\rho,u)$. Here α_τ is a sequence tending to 0 with τ and ζ_α is the regularizing kernel defined by

$$\zeta_\alpha(x,\rho,u) = \frac{1}{\alpha^3}\bar{\zeta}\left(\frac{x}{\alpha}\right)\bar{\zeta}\left(\frac{\rho}{\alpha}\right)\bar{\zeta}\left(\frac{u}{\alpha}\right),$$

where $\bar{\zeta}$ is the non-negative smooth density of a probability measure. To define the convolution product with respect to ρ in (3.42) we have set $\sigma_k^\varepsilon(x,\rho,u) = 0$ for $\rho \leq 0$: this is consistent with the bound (3.2) which gives $\sigma_k^\varepsilon(x,\rho,u) = 0$ for $\rho = 0$. We assume furthermore that $\bar{\zeta}$ is compactly supported in $\mathbb{R}_+$. Then, by (3.2), we have, for α_τ small enough,

$$\mathbf{G}^{\varepsilon,\tau}(x,\rho,u) := \left(\sum_{k \geq 1}|\sigma_k^{\varepsilon,\tau}(x,\rho,u)|^2\right)^{1/2} \leq 2A_0\rho\left[1 + u^2 + \rho^{2\theta}\right]^{1/2}, \qquad (3.43)$$

for all $x \in \mathbb{T}$, $\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$. Let us explain our arguments for obtaining (3.43). By the Jensen inequality, we have

$$|\sigma_k^{\varepsilon,\tau}(x,\rho,u)|^2 \leq \iiint \zeta_{\alpha_\tau}(x-y,\rho-r,u-v)|\sigma_k(y,r,v)|^2 dy dr dv.$$

Summing over k and using (3.2), we obtain

$$|\mathbf{G}^{\varepsilon,\tau}(x,\rho,u)|^2 \leq A_0^2 \iiint \zeta_{\alpha_\tau}(x-y,\rho-r,u-v)r^2(1+v^2+r^{2\theta})dy dr dv. \qquad (3.44)$$

As $\bar{\zeta}$ is compactly supported in $\mathbb{R}_+$, say in $[0,b]$, the integral with respect to r in (3.44) is on $r \leq \rho$. We also have $|v| \leq |u| + b\alpha_\tau$, hence $v^2 \leq 2u^2 + 2b^2\alpha_\tau^2$. It follows that $|\mathbf{G}^{\varepsilon,\tau}(x,\rho,u)|^2 \leq A_0^2\rho^2(1 + 2b^2\alpha_\tau^2 + 2u^2 + \rho^{2\theta})$. This gives (3.43) for α_τ small enough. By (3.3), we have also, for α_τ small enough with respect to $\varkappa_\varepsilon$,

$$\text{supp}(\mathbf{G}^{\varepsilon,\tau}) \subset \mathbb{T}_x \times \Lambda_{2\varkappa_\varepsilon}. \qquad (3.45)$$

Additionally, it follows from (3.4) and (3.5) that

$$|\mathbf{G}^{\varepsilon,\tau}(x,\mathbf{U})| \leq M(\varkappa_\varepsilon), \qquad (3.46)$$

and

$$\sum_{k \geq 1}|\sigma_k^{\varepsilon,\tau}(x,\mathbf{U}_1) - \sigma_k^{\varepsilon,\tau}(x,\mathbf{U}_2)|^2 \leq C(\varepsilon,R)|\mathbf{U}_1 - \mathbf{U}_2|^2, \qquad (3.47)$$

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or all $x \in \mathbb{T}$, $\mathbf{U}_1, \mathbf{U}_2 \in \mathbb{R}_+ \times \mathbb{R}$.

For further use, we note here that (3.43) gives

$$|\mathbf{G}^{\varepsilon,\tau}(x,\mathbf{U})|^2 \leq \rho A_0^\sharp(\eta_0(\mathbf{U}) + \eta_E(\mathbf{U})), \tag{3.48}$$

where $A_0^\sharp$ depends on A_0 and γ only (compare to (2.14)).

Let us define the following indicator functions

$$\mathbf{1}_{\mathrm{det}} = \sum_{k \geq 0} \mathbf{1}_{[t_{2k}, t_{2k+1})}, \quad \mathbf{1}_{\mathrm{sto}} = 1 - \mathbf{1}_{\mathrm{det}}, \tag{3.49}$$

which will be used to localize various estimates below.

Definition 3.2 (Pathwise solution to the splitting approximation). Let $\mathbf{U}_0 \in L^\infty(\mathbb{T})$ satisfy $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$, for a positive constant c_0. Let $T > 0$. A process $(\mathbf{U}(t))_{t \in [0,T]}$ with values in $L^2(\mathbb{T})$ is said to be a pathwise solution to (3.40)-(3.41) if it is a predictable process such that

1. almost surely, $\mathbf{U} \in C([0,T]; L^2(\mathbb{T}))$,

2. there exists some random variables c_{min}, C_{max} with values in $(0, +\infty)$ such that, almost surely,
$$c_{\mathrm{min}} \leq \rho \leq C_{\mathrm{max}}, \quad |q| \leq C_{\mathrm{max}} \text{ a.e. in } Q_T, \tag{3.50}$$

3. almost surely, for all $t \in [0,T]$, for all test function $\varphi \in C^2(\mathbb{T}; \mathbb{R}^2)$, the following equation is satisfied:

$$\langle \mathbf{U}(t), \varphi \rangle = \langle \mathbf{U}_0, \varphi \rangle + 2 \int_0^t \mathbf{1}_{\mathrm{det}}(s) \left[\langle \mathbf{F}(\mathbf{U}), \partial_x \varphi \rangle + \varepsilon \langle \mathbf{U}, \partial_{xx}^2 \varphi \rangle \right] ds$$
$$+ \sqrt{2} \int_0^t \mathbf{1}_{\mathrm{sto}}(s) \langle \mathbf{\Psi}^\varepsilon(\mathbf{U}) \, dW(s), \varphi \rangle. \tag{3.51}$$

Note that item 2. in Definition 3.2 implies a property similar to item 2. in Definition 3.1: $\mathbf{U} \in D_R$ with high probability. Indeed, if $\alpha > 0$ is given and R is chosen such that

$$\mathbb{P}(c_{\mathrm{min}} < R^{-1}) \leq \alpha, \quad \mathbb{P}(C_{\mathrm{max}} > R) \leq \alpha,$$

then the event "for all $t \in [0,T]$, $\mathbf{U}(t) \in D_R$" has probability greater than $1 - 2\alpha$.

Proposition 3.4 (Pathwise solution to the splitting approximation). *Let $T > 0$, let $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ satisfy $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume that (2.2) is satisfied. Then there exists a unique pathwise solution $\mathbf{U}^\tau$ to (3.40)-(3.41). Let $g \in C^2(\mathbb{R})$ be a convex function. Given an entropy-entropy flux pair (η, H) defined by (2.8)-(2.10), $\mathbf{U}^\tau$ satisfies the following entropy balance equation: almost surely, for all $t \in [0,T]$, for all test function $\varphi \in C^2(\mathbb{T})$,*

$$\langle \eta(\mathbf{U}^\tau(t)), \varphi \rangle = \langle \eta(\mathbf{U}_0), \varphi \rangle + 2 \int_0^t \mathbf{1}_{\mathrm{det}}(s) \left[\langle H(\mathbf{U}^\tau), \partial_x \varphi \rangle + \varepsilon \langle \eta(\mathbf{U}^\tau), \partial_{xx}^2 \varphi \rangle \right] ds$$
$$- 2\varepsilon \int_0^t \mathbf{1}_{\mathrm{det}}(s) \langle \eta''(\mathbf{U}^\tau) \cdot (\partial_x \mathbf{U}^\tau, \partial_x \mathbf{U}^\tau), \varphi \rangle ds$$
$$+ \sqrt{2} \int_0^t \mathbf{1}_{\mathrm{sto}}(s) \langle \eta'(\mathbf{U}^\tau) \mathbf{\Psi}^{\varepsilon,\tau}(\mathbf{U}^\tau) \, dW(s), \varphi \rangle$$
$$+ \int_0^t \mathbf{1}_{\mathrm{sto}}(s) \langle \mathbf{G}^{\varepsilon,\tau}(\mathbf{U}^\tau)^2 \partial_{qq}^2 \eta(\mathbf{U}^\tau), \varphi \rangle ds. \tag{3.52}$$

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Proof. The deterministic problem (3.40) is solved in [LPS96]: for Lipschitz continuous initial data (ρ_0, q_0) with an initial density ρ_0 uniformly positive, say $\rho_0 \geq c_0 > 0$ on $\mathbb{T}$, the Problem (3.40) admits a unique solution $\mathbf{U}$ in the class of functions

$$\mathbf{U} \in L^\infty(0, \tau, W^{1,\infty}(\mathbb{T})) \cap C([0, t_1]; L^2(\mathbb{T})); \quad \rho \geq c_1 \text{ on } \mathbb{T} \times [0, t_1].$$

Here $c_1 > 0$ is a constant depending continuously on t_1 and on c_0, $\|\rho_0\|_{L^\infty(\mathbb{T})}$, $\|q_0\|_{L^\infty(\mathbb{T})}$ (see Theorem A.1 and Remark A.1 in this paper for more details about this positivity result). By (3.26), we have $\mathbf{U}(t_1) \in W^{2,2}(\mathbb{T})$.

In a second step, we solve the stochastic problem (3.41) on the interval $[t_1, t_2]$. It is an ordinary stochastic differential equation. The coefficients of the noise in (3.42) are functions with bounded derivatives at all orders. Since $x \mapsto \rho^\tau(x, t_1)$ is in $W^{2,2}(\mathbb{T})$, we may rewrite the second equation of (3.41) as

$$dq = \sum_{k=1}^{K^\tau} g_k(x, q) d\beta_k(t), \tag{3.53}$$

where g_k satisfies

$$\partial_x^m \partial_q^l g_k \in L^\infty(\mathbb{R}; L^2(\mathbb{T})), \tag{3.54}$$

for all $l \geq 0$, $m \in \{0, 1, 2\}$. The existence of a solution to (3.53) on (t_1, t_2) with initial datum $q(x, t_1)$ at $t = t_1$ is ensured by a classical fixed point theorem, in the space of adapted functions

$$q \in C([t_1, t_2]; L^2(\Omega \times \mathbb{T})).$$

By differentiating once with respect to x in (3.53), we obtain

$$d(\partial_x q) = \sum_{k=1}^{K^\tau} \big(\partial_x g_k(x, q) + \partial_q g_k(x, q)(\partial_x q)\big) d\beta_k(t).$$

By the Itō Formula and the Gronwall Lemma, it follows that

$$\sup_{t \in [t_1, t_2]} \mathbb{E}\|\partial_x q\|_{L^p(\mathbb{T})}^p \leq C \mathbb{E}\|\partial_x q(t_1)\|_{L^p(\mathbb{T})}^p, \quad p \geq 2, \tag{3.55}$$

where the constant C depends on the function g_k's, on p and on τ. By differentiating again in (3.53), we have

$$d(\partial_{xx}^2 q) = \sum_{k=1}^{K^\tau} \big(\partial_{xx}^2 g_k(x, q) + 2\partial_{xq}^2 g_k(x, q)(\partial_x q) + \partial_{qq}^2 g_k(x, q)|\partial_x q|^2 + \partial_q g_k(x, q)(\partial_{xx}^2 q)\big) d\beta_k(t). \tag{3.56}$$

Using (3.55) with $p = 2$ and $p = 4$, the Itō Formula and the Gronwall Lemma, we obtain

$$\sup_{t \in [t_1, t_2]} \mathbb{E}\|\partial_{xx}^2 q\|_{L^2(\mathbb{T})}^2 \leq C\big(\mathbb{E}\|\partial_{xx}^2 q(t_1)\|_{L^2(\mathbb{T})}^2 + \mathbb{E}\|\partial_x q(t_1)\|_{L^2(\mathbb{T})}^2 + \mathbb{E}\|\partial_x q(t_1)\|_{L^4(\mathbb{T})}^4\big), \tag{3.57}$$

where the constant C depends on the function g_k's and on τ. By the Doob's Martingale Inequality, we have therefore

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \left\| \int_{t_1}^t \partial_q g_k(x, q(s)) \partial_{xx}^2 q(s) d\beta_k(s) \right\|_{L^2(\mathbb{T})}^2$$

$$\leq 4\mathbb{E} \left\| \int_{t_1}^{t_2} \partial_q g_k(x, q(s)) \partial_{xx}^2 q(s) d\beta_k(s) \right\|_{L^2(\mathbb{T})}^2$$

$$\leq C\big(\mathbb{E}\|\partial_{xx}^2 q(t_1)\|_{L^2(\mathbb{T})}^2 + \mathbb{E}\|\partial_x q(t_1)\|_{L^2(\mathbb{T})}^2 + \mathbb{E}\|\partial_x q(t_1)\|_{L^4(\mathbb{T})}^4\big).$$

Returning to (3.56), we deduce that

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \|\partial_{xx}^2 q\|_{L^2(\mathbb{T})}^2 \le C \big(\mathbb{E} \|\partial_{xx}^2 q(t_1)\|_{L^2(\mathbb{T})}^2 + \mathbb{E} \|\partial_x q(t_1)\|_{L^2(\mathbb{T})}^2 + \mathbb{E} \|\partial_x q(t_1)\|_{L^4(\mathbb{T})}^4 \big). \tag{3.58}$$

By a similar argument, using Doob's Martingale Inequality, we can improve (3.55) into

$$\mathbb{E} \sup_{t \in [t_1, t_2]} \|\partial_x q\|_{L^p(\mathbb{T})}^p \le C \mathbb{E} \|\partial_x q(t_1)\|_{L^p(\mathbb{T})}^p, \quad p \ge 2. \tag{3.59}$$

Note that differentiation in (3.53) has to be justified. The argument is standard: to obtain a solution to (3.53) which satisfies (3.59) and (3.58), we simply prove existence by using a fixed-point method in a smaller space, incorporating the bounds (3.59) and (3.58). By (3.58), we conclude that $\mathbf{U}(t_2) \in W^{2,2}(\mathbb{T})$. Of course $\rho(t_2) = \rho(t_1) \ge c_1$ a.e. on $\mathbb{T}$. The initial datum $\mathbf{U}(t_2)$ is therefore admissible with regard to the resolution of the deterministic problem (3.40) on Q_{t_2, t_3}. By iterating the procedure, we build $\mathbf{U}^\tau$ on the whole interval $[0, T]$. On intervals $[t_{2k+1}, t_{2k+2}]$ (stochastic evolution), the density ρ is unchanged. On intervals $[t_{2k}, t_{2k+1}]$ the positivity of ρ at $t = t_{2k}$ is preserved by Theorem A.1 and Remark A.1. Therefore there exists a random variable $c_{\min}$ (the possibility that it depends on τ is not excluded at this stage of the proof) such that, almost surely $\rho^\tau \ge c_{\min}$ a.e. on Q_T.

Regarding the measurability of $\mathbf{U}^\tau$, we observe that the function $\mathbf{U}^\tau(t_2)$ is $\mathcal{F}_{t_2}$-measurable. Since $\mathbf{U}^\tau(t_2) \mapsto (\mathbf{U}^\tau(t))_{t \in [t_2, t_3]}$ is Lipschitz continuous from $L^2(\mathbb{T})^2$ into $C([t_2, t_3]; L^2(\mathbb{T})^2)$ by Remark 3.1, the random variable $\mathbf{U}^\tau(t)$ is $\mathcal{F}_{t_2}$-measurable for every $t \in [t_2, t_3]$. In particular, $\mathbf{U}^\tau(t)$ is adapted on $[t_2, t_3]$. Repeating the argument, we obtain that $\mathbf{U}^\tau(t)$ is adapted. Since $\mathbf{U}^\tau$ is almost surely in $C([0, T]; L^2(\mathbb{T}))$, it has a modification which is predictable. For $2 \le m < +\infty$, let E be any of the Banach spaces $W^{1,2}(\mathbb{T})$ and $L^m(\mathbb{T})$. We can view $(\mathbf{U}^\tau(t))$ as a process taking values in E. By Pettis' Theorem [Yo80, Theorem p. 131], and since E is separable, the predictability of $(\mathbf{U}^\tau(t))$ is characterized by the fact that $\langle \mathbf{U}^\tau(t), \varphi \rangle$ is predictable for all φ in a dense subset D of the topological dual E^* of E. Set $D = L^2(\mathbb{T})$. Then D is dense in E^* whatever the choice of E. The fact that $(\mathbf{U}^\tau(t))$ is predictable as a process with values in $L^2(\mathbb{T})$ implies therefore that $(\mathbf{U}^\tau(t))$ is predictable when viewed as a process with values in $W^{1,2}(\mathbb{T})$ or $L^m(\mathbb{T})$, $m > 2$.

This achieves the proof of the existence of a pathwise solution $\mathbf{U}^\tau$ to (3.40)-(3.41). The uniqueness is a consequence of the uniqueness properties for the deterministic and the stochastic problems. Similarly, the entropy balance equation (3.52) is obtained by using the following entropy balance law on $[t_{2k}, t_{2k+1}]$:

$$\langle \eta(\mathbf{U}^\tau(t)), \varphi \rangle = \langle \eta(\mathbf{U}^\tau(t_{2k})), \varphi \rangle + 2 \int_{t_{2k}}^t \mathbf{1}_{\det}(s) \left[\langle H(\mathbf{U}^\tau), \partial_x \varphi \rangle + \varepsilon \langle \eta(\mathbf{U}^\tau), \partial_{xx}^2 \varphi \rangle \right] ds$$

$$- 2\varepsilon \int_{t_{2k}}^t \mathbf{1}_{\det}(s) \langle \eta''(\mathbf{U}^\tau) \cdot (\partial_x \mathbf{U}^\tau, \partial_x \mathbf{U}^\tau), \varphi \rangle ds, \tag{3.60}$$

for all $t \in [t_{2k}, t_{2k+1}]$, and by combining (3.60) with the identity

$$\langle \eta(\mathbf{U}^\tau(t)), \varphi \rangle = \langle \eta(\mathbf{U}^\tau(t_{2k+1})), \varphi \rangle + \sqrt{2} \int_{t_{2k+1}}^t \mathbf{1}_{\text{sto}}(s) \langle \eta'(\mathbf{U}^\tau) \mathbf{\Psi}^{\varepsilon, \tau}(\mathbf{U}^\tau) \, dW(s), \varphi \rangle$$

$$+ \int_{t_{2k+1}}^t \mathbf{1}_{\text{sto}}(s) \langle \mathbf{G}^{\varepsilon, \tau}(\mathbf{U}^\tau)^2 \partial_{qq}^2 \eta(\mathbf{U}^\tau), \varphi \rangle ds, \tag{3.61}$$

for all $t \in [t_{2k+1}, t_{2k+2}]$. We deduce (3.61) from the stochastic equation (3.41) (where x is a parameter) and the Itō Formula, which we sum against φ. This concludes the proof of the proposition. ∎

3.2.2 Entropy bounds

If $\eta \in C(\mathbb{R}^2)$ is an entropy and $\mathbf{U} \colon \mathbb{T} \to \mathbb{R}^2$, we denote by

$$\Gamma_\eta(\mathbf{U}) := \int_\mathbb{T} \eta(\mathbf{U}(x))dx$$

the averaged entropy of $\mathbf{U}$.

Proposition 3.5 (Entropy bounds)**.** *Let $m \in \mathbb{N}$. Let η_m be the entropy given by (2.8) with $g(\xi) = \xi^{2m}$. Let $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ be such that $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume that the growth condition (3.2) is satisfied and that*

$$\mathbb{E} \int_\mathbb{T} \left(\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2m}(\mathbf{U}_{\varepsilon 0}) \right) dx < +\infty.$$

Then the solution $\mathbf{U}^\tau$ to (3.40)-(3.41) satisfies the estimate

$$\mathbb{E} \sup_{t \in [0,T]} \Gamma_{\eta_m}(\mathbf{U}^\tau(t)) + 2\varepsilon \mathbb{E} \iint_{Q_T} \mathbf{1}_{\det} \eta_m''(\mathbf{U}^\tau) \cdot (\partial_x \mathbf{U}^\tau, \partial_x \mathbf{U}^\tau)dxdt = \mathcal{O}(1), \tag{3.62}$$

where the quantity denoted by $\mathcal{O}(1)$ depends only on T, γ, on the constant A_0 in (3.2), on m and on the initial quantities $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$ for $\eta \in \{\eta_0, \eta_{2m}\}$.

Proof. To prove Proposition 3.5 we will use the following result.

Lemma 3.6. *Let $m, n \in \mathbb{N}$. Then*

$$\rho(u^{2m} + \rho^{2m\theta}) = \mathcal{O}(1)\eta_m(\mathbf{U}), \quad \eta_m(\mathbf{U}) = \mathcal{O}(1)\left[\rho(u^{2m} + \rho^{2m\theta}) \right], \tag{3.63}$$

where $\mathcal{O}(1)$ depends on m;

$$\eta_m(\mathbf{U}) \cdot \eta_n(\mathbf{U}) = \mathcal{O}(1)\left[\rho\eta_{m+n}(\mathbf{U}) \right], \tag{3.64}$$

where $\mathcal{O}(1)$ depends on m and n;

$$\rho\eta_m(\mathbf{U}) = \mathcal{O}(1)\left[\eta_m(\mathbf{U}) + \eta_p(\mathbf{U}) \right], \tag{3.65}$$

for any $p \geq m + \frac{1}{2\theta}$, where $\mathcal{O}(1)$ depends on m and p, and

$$\eta_n(\mathbf{U}) = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right], \tag{3.66}$$

where $\mathcal{O}(1)$ depends on m and n if $0 \leq n \leq m$. Besides, Hypothesis (3.43) gives the following bounds:

$$\mathbf{G}^{\varepsilon,\tau}(\mathbf{U})^2|\partial_q \eta_m(\mathbf{U})|^2 = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_{2m}(\mathbf{U}) \right], \tag{3.67}$$

and

$$\mathbf{G}^{\varepsilon,\tau}(\mathbf{U})^2 \partial_{qq}^2 \eta_m(\mathbf{U}) = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_m(\mathbf{U}) \right]. \tag{3.68}$$

Proof. The second estimate in (3.63), the estimates (3.64) and (3.66), are all obtained by repeated applications of the Young Inequality. The first estimate in (3.63) is proved by developing the term $g(u + z\rho^\theta)$ in (2.11):

$$\eta_m(\mathbf{U}) = \rho c_\lambda \sum_{j=0}^{2m} \binom{2m}{j} \int_{-1}^1 u^j z^{2m-j} \rho^{\theta(2m-j)} (1-z^2)_+^\lambda dz. \tag{3.69}$$

The terms with odd index j in the sum in the right-hand side of (3.69) all vanish. Therefore only non-negative terms remain:

$$\eta_m(\mathbf{U}) \geq \rho c_\lambda \sum_{j \in \{0,2m\}} \binom{2m}{j} \int_{-1}^{1} u^j z^{2m-j} \rho^{\theta(2m-j)} (1-z^2)_+^\lambda dz$$
$$= \rho \left(d_\lambda(m) \rho^{2\theta m} + u^{2m} \right),$$

where the coefficient $d_\lambda(m)$ is given by

$$d_\lambda(m) = c_\lambda \int_{-1}^{1} z^{2m} (1-z^2)_+^\lambda dz.$$

Let us now give the details of the proof of (3.65): using (3.63), it is sufficient to get an estimate on $\rho^2(u^{2m} + \rho^{2m\theta})$. If $\rho \leq 1$, then $\eta_m(\mathbf{U})$ will provide an upper bound by (3.63) again. If $\rho \geq 1$, then $\rho^{2m\theta+1} \leq \rho^{2p\theta}$ and

$$\rho u^{2m} \leq \frac{\rho^\alpha}{\alpha} + \frac{u^{2m\beta}}{\beta}, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

Taking $\beta = \frac{p}{m}$ gives $\alpha = \frac{p}{p-m} \leq 2p\theta$, hence

$$\rho u^{2m} = \mathcal{O}(1) \left[u^{2p} + \rho^{2p\theta} \right]$$

since $\rho \geq 1$. We conclude to (3.65). To obtain (3.67) and (3.68), we observe that (3.43) is equivalent to

$$\mathbf{G}^{\varepsilon,\tau}(\mathbf{U})^2 = \mathcal{O}(1) \left[\rho \left(\eta_0(\mathbf{U}) + \eta_1(\mathbf{U}) \right) \right]. \tag{3.70}$$

Also, by (2.8) and (3.63), we have

$$|\partial_q \eta_m(\mathbf{U})|^2 = \mathcal{O}(1) \left[\frac{1}{\rho^2} \eta_{2m-1}(\mathbf{U}) \right], \quad \partial_{qq}^2 \eta_m(\mathbf{U}) = \mathcal{O}(1) \left[\frac{1}{\rho^2} \eta_{m-1}(\mathbf{U}) \right].$$

Using (3.64), (3.66), we deduce (3.67) and (3.68). ∎

We go on now with the proof of Proposition 3.5: we apply the entropy balance equation (3.52) to $\mathbf{U}^\tau$ with $\varphi \equiv 1$ and take expectation in both sides. This gives

$$\mathbb{E}\Gamma_{\eta_m}(\mathbf{U}^\tau(t)) + 2\varepsilon \mathbb{E} \iint_{Q_t} \mathbf{1}_{\det} \eta''(\mathbf{U}^\tau) \cdot (\partial_x \mathbf{U}^\tau, \partial_x \mathbf{U}^\tau) dx ds = \mathbb{E}\Gamma_{\eta_m}(\mathbf{U}_0^\tau) + \mathbb{E}R_{\eta_m}(t),$$

where

$$R_{\eta_m}(t) := \iint_{Q_t} \mathbf{1}_{\mathrm{sto}} \mathbf{G}^{\varepsilon,\tau}(\mathbf{U}^\tau)^2 \partial_{qq}^2 \eta_m(\mathbf{U}^\tau) dx ds$$

is the Itō correction term. If $m = 0$, then $\partial_{qq}^2 \eta = 0$ and we obtain (note the difference with (3.62) in the first term)

$$\sup_{t \in [0,T]} \mathbb{E}\Gamma_{\eta_0}(\mathbf{U}^\tau(t)) + 2\varepsilon \mathbb{E} \iint_{Q_T} \mathbf{1}_{\det} \eta_0''(\mathbf{U}^\tau) \cdot (\partial_x \mathbf{U}^\tau, \partial_x \mathbf{U}^\tau) dx dt = \mathcal{O}(1). \tag{3.71}$$

If $m \geq 1$, then (3.68) gives

$$\mathbb{E}R_{\eta_m}(t) = \mathcal{O}(1) \left[\int_0^t \mathbb{E}(\Gamma_{\eta_m}(\mathbf{U}^\tau(s)) + \Gamma_{\eta_0}(\mathbf{U}^\tau(s))) ds \right]. \tag{3.72}$$

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We use Gronwall's Lemma and (3.71) and deduce the following preliminary estimate

$$\sup_{t\in[0,T]} \mathbb{E}\Gamma_{\eta_m}(\mathbf{U}^\tau(t)) + 2\varepsilon\mathbb{E}\iint_{Q_T} \mathbf{1}_{\det}\eta_m''(\mathbf{U}^\tau)\cdot(\partial_x\mathbf{U}^\tau,\partial_x\mathbf{U}^\tau)dxdt = \mathcal{O}(1). \tag{3.73}$$

To prove (3.62), we have to take into account the noise term, *i.e.* we apply the entropy balance equation (3.52) to $\mathbf{U}^\tau$ with $\varphi \equiv 1$ and do not take expectation this time: we have then

$$0 \le \Gamma_{\eta_m}(\mathbf{U}^\tau(t)) = \Gamma_{\eta_m}(\mathbf{U}_0^\tau) + M_{\eta_m}(t) + R_{\eta_m}(t) - \varepsilon D_{\eta_m}(t) \tag{3.74}$$

where

$$M_{\eta_m}(t) = \sqrt{2}\sum_{k\ge 1}\int_0^t \mathbf{1}_{\mathrm{sto}}(s)\langle\sigma_k^{\varepsilon,\tau}(\mathbf{U}^\tau(s)),\partial_q\eta_m(\mathbf{U}^\tau(s))\rangle_{L^2(\mathbb{T})}d\beta_k(s)$$

and

$$D_{\eta_m}(t) = 2\iint_{Q_t} \mathbf{1}_{\det}\eta_m''(\mathbf{U}^\tau)\cdot(\partial_x\mathbf{U}^\tau,\partial_x\mathbf{U}^\tau)dxds.$$

Since $D_{\eta_m} \ge 0$, (3.74) gives

$$0 \le \Gamma_{\eta_m}(\mathbf{U}^\tau(t)) \le \Gamma_{\eta_m}(\mathbf{U}_0^\tau) + M_{\eta_m}(t) + R_{\eta_m}(t).$$

Similarly as for (3.72), we have

$$\mathbb{E}\sup_{t\in[0,T]}|R_{\eta_m}(t)| = \mathcal{O}(1)\left[\int_0^T \mathbb{E}(\Gamma_{\eta_m}(\mathbf{U}^\tau(s)) + \Gamma_{\eta_0}(\mathbf{U}^\tau(s)))ds\right],$$

and therefore, by (3.73), the last term R_{η_m} satisfies the bound

$$\mathbb{E}\sup_{t\in[0,T]}|R_{\eta_m}(t)| = \mathcal{O}(1).$$

By the Doob's Martingale Inequality, we also have

$$\mathbb{E}\sup_{t\in[0,T]}|M_{\eta_m}(t)| \le C\mathbb{E}\left(\int_0^T \sum_{k\ge 1}\langle\sigma_k^{\varepsilon,\tau}(\mathbf{U}^\tau(s)),\partial_q\eta_m(\mathbf{U}^\tau(s))\rangle_{L^2(\mathbb{T})}^2 ds\right)^{1/2}$$

$$\le C\mathbb{E}\left(\iint_{Q_T} \mathbf{G}^{\varepsilon,\tau}(\mathbf{U}^\tau)^2|\partial_q\eta_m(\mathbf{U}^\tau)|^2 dxds\right)^{1/2}$$

for a given constant C. By (3.67) and (3.73) (with $2m$ instead of m) we obtain

$$\mathbb{E}\sup_{t\in[0,T]}|M_\eta(t)| = \mathcal{O}(1).$$

This concludes the proof of the proposition. ∎

Corollary 3.7 (Bounds on the moments). *Let $m \in \mathbb{N}$. Let η_m be the entropy given by (2.8) with $g(\xi) = \xi^{2m}$. Let $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ be such that $\rho_0 \ge c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume that the growth condition (3.2) is satisfied and that*

$$\mathbb{E}\int_{\mathbb{T}} (\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2m}(\mathbf{U}_{\varepsilon 0}))\,dx < +\infty.$$

Then, the solution $\mathbf{U}^\tau$ to (3.40)-(3.41) satisfies:

$$\mathbb{E} \sup_{t \in [0,T]} \int_\mathbb{T} \left(|u^\tau|^{2m} + |\rho^\tau|^{m(\gamma-1)} \right) \rho^\tau \, dx = \mathcal{O}(1), \tag{3.75}$$

where $\mathcal{O}(1)$ depends only on T, γ, on the constant A_0 in (3.2), on m and on the initial quantities $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$ for $\eta \in \{\eta_0, \eta_{2m}\}$.

To conclude this part we complete Lemma 3.6 with the following result, which will be used later, in particular to get some estimates on the moments of entropy-entropy flux pairs.

Lemma 3.8. *For $m \in \mathbb{N}$, let (η_m, H_m) be the entropy-entropy flux pair associated to the function $g(\xi) = \xi^{2m}$ by (2.8)-(2.10). Let $s \geq 1$. Then*

$$|\eta_m(\mathbf{U})|^s = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})\right], \quad p \geq ms + \frac{s-1}{2\theta},$$

$$|H_m(\mathbf{U})|^s = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})\right], \quad p \geq (m+1/2)s + \frac{s-1}{2\theta},$$

$$|u\eta_m(\mathbf{U})|^s = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})\right], \quad p \geq (m+1/2)s + \frac{s-1}{2\theta},$$

$$|uH_m(\mathbf{U})|^s = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})\right], \quad p \geq (m+1)s + \frac{s-1}{2\theta},$$

where $\mathcal{O}(1)$ depends on m, s and the exponent p of each equation.

Proof. By (3.63), $|\eta_m(\mathbf{U})|^s = \mathcal{O}(1)\left[\rho^s|u|^{2ms} + \rho^{s+2m\theta s}\right]$. Let $\tilde{s} \geq ms$. By the Young Inequality, we have

$$\rho^s|u|^{2ms} \leq C_{s,\tilde{s}}\rho\left(|u|^{2\tilde{s}} + \rho^{\frac{(s-1)\tilde{s}}{\tilde{s}-ms}}\right). \tag{3.76}$$

Let $\tilde{s} = ms + \frac{s-1}{2\theta}$. If $p \geq \tilde{s}$, then

$$\frac{(s-1)\tilde{s}}{\tilde{s}-ms} = 2\theta\tilde{s} \leq 2\theta p$$

and (3.76) gives

$$\rho^s|u|^{2ms} = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})\right].$$

We also have

$$\rho^{s+2m\theta s} = \rho\rho^{2\theta\tilde{s}} = \mathcal{O}(1)\left[\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})\right]$$

and the first estimate follows. The proof of the three other estimates is similar. ∎

3.2.3 L^∞ estimates

Proposition 3.9 (L^∞ estimates). *Let $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ be such that $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume that the growth condition (3.2) and the localization condition (3.3) are satisfied and that $\mathbf{U}_0 \in \Lambda_{\varkappa_\varepsilon}$. Then the solution $\mathbf{U}^\tau$ to (3.40)-(3.41) satisfies: almost surely, for all $t \in [0,T]$, $\mathbf{U}^\tau(t) \in \Lambda_{2\varkappa_\varepsilon}$. In particular, almost surely, $\|u^\tau\|_{L^\infty(Q_T)} \leq 2\varkappa_\varepsilon$ and $\|\rho^\tau\|^\theta_{L^\infty(Q_T)} \leq 2\varkappa_\varepsilon$.*

Proof. We refer to [DiP83b, Section 4.] and [CCS77] for the proof that $\Lambda_\varkappa$ is an invariant region for (3.40). See also [Ser96] for a presentation of invariant regions. In (3.41), $\rho(t)$ is constant. Dividing by ρ the equation on $q = \rho u$, we deduce from (3.41) a stochastic differential equation on u. Using again that $\rho(t)$ is constant, this gives a stochastic differential equation on w with x as a parameter and similarly for z. By Hypothesis (3.3), we have the localization property (3.45) and the region $\Lambda_{2\varkappa_\varepsilon}$ is also an invariant domain for (3.41). It follows that, a.s., for all $t \in [0,T]$, $\mathbf{U}^\tau(t) \in \Lambda_{2\varkappa_\varepsilon}$. ∎

3.2.4 Gradient estimates

In Proposition 3.5 above, we have obtained an estimate on $\mathbf{U}_x^\tau$. In the case where $\eta = \eta_E$ is the energy (this corresponds to $g(\xi) = \frac{1}{2}\xi^2$), we have

$$\eta_E''(\mathbf{U}) \cdot (\partial_x \mathbf{U}, \partial_x \mathbf{U}) = \theta^2 |\rho|^{\gamma-2} |\partial_x \rho|^2 + \rho |\partial_x u|^2. \tag{3.77}$$

More generally, we prove the following weighted estimates (see in particular Corollary 3.11 below).

Proposition 3.10 (Gradient bounds). *Let $m \in \mathbb{N}$. Let η_m be the entropy given by (2.8) with $g(\xi) = \xi^{2m}$. Let $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ be such that $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume that the growth condition (3.2) is satisfied and that*

$$\mathbb{E} \int_{\mathbb{T}} (\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2m}(\mathbf{U}_{\varepsilon 0}))\, dx < +\infty.$$

Then, the solution $\mathbf{U}^\tau$ to (3.40)-(3.41) satisfies the estimate

$$\varepsilon \mathbb{E} \iint_{Q_T} \mathbf{1}_{\det}(t)\, G^{[2]}(\rho^\tau, u^\tau) \big[\theta^2 |\rho^\tau|^{\gamma-2} |\partial_x \rho^\tau|^2 + \rho^\tau |\partial_x u^\tau|^2\big] dx dt$$

$$\leq \varepsilon \mathbb{E} \iint_{Q_T} \mathbf{1}_{\det}(t)\, G^{[1]}(\rho^\tau, u^\tau) \big[2\theta |\rho^\tau|^{\frac{\gamma-2}{2}} |\partial_x \rho^\tau| \cdot |\rho^\tau|^{1/2} |\partial_x u^\tau|\big] dx dt + \mathcal{O}(1), \tag{3.78}$$

where

$$G^{[2]}(\rho, u) = c_\lambda \int_{-1}^1 g''(u + z\rho^\theta)(1 - z^2)_+^\lambda dz,$$

$$G^{[1]}(\rho, u) = c_\lambda \int_{-1}^1 |z| g''(u + z\rho^\theta)(1 - z^2)_+^\lambda dz,$$

and $\mathcal{O}(1)$ depends on T, γ, on the constant A_0 in (3.2) and on the initial quantities $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$ for $\eta \in \{\eta_0, \eta_{2m}\}$.

Proof. We introduce the probability measure

$$dm_\lambda(z) = c_\lambda(1 - z^2)_+^\lambda dz$$

and the 2×2 matrix

$$S = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

which satisfies

$$\partial_x \mathbf{U} = S\mathbf{W}, \quad \mathbf{W} := \begin{pmatrix} \partial_x \rho \\ \rho \partial_x u \end{pmatrix}. \tag{3.79}$$

By (3.62), we then have

$$\varepsilon \int_0^T \mathbb{E} \int_{\mathbb{T}} \mathbf{1}_{\det}(t) \langle S^* \eta''(\mathbf{U}^\tau) S\mathbf{W}, \mathbf{W} \rangle dx dt = \mathcal{O}(1), \tag{3.80}$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product on $\mathbb{R}^2$ and S^* is the adjoint of S for this scalar product. We compute

$$\eta''(\mathbf{U}) = \frac{1}{\rho} \int_{\mathbb{R}} \big[A(z)g'\left(u + z\rho^\theta\right) + B(z)g''\left(u + z\rho^\theta\right)\big] dm_\lambda(z),$$

where

$$A(z) = \begin{pmatrix} \frac{\gamma^2-1}{4} z \rho^\theta & 0 \\ 0 & 0 \end{pmatrix}, \quad B(z) = \begin{pmatrix} \left[-u + \theta z \rho^\theta\right]^2 & -u + \theta z \rho^\theta \\ -u + \theta z \rho^\theta & 1 \end{pmatrix}.$$

In particular

$$S^* A S(z) = \begin{pmatrix} \frac{\gamma^2-1}{4} z \rho^\theta & 0 \\ 0 & 0 \end{pmatrix}, \quad S^* B S(z) = \begin{pmatrix} \theta^2 z^2 \rho^{2\theta} & \theta z \rho^\theta \\ \theta z \rho^\theta & 1 \end{pmatrix},$$

and (3.79)-(3.80) give

$$\varepsilon \mathbb{E} \iint_{Q_T} \mathbf{1}_{\det}(t) \left(\mathbf{I} |\partial_x \rho^\tau|^2 + \mathbf{J} \partial_x \rho^\tau \cdot |\rho^\tau|^{1/2} \partial_x u^\tau + \mathbf{K} \rho^\tau |\partial_x u^\tau|^2 \right) dx dt = \mathcal{O}(1), \qquad (3.81)$$

where

$$\mathbf{I} = |\rho^\tau|^{2\theta-1} \int_{\mathbb{R}} \theta^2 z^2 g'' \left(u^\tau + z |\rho^\tau|^\theta \right) dm_\lambda(z) + |\rho^\tau|^{\theta-1} \int_{\mathbb{R}} \frac{\gamma^2-1}{4} z g' \left(u^\tau + z |\rho^\tau|^\theta \right) dm_\lambda(z),$$

and

$$\mathbf{J} = 2|\rho^\tau|^{\theta-\frac{1}{2}} \int_{\mathbb{R}} \theta z g'' \left(u^\tau + z |\rho^\tau|^\theta \right) dm_\lambda(z), \quad \mathbf{K} = \int_{\mathbb{R}} g'' \left(u^\tau + z |\rho^\tau|^\theta \right) dm_\lambda(z).$$

We observe that $2z \, dm_\lambda(z) = -\frac{c_\lambda}{\lambda+1} d(1-z^2)_+^{\lambda+1}$. By integration by parts, the second term in $\mathbf{I}$ can therefore be written

$$\frac{1}{\lambda+1} |\rho^\tau|^{2\theta-1} \int_{\mathbb{R}} \frac{\gamma^2-1}{8} (1-z^2) g'' \left(u^\tau + z |\rho^\tau|^{\frac{\gamma-1}{2}} \right) dm_\lambda(z).$$

Since $\frac{\gamma^2-1}{8} \frac{1}{\lambda+1} = \theta^2$, we have

$$\mathbf{I} = |\rho^\tau|^{2\theta-1} \int_{\mathbb{R}} \theta^2 g'' \left(u^\tau + z |\rho^\tau|^{\frac{\gamma-1}{2}} \right) dm_\lambda(z).$$

This gives (3.78). ∎

We apply (3.78) with $g(\xi) := |\xi|^{2m+2}$ and $\eta = \eta_{m+1}$ given by (2.8). Then we compute explicitly

$$G^{[2]}(\rho, u) - G^{[1]}(\rho, u) = (2m+2)(2m+1) \sum_{k=0}^{m} \binom{2m}{2k} a_k \rho^{2\theta k} u^{2(m-k)},$$

where the coefficients

$$a_k = c_\lambda \int_{-1}^{1} |z|^{2k} (1-|z|)(1-z^2)_+^\lambda dz$$

are positive. By letting m vary, we obtain the following result.

Corollary 3.11. *Let $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ be such that $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Let η_m be the entropy given by (2.8) with $g(\xi) = \xi^{2m}$. Assume that the growth condition (3.2) is satisfied and that*

$$\mathbb{E} \int_{\mathbb{T}} \left(\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2m}(\mathbf{U}_{\varepsilon 0}) \right) dx < +\infty.$$

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Then, the solution $\mathbf{U}^\tau$ *to (3.40)-(3.41) satisfies the estimate*

$$\varepsilon\mathbb{E}\iint_{Q_T} \mathbf{1}_{\det}(t)\left(|u^\tau|^{2m} + |\rho^\tau|^{2m\theta}\right)|\rho^\tau|^{\gamma-2}|\partial_x\rho^\tau|^2 dxdt = \mathcal{O}(1), \tag{3.82}$$

and

$$\varepsilon\mathbb{E}\iint_{Q_T} \mathbf{1}_{\det}(t)\left(|u^\tau|^{2m} + |\rho^\tau|^{2m\theta}\right)\rho^\tau|\partial_x u^\tau|^2 dxdt = \mathcal{O}(1), \tag{3.83}$$

for all $m \in \mathbb{N}$, *where* $\mathcal{O}(1)$ *depends on* T, γ, *on the constant* A_0 *in (2.2) and on the initial quantities* $\mathbb{E}\Gamma_\eta(\mathbf{U}_0)$ *for* $\eta \in \{\eta_0, \eta_{2m+2}\}$.

3.2.5 Positivity of the density

Proposition 3.12 (Positivity). *Let* $\mathbf{U}^\tau$ *be the solution to (3.40)-(3.41) with initial datum* $\mathbf{U}_0 = (\rho_0, q_0)$ *and assume that* ρ_0 *is uniformly positive: there exists* $c_0 > 0$ *such that* $\rho_0 \geq c_0$ *a.e. on* $\mathbb{T}$. *Let* $m > 3$. *Then there is a random variable* $c_{\min}$ *with values in* $(0, +\infty)$ *depending on* c_0, T,

$$\iint_{Q_T} \mathbf{1}_{\det}(t)\rho^\tau|\partial_x u^\tau|^2 dxdt \ \text{and} \ \iint_{Q_T} |u^\tau|^m dxdt \tag{3.84}$$

only (in the sense that $c_{\min}^{-1}$ *is a continuous non-decreasing function of these former quantities), such that, almost surely,*

$$\rho^\tau \geq c_{\min} \tag{3.85}$$

a.e. in $\mathbb{T} \times [0, T]$.

Proof. We apply Theorem A.1 proved in Appendix A. ∎

3.2.6 Regularity of U^τ

Proposition 3.9 and Corollary 3.11 give a control (on the expectancy) of the two quantities in (3.84) in Proposition 3.12. By the Markov inequality, we have

$$\mathbb{P}\left(\iint_{Q_T}\mathbf{1}_{\det}(t)\rho^\tau|\partial_x u^\tau|^2 dxdt \geq M \text{ or } \|u^\tau\|_{L^m(Q_T)} \geq M\right) \leq \frac{C(\varepsilon)}{M}$$

where the constant $C(\varepsilon)$ depend on ε and also on T, γ, on the constant A_0 in (2.2), and on $\|\mathbf{U}_0\|_{L^\infty(\mathbb{T})}$. Let $\alpha > 0$. Let $M > 0$ be such that $\frac{C(\varepsilon)}{M} \leq \alpha$. Let $R > 0$ be such that

$$\iint_{Q_T}\mathbf{1}_{\det}(t)\rho^\tau|\partial_x u^\tau|^2 dxdt \leq M \text{ and } \|u^\tau\|_{L^m(Q_T)} \leq M$$

implies $c_{\min} > R^{-1}$. Note that R is independent on τ. Without loss of generality, we can also assume $R \geq R_0$, where

$$R_0 = \max\left((2\varkappa_\varepsilon)^{\frac{1}{\theta}}, (2\varkappa_\varepsilon)^{1+\frac{1}{\theta}}\right).$$

From the bounds from above obtained in Proposition 3.9, we deduce the following result.

Proposition 3.13 ($\mathbf{U}^\tau$ is a bounded solution). *Let* $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ *be such that* $\rho_0 \geq c_0$ *a.e. in* $\mathbb{T}$ *for a given constant* $c_0 > 0$. *Assume that the growth condition (3.2) and the localization condition (3.3) are satisfied and that* $\mathbf{U}_0 \in \Lambda_{\varkappa_\varepsilon}$. *Then* $\mathbf{U}^\tau \in D_R$ *with high probability, uniformly with respect to* τ: *for all* $\alpha > 0$, *there exists* $R > 0$ *independent on* τ, *such that the event "for all* $t \in [0, T]$, $\mathbf{U}^\tau(t) \in D_R$" *has probability greater then* $1 - \alpha$.

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We use Proposition 3.13 in particular to obtain some estimates on Hölder or Sobolev norms of $\mathbf{U}^\tau$ independent on τ. We let T_R denote the exit time

$$T_R = \inf\left\{t \in [0,T]; \mathbf{U}^\tau(t) \notin D_R\right\}, \tag{3.86}$$

where D_R is defined in the paragraph that follows (3.6). By Proposition 3.13, we have

$$\lim_{R \to +\infty} \mathbb{P}(T_R = T) = 1 \tag{3.87}$$

uniformly in τ.

Proposition 3.14 (Regularity of $\mathbf{U}^\tau$). *Let $\mathbf{U}_0 \in (W^{2,2}(\mathbb{T}))^2$ be such that $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume that the growth condition (3.2) and the localization condition (3.3) are satisfied and that $\mathbf{U}_0 \in \Lambda_{\varkappa_\varepsilon}$. Let $\mathbf{U}^\tau$ be the solution to (3.40)-(3.41). Then, for all $\alpha \in (0, 1/4)$, $\mathbf{U}^\tau(\cdot \wedge T_R)$ has a modification whose trajectories are almost surely in $C^\alpha([0,T]; L^2(\mathbb{T}))$ and such that*

$$\mathbb{E}\|\mathbf{U}^\tau(\cdot \wedge T_R)\|^2_{C^\alpha([0,T]; L^2(\mathbb{T}))} \leq C(R, \varepsilon, T, \alpha, \mathbf{U}_0), \tag{3.88}$$

where $C(R, \varepsilon, T, \alpha)$ is a constant depending on R, T, ε, α, $\|\mathbf{U}_0\|_{W^{1,2}(\mathbb{T})}$ but not on τ. Furthermore, for every $s \in [0, 1)$, $\mathbf{U}^\tau$ satisfies the estimate

$$\sup_{t \in [0,T]} \mathbb{E}\|\mathbf{U}^\tau(t \wedge T_R)\|^2_{W^{s,2}(\mathbb{T})} \leq C(R, \varepsilon, T, s, \mathbf{U}_0), \tag{3.89}$$

where $C(R, \varepsilon, T, s, \mathbf{U}_0)$ is a constant depending on R, T, ε, s and $\|\mathbf{U}_0\|_{W^{1,2}(\mathbb{T})}$ but not on τ. If additionally $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ and the Lipschitz condition (3.5) is satisfied, then

$$\mathbb{E}\|\mathbf{U}^\tau(t \wedge T_R)\|^2_{C^\alpha([0,T]; W^{1,2}(\mathbb{T}))} \leq C(R, \varepsilon, T, \alpha, \mathbf{U}_{\varepsilon 0}), \tag{3.90}$$

for all $\alpha \in [0, 1/4)$, and

$$\sup_{t \in [0,T]} \mathbb{E}\|\mathbf{U}^\tau(t \wedge T_R)\|^2_{W^{2,2}(\mathbb{T})} \leq C(R, \varepsilon, T, \mathbf{U}_{\varepsilon 0}), \tag{3.91}$$

for some constants $C(R, \varepsilon, T, \alpha, \mathbf{U}_{\varepsilon 0})$ and $C(R, \varepsilon, T, \mathbf{U}_{\varepsilon 0})$ depending on α, R, T, ε, on the constant $C(\varepsilon, R)$ in (3.5), on $\|\mathbf{U}_{\varepsilon 0}\|_{W^{2,2}(\mathbb{T})}$, but not on τ.

Proof. We only give the sketch of the proof since this is very similar to the proof of Proposition 3.3. First, we establish, for $\mathbf{U}^\tau$, an identity analogous to (3.14). For Problem (3.40) we have the mild formulation

$$\mathbf{U}^\tau(t) = S_{2\varepsilon}(t - t_{2n})\mathbf{U}^\tau(t_{2n}) - 2\int_{t_{2n}}^t \partial_x S_{2\varepsilon}(t-s)\mathbf{F}(\mathbf{U}^\tau(s))ds \tag{3.92}$$

for $t_{2n} \leq t \leq t_{2n+1}$, and, for Problem (3.41), we have the integral formulation

$$\mathbf{U}^\tau(t) = \mathbf{U}^\tau(t_{2n+1}) + \sqrt{2}\int_{t_{2n+1}}^t \boldsymbol{\Psi}^{\varepsilon,\tau}(\mathbf{U}^\tau(s))\,dW(s), \tag{3.93}$$

for $t_{2n+1} \leq t \leq t_{2n+2}$. By combining (3.92) and (3.93), we obtain the identity

$$\mathbf{U}^\tau(t) = S_\varepsilon(t_\sharp)\mathbf{U}_0 - \int_0^{t_\sharp} \partial_x S_\varepsilon(t_\sharp - s)\mathbf{F}(\mathbf{U}^\tau(s_\flat))ds$$
$$+ \sqrt{2}\int_0^t \mathbf{1}_{\text{sto}}(s)S_\varepsilon(t_\sharp - s_\sharp)\boldsymbol{\Psi}^{\varepsilon,\tau}(\mathbf{U}^\tau(s))\,dW(s), \quad (3.94)$$

where we have set

$$t_\sharp = \min(2t - t_{2n}, t_{2n+2}), \quad t_\flat = \frac{t + t_{2n}}{2}, \quad t_{2n} \le t < t_{2n+2}.$$

Then we proceed as in the proof of Proposition 3.3. Note that $t \mapsto t_\sharp$ is 2-Lipschitz continuous and that we have the control (3.46), therefore (compare with (3.31)), $\mathbf{U}^\tau$ satisfies

$$\mathbb{E}\|\mathbf{U}^\tau(t \wedge T_R) - \mathbf{U}^\tau(s \wedge T_R)\|_{L^2(\mathbb{T})}^{2k} \le C(R, \varepsilon, T, k) \max\left((t-s)^{k/2}, (t-s)^k\right), \qquad (3.95)$$

for all $0 \le s \le t \le T$, where $C(R, \varepsilon, T, k, \mathbf{U}_{\varepsilon 0})$ is a constant depending on R, T, ε, k, $\|\mathbf{U}_0\|_{W^{1,2}(\mathbb{T})}$ but not on τ. This gives (3.88) by the Kolmogorov's criterion.

To obtain the regularity in x (3.89), we also proceed as in the proof of Proposition 3.3. Let $s \in [0, 1)$. The estimates (3.33)-(3.34) hold true here: the dependence on time being slightly different between (3.14) and (3.94), the bounds (3.35) and (3.36) have to be replaced by

$$\int_0^{t_\sharp} (t_\sharp - r)^{-\frac{1+s}{2}} dr \ C(R, \varepsilon), \qquad (3.96)$$

and

$$\int_0^t \mathbf{1}_{\mathrm{sto}}(r)(t_\sharp - r_\sharp)^{-s} dr \ C(R, \varepsilon), \qquad (3.97)$$

respectively. In (3.96), we have

$$\int_0^{t_\sharp} (t_\sharp - r)^{-\frac{1+s}{2}} dr \le \frac{2}{1-s} T^{\frac{1-s}{2}},$$

while, for $t_{2n} \le t \le t_{2n+2}$ (and thus $2n\tau \le T$), the integral term in (3.97) is

$$\int_0^t \mathbf{1}_{\mathrm{sto}}(r)(t_\sharp - r_\sharp)^{-s} dr = \sum_{k=1}^n \tau(t_{2k})^{-s} \le C(s) T^{1-s},$$

where $C(s)$ depends on s only. The proof of (3.90)-(3.91) is similar to the proof of the estimates (3.25)-(3.26) for the solution to (3.1), *cf.* the proof of Proposition 3.3. ∎

3.2.7 Compactness argument

We introduce the independent processes $X_1^\tau, X_2^\tau, \ldots$ defined by

$$X_k^\tau(t) = \sqrt{2} \int_0^t \mathbf{1}_{\mathrm{sto}}(s) d\beta_k(s)$$

and set

$$W^\tau(t) = \sum_{k \ge 1} X_k^\tau(t) e_k. \qquad (3.98)$$

The random variable $X_k(t)$ is Gaussian, with mean 0 and variance given by

$$\sigma_\tau^2(t) = t_{2n+1} + 2(t - t_{2n+2}), \quad t \in [t_{2n+1}, t_{2n+2}].$$

Let $0 \le s_1 \le \ldots \le s_m \le T$ be m given points in $[0, T]$. We have $|\sigma_\tau^2(t) - t| \le \tau$ for all $t \in [0, T]$, therefore the finite dimensional distribution of $(X_1^\tau(s_j))_{1,m}$ converges in law to the distribution

of $(\beta_1(s_i))_{1,m}$ when $\tau \to 0$. Besides, (X_1^τ) is tight in $C([0,T])$ since $\mathbb{E}\|X_1^\tau\|_{C^\alpha([0,T])}$ is bounded uniformly with respect to τ for any $\alpha < 1/2$. By [Bil99, Theorem 8.1], (X_1^τ) converges in law to β_1 on $C([0,T])$. We have the same result $X_k^\tau \to \beta_k$ in law for each $k \geq 2$, since the distributions are all the same.

Let $\mathfrak{U}_0$ be defined by (2.5) and let

$$\mathcal{X}_W = C([0,T];\mathfrak{U}_0) \tag{3.99}$$

denote the path space of W^τ. Since the embedding $\mathfrak{U} \hookrightarrow \mathfrak{U}_0$ is Hilbert-Schmidt, the $\mathcal{X}_W$-valued process W^τ converges in law to W when $\tau \to 0$ (again, we can use [Bil99, Theorem 8.1]).

Define the path space $\mathcal{X} = \mathcal{X}_\mathbf{U} \times \mathcal{X}_W$, where

$$\mathcal{X}_\mathbf{U} = C([0,T];L^2(\mathbb{T})).$$

Let us denote by $\mu_\mathbf{U}^\tau$ the law of $\mathbf{U}^\tau$ on $\mathcal{X}_\mathbf{U}$. The joint law of $\mathbf{U}^\tau$ and W^τ on $\mathcal{X}$ is denoted by μ^τ.

Proposition 3.15 (Tightness of (μ^τ)). *Let $\mathbf{U}_0 \in W^{2,2}(\mathbb{T})$ be such that $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ for a given constant $c_0 > 0$. Assume that the growth condition (3.2) and the localization condition (3.3) are satisfied and that $\mathbf{U}_0 \in \Lambda_{\varkappa_\varepsilon}$. Let $\mathbf{U}^\tau$ be the solution to (3.40)-(3.41). Then the set $\{\mu^\tau; \tau \in (0,1)\}$ is tight and therefore relatively weakly compact in the set of probability measures on $\mathcal{X}$.*

Proof. First, we prove tightness of $\{\mu_\mathbf{U}^\tau; \tau \in (0,1)\}$ in $\mathcal{X}_\mathbf{U}$. Let $\alpha \in (0,1/4)$ and $s \in (0,1)$. Then

$$K_M := \left\{\mathbf{U} \in \mathcal{X}_\mathbf{U}; \|\mathbf{U}\|_{C^\alpha([0,T];L^2(\mathbb{T}))} + \|\mathbf{U}\|_{L^2([0,T];W^{s,2}(\mathbb{T}))} \leq M\right\}$$

is compact in $\mathcal{X}_\mathbf{U}$ [Sim87]. Recall that the stopping time T_R is defined by (3.86). Note also that a consequence of the L_t^∞-estimate (3.89), is the L_t^2-estimate

$$\mathbb{E}\int_0^T \|\mathbf{U}^\tau(t \wedge T_R)\|_{W^{s,2}(\mathbb{T})}^2 dt \leq C(R,\varepsilon,T,s,\mathbf{U}_0),$$

which gives

$$\mathbb{E}\|\mathbf{U}^\tau(t \wedge T_R)\|_{L^2(0,T;W^{s,2}(\mathbb{T}))}^2 \leq C(R,\varepsilon,T,s,\mathbf{U}_0), \tag{3.100}$$

by the Fubini Theorem. By (3.88), (3.100) and the Markov inequality, we obtain the estimate

$$\mathbb{P}(\mathbf{U}^\tau \notin K_M) \leq \mathbb{P}(T_R < T) + \mathbb{P}(\mathbf{U}^\tau \notin K_M \ \& \ T_R = T)$$
$$\leq \mathbb{P}(T_R < T) + \frac{C(R,\varepsilon,T,\alpha,s,\mathbf{U}_0)}{M^2}.$$

By (3.87), given $\eta > 0$ there exists $R, M > 0$ independent on τ such that

$$\mu_\mathbf{U}^\tau(K_M) \geq 1 - \eta,$$

i.e. $(\mu_\mathbf{U}^\tau)$ is tight. We have proved above that the law μ_{W^τ} is tight. The set of the joint laws $\{\mu^\tau; \tau \in (0,1)\}$ is therefore tight. By Prohorov's theorem, it is relatively weakly compact. ∎

Let now (τ_n) be a sequence decreasing to 0. Up to a subsequence, there is a probability measure μ_ε on $\mathcal{X}$ such that (μ^{τ_n}) converges weakly to μ. By the Skorohod Theorem [Bil99, p. 70], we can assume almost sure convergence of the random variables by changing the probability space.

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Proposition 3.16. *There exists a probability space $(\tilde{\Omega}^\varepsilon, \tilde{\mathcal{F}}^\varepsilon, \tilde{\mathbb{P}}^\varepsilon)$, a sequence of $\mathcal{X}$-valued random variables $(\tilde{\mathbf{U}}^{\tau_n}, \tilde{W}^{\tau_n})_{n \in \mathbb{N}}$ and a $\mathcal{X}$-valued random variable $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$ such that*

   1. the laws of $(\tilde{\mathbf{U}}^{\tau_n}, \tilde{W}^{\tau_n})$ and $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$ under $\tilde{\mathbb{P}}^\varepsilon$ coincide with μ^{τ_n} and μ_ε respectively,

   2. $(\tilde{\mathbf{U}}^{\tau_n}, \tilde{W}^{\tau_n})$ converges $\tilde{\mathbb{P}}^\varepsilon$-almost surely to $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$ in the topology of $\mathcal{X}$.

We had dropped the variable ε in most of the quantities defined by the splitting scheme, in particular $\mathbf{U}^\tau$, W^τ, etc. We reintroduce the dependence on ε for the limits $\mathbf{U}_\varepsilon$, W_ε etc. to indicate the relation to Problem (3.1).

3.2.8   Identification of the limit

Our aim in this section is to identify the limit $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$ given by Proposition 3.16.

Let $(\tilde{\mathcal{F}}_t^\varepsilon)$ be the $\tilde{\mathbb{P}}^\varepsilon$-augmented canonical filtration of the process $(\tilde{\mathbf{U}}_\varepsilon, \tilde{W}_\varepsilon)$, *i.e.*

$$\tilde{\mathcal{F}}_t^\varepsilon = \sigma\big(\sigma\big(\varrho_t \tilde{\mathbf{U}}_\varepsilon, \varrho_t \tilde{W}_\varepsilon\big) \cup \big\{N \in \tilde{\mathcal{F}}^\varepsilon; \; \tilde{\mathbb{P}}^\varepsilon(N) = 0\big\}\big), \quad t \in [0, T],$$

where ϱ_t is the operator of restriction to the interval $[0, t]$ defined as follows: if E is a Banach space and $t \in [0, T]$, then

$$\begin{aligned} \varrho_t : C([0, T]; E) &\longrightarrow C([0, t]; E) \\ k &\longmapsto k|_{[0, t]}. \end{aligned} \tag{3.101}$$

Proposition 3.17 (Martingale solution to (3.1))**.** *The sextuplet*

$$\big(\tilde{\Omega}^\varepsilon, \tilde{\mathcal{F}}^\varepsilon, (\tilde{\mathcal{F}}_t^\varepsilon), \tilde{\mathbb{P}}^\varepsilon, \tilde{W}_\varepsilon, \tilde{\mathbf{U}}_\varepsilon\big)$$

is a martingale bounded solution to (3.1).

By martingale bounded solution, we mean the following:

$$\big(\tilde{\Omega}^\varepsilon, \tilde{\mathcal{F}}^\varepsilon, (\tilde{\mathcal{F}}_t^\varepsilon), \tilde{\mathbb{P}}^\varepsilon, \tilde{W}_\varepsilon\big)$$

is a stochastic basis and $\tilde{\mathbf{U}}_\varepsilon$ is a bounded solution, in the sense of Definition 3.1, to (3.1) after the substitution

$$\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, \mathbf{U}_\varepsilon\big) \leftarrow \big(\tilde{\Omega}^\varepsilon, \tilde{\mathcal{F}}^\varepsilon, (\tilde{\mathcal{F}}_t^\varepsilon), \tilde{\mathbb{P}}^\varepsilon, \tilde{W}_\varepsilon, \tilde{\mathbf{U}}_\varepsilon\big).$$

This substitution leaves invariant the law of the resulting process $(\mathbf{U}_\varepsilon(t))$.

The proof of Proposition 3.17 uses a method of construction of martingale solutions to SPDEs that avoids in part the use of representation Theorem. This technique has been developed in Ondreját [Ond10], Brzeźniak, Ondreját [BO11] and used in particular in Hofmanová, Seidler [HS12] and in [Hof13, DHV15].

Note first that item *1.* in Definition 3.1 is satisfied by the choice of the path space $\mathcal{X}_{\mathbf{U}}$ and that item *2.* follows from the convergence of $(\tilde{\mathbf{U}}^{\tau_n})$ to $(\tilde{\mathbf{U}}_\varepsilon)$ $\tilde{\mathbb{P}}^\varepsilon$-almost surely in the topology of $\mathcal{X}_{\mathbf{U}}$ and from (3.87). Our objective is now to prove item *3.* in Definition 3.1.

Recall that $\mathbf{F}$, the flux function in Equation (1.1), is defined by (2.6). Let us define for all $t \in [0, T]$ and a test function $\varphi = (\varphi_1, \varphi_2) \in C^\infty(\mathbb{T}; \mathbb{R}^2)$,

$$M^\tau(t) = \langle \mathbf{U}^\tau(t), \varphi \rangle - \langle \mathbf{U}_{\varepsilon 0}, \varphi \rangle - 2 \int_0^t \mathbf{1}_{\det}(s) \langle \mathbf{F}(\mathbf{U}^\tau), \partial_x \varphi \rangle + \varepsilon \langle \mathbf{U}^\tau, \partial_{xx}^2 \varphi \rangle \, ds,$$

$$\tilde{M}^\tau(t) = \langle \tilde{\mathbf{U}}^\tau(t), \varphi \rangle - \langle \tilde{\mathbf{U}}_{\varepsilon 0}, \varphi \rangle - 2 \int_0^t \mathbf{1}_{\det}(s) \langle \mathbf{F}(\tilde{\mathbf{U}}^\tau), \partial_x \varphi \rangle + \varepsilon \langle \tilde{\mathbf{U}}^\tau, \partial_{xx}^2 \varphi \rangle \, ds,$$

$$\tilde{M}_\varepsilon(t) = \langle \tilde{\mathbf{U}}_\varepsilon(t), \varphi \rangle - \langle \tilde{\mathbf{U}}_{\varepsilon 0}, \varphi \rangle - \int_0^t \langle \mathbf{F}(\tilde{\mathbf{U}}_\varepsilon), \partial_x \varphi \rangle + \langle \tilde{\mathbf{U}}_\varepsilon, \partial_{xx}^2 \varphi \rangle \, ds.$$

The proof of Proposition 3.17 will be a consequence of the following two lemmas.

Lemma 3.18. *The process $\tilde{W}_\varepsilon$ has a modification which is a $(\tilde{\mathcal{F}}_t^\varepsilon)$-adapted $\mathfrak{U}_0$-cylindrical Wiener process, and there exists a collection of mutually independent real-valued $(\tilde{\mathcal{F}}_t^\varepsilon)$-Wiener processes $\{\tilde{\beta}_k^\varepsilon\}_{k\geq 1}$ such that*

$$\tilde{W}_\varepsilon = \sum_{k\geq 1} \tilde{\beta}_k^\varepsilon e_k \tag{3.102}$$

in $C([0,T]; \mathfrak{U}_0)$

Proof. it is clear that $\tilde{W}_\varepsilon$ is a $\mathfrak{U}_0$-cylindrical Wiener process (this notion is stable by convergence in law; actually it can be characterized in terms of the law of $\tilde{W}_\varepsilon$ uniquely if we drop the usual hypothesis of a.s. continuity of the trajectories. This latter can be recovered, after a possible modification of the process, by using the Kolmogorov's Theorem). Also $\tilde{W}_\varepsilon$ is $(\tilde{\mathcal{F}}_t^\varepsilon)$-adapted by definition of the filtration $(\tilde{\mathcal{F}}_t^\varepsilon)$. By [DPZ92, Proposition 4.1], we obtain the decomposition (3.102). ∎

Lemma 3.19. *The processes $\tilde{M}_\varepsilon$,*

$$\tilde{M}_\varepsilon^2 - \sum_{k\geq 1} \int_0^\cdot \left\langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon), \varphi_2 \right\rangle^2 dr \quad and \quad \tilde{M}_\varepsilon \tilde{\beta}_k^\varepsilon - \int_0^\cdot \left\langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon), \varphi_2 \right\rangle dr,$$

are $(\tilde{\mathcal{F}}_t^\varepsilon)$-martingales.

Proof. We fix some times $s,t \in [0,T]$, $s \leq t$, and a continuous function

$$\gamma : C\big([0,s]; L^2(\mathbb{T})\big) \times C\big([0,s]; \mathfrak{U}_0\big) \longrightarrow [0,1].$$

Let us set $\tilde{X}_k^\tau = \langle \tilde{W}^\tau, e_k \rangle_{\mathfrak{U}_\circ}$. For all $\tau \in (0,1)$, the process

$$M^\tau = \sum_{k\geq 1} \int_0^\cdot \left\langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \right\rangle dX_k^\tau(r)$$

is a square integrable $(\mathcal{F}_t)$-martingale and therefore

$$M_2^\tau := (M^\tau)^2 - \sum_{k\geq 1} \int_0^\cdot \left\langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \right\rangle^2 d\langle\!\langle X^\tau \rangle\!\rangle(r),$$

and

$$M_3^\tau := M^\tau \beta_k - \int_0^\cdot \left\langle \sigma_k^\tau(\mathbf{U}^\tau), \varphi_2 \right\rangle d\langle\!\langle X^\tau \rangle\!\rangle(r)$$

are $(\mathcal{F}_t)$-martingales, where we have denoted by

$$\langle\!\langle X^\tau \rangle\!\rangle(t) = 2 \int_0^t \mathbf{1}_{\mathrm{sto}}(r) dr$$

the quadratic variation of X_k^τ (note that $\langle\!\langle X^\tau \rangle\!\rangle(t) \to t$ when $\tau \to 0$). Besides, it follows from the equality of laws that

$$\tilde{\mathbb{E}}^\varepsilon \gamma\big(\varrho_s \tilde{\mathbf{U}}^\tau, \varrho_s \tilde{W}^\tau\big)\big[\tilde{M}^\tau(t) - \tilde{M}^\tau(s)\big] = \mathbb{E} \gamma\big(\varrho_s \mathbf{U}^\tau, \varrho_s W^\tau\big)\big[M^\tau(t) - M^\tau(s)\big].$$

hence

$$\tilde{\mathbb{E}}^\varepsilon \, \gamma\big(\varrho_s \tilde{\mathbf{U}}^{\tau_n}, \varrho_s \tilde{W}^{\tau_n}\big)\big[\tilde{M}^{\tau_n}(t) - \tilde{M}^{\tau_n}(s)\big] = 0,$$

for all n. We can pass to the limit in this equation, due to the moment estimates (3.75) and the Vitali convergence theorem. We obtain

$$\tilde{\mathbb{E}}^\varepsilon \, \gamma\big(\varrho_s \tilde{\mathbf{U}}_\varepsilon, \varrho_s \tilde{W}_\varepsilon\big)\big[\tilde{M}_\varepsilon(t) - \tilde{M}_\varepsilon(s)\big] = 0,$$

i.e. $\tilde{M}_\varepsilon$ is a $(\tilde{\mathcal{F}}_t^\varepsilon)$-martingale. We proceed similarly to show that

$$\tilde{M}_{\varepsilon 2} := \tilde{M}_\varepsilon^2 - \sum_{k \geq 1} \int_0^{\cdot} \big\langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon), \varphi_2 \big\rangle^2 \, dr$$

is a $(\tilde{\mathcal{F}}_t^\varepsilon)$-martingale, by passing to the limit in the identity

$$\tilde{\mathbb{E}}^\varepsilon \, \gamma\big(\varrho_s \tilde{\mathbf{U}}^\tau, \varrho_s \tilde{W}^\tau\big)\big[\tilde{M}_2^\tau(t) - \tilde{M}_2^\tau(s)\big] = 0,$$

and again similarly for

$$\tilde{M}_{\varepsilon 3} := \tilde{M}_\varepsilon \tilde{\beta}_k^\varepsilon - \int_0^{\cdot} \big\langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon), \varphi_2 \big\rangle \, dr.$$

This concludes the proof of Lemma 3.19. ■

Proof of Proposition 3.17. Once the above lemmas are established, we can show that

$$\tilde{\mathbb{E}}^\varepsilon \left[\Big(\tilde{M}_\varepsilon(t) - \tilde{M}_\varepsilon(s)\Big) \int_s^t \langle h \, d\tilde{W}_\varepsilon(\sigma), \varphi_2 \rangle - \sum_{k \geq 1} \int_s^t \langle h(\sigma) e_k, \varphi_2 \rangle \langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon)(\sigma), \varphi_2 \rangle d\sigma \,\Big|\, \tilde{\mathcal{F}}_s^\varepsilon \right] = 0,$$

$$(3.103)$$

for all $(\tilde{\mathcal{F}}_t^\varepsilon)$-predictable, $L_2(\mathfrak{U}, L^2(\mathbb{T}))$-valued process satisfying

$$\int_0^T \|h(\sigma)\|_{L_2(\mathfrak{U}, L^2(\mathbb{T}))}^2 \, d\sigma < +\infty. \tag{3.104}$$

Here, if H is a given Hilbert space, $L_2(\mathfrak{U}, H)$ is the set of Hilbert-Schmidt operators $\mathfrak{U} \to H$. In particular, in (3.104), we have

$$\|h(\sigma)\|_{L_2(\mathfrak{U}, L^2(\mathbb{T}))}^2 = \sum_{k \geq 1} \|h(\sigma) e_k\|_{L^2(\mathbb{T})}^2.$$

Equation (3.103) is proved in [Hof13, Proposition A.1]. Taking $s = 0$ and $h = \Phi^\varepsilon(\tilde{\mathbf{U}}_\varepsilon)$ in (3.103), we obtain

$$\tilde{\mathbb{E}}^\varepsilon \sum_{k \geq 1} \left[\tilde{M}_\varepsilon(t) \int_0^t \langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon), \varphi_2 \rangle ds - \int_0^t \langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon), \varphi_2 \rangle^2 ds \right] = 0.$$

This shows that

$$\tilde{\mathbb{E}}^\varepsilon \left[\tilde{M}_\varepsilon(t) - \sum_{k \geq 1} \int_0^t \big\langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon) \, d\tilde{\beta}_k^\varepsilon(s), \varphi_2 \big\rangle \right]^2 = 0. \tag{3.105}$$

Accordingly, we have

$$\langle \tilde{\mathbf{U}}_\varepsilon(t), \varphi \rangle = \langle \tilde{\mathbf{U}}_{\varepsilon 0}, \varphi \rangle + \int_0^t \langle \mathbf{F}(\tilde{\mathbf{U}}_\varepsilon), \partial_x \varphi \rangle + \langle \tilde{\mathbf{U}}_\varepsilon, \partial_x^2 \varphi \rangle \, ds$$

$$+ \sum_{k \geq 1} \int_0^t \langle \sigma_k^\varepsilon(\tilde{\mathbf{U}}_\varepsilon) \, d\tilde{\beta}_k^\varepsilon, \varphi_2 \rangle, \quad t \in [0, T], \ \ \tilde{\mathbb{P}}^\varepsilon\text{-a.s.},$$

and this gives the weak formulation (3.7) $\tilde{\mathbb{P}}^\varepsilon$-almost surely. By Proposition 3.13, we have point 2. of Definition 3.1. This concludes the proof of Proposition 3.17. ∎

3.2.9 Proof of Theorem 3.1

We apply the Gyöngy-Krylov argument [GK96], see also [Hof13, Section 4.5], which shows that the existence of a martingale solution and uniqueness of pathwise solutions (Theorem 3.2) give existence and uniqueness of pathwise solutions and convergence in probability in $\mathcal{X}_\mathbf{U} = C([0,T); L^2(\mathbb{T}))$ of the whole sequence $(\mathbf{U}^{\tau_n})$ to $\mathbf{U}_\varepsilon$. If $\mathbf{U} \mapsto J(\mathbf{U}) \in [0, +\infty]$ is a lower semi-continuous functional on the space $\mathcal{X}$, then $\mathbf{U} \mapsto \mathbb{E}J(\mathbf{U})$ is a lower semi-continuous functional on the space $L^1(\Omega; \mathcal{X})$ endowed with the topology of convergence in probability. To prove this fact we apply the inequality

$$\mathbb{E}J(\mathbf{U}) \leq \mathbb{E}\left(\mathbf{1}_{\|\mathbf{U}-\mathbf{U}^n\| \leq \varepsilon} J(U)\right) + \mathbb{P}\left(\|\mathbf{U} - \mathbf{U}^n\| > \varepsilon\right).$$

In particular the moment estimate (3.10) follows from the moment estimate (3.75) for $\mathbf{U}^\tau$ and the gradient estimates (3.11) and (3.12) are deduced from the corresponding estimates (3.82) and (3.83) satisfied by $\mathbf{U}^\tau$. Also we have the regularity (3.25)-(3.26) as a consequence of (3.90)-(3.91). By (3.90)-(3.91) we also have, up to a subsequence, and in probability, convergence of $\mathbf{U}^{\tau_n}$ to $\mathbf{U}_\varepsilon$ in $C([0,T]; W^{1,2}(\mathbb{T}))$. This convergence is strong enough to obtain the entropy balance equation (3.13) by taking the limit in Equation (3.52). This concludes the proof of Theorem 3.1.

3.3 Additional estimates

3.3.1 Moments of the entropy and entropy flux

Proposition 3.20 (Moments of the entropy and entropy flux). *Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Let $p \in \mathbb{N}$ satisfy $p \geq 4 + \frac{1}{2\theta}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E}\int_{\mathbb{T}} \left(\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2p}(\mathbf{U}_{\varepsilon 0})\right) dx \tag{3.106}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1). Let g be a subquadratic convex function (i.e. g satisfies (5.1)) and let (η, H) be the entropy-entropy flux pair associated to g by (2.8)-(2.10). Let $s \geq 1$ satisfy

$$p \geq \frac{3}{2}s + \frac{s-1}{2\theta}. \tag{3.107}$$

Then $(\eta(\mathbf{U}_\varepsilon))$ and $(H(\mathbf{U}_\varepsilon))$ are uniformly bounded in $L^s(\Omega; L^s(Q_T))$

Proof. Let $s_0 = 2\frac{4\theta+1}{3\theta+1}$. Note that $s_0 > 2$. By Lemma 3.8, we have, under condition (3.107),

$$|\eta(\mathbf{U})|^s, \ |H(\mathbf{U})|^s \leq C_s(\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})), \quad 1 \leq s \leq s_0, \tag{3.108}$$

for all $\mathbf{U} \in \mathbb{R}_+ \times \mathbb{R}$, where C_s is constant depending on γ, s, p only. By (3.106) and the estimate (3.10) on the moments of $\mathbf{U}_\varepsilon$, we deduce that $\eta(\mathbf{U}_\varepsilon)$ and $H(\mathbf{U}_\varepsilon)$ are uniformly bounded in $L^s(\Omega; L^s(Q_T))$. ∎

3.3.2 Bound in $C([0,T]; W^{-2,2}(\mathbb{T}))$

In the following statement, $W^{-2,2}(\mathbb{T})$ denotes the dual to the space $W^{2,2}(\mathbb{T})$.

Proposition 3.21 (Additional continuity estimate). *Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Let $p \in \mathbb{N}$ satisfy $p \geq 4 + \frac{1}{2\theta}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} \left(\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2p}(\mathbf{U}_{\varepsilon 0}) \right) dx \tag{3.109}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1). Let g be a subquadratic convex function (i.e. g satisfies (5.1)) and let (η, H) be the entropy-entropy flux pair associated to g by (2.8)-(2.10). Let $B_\varepsilon(t)$ be the random distribution

$$
\begin{aligned}
B_\varepsilon(t) = & \eta(\mathbf{U}_{\varepsilon 0}) + \int_0^t \left[-\partial_x H(\mathbf{U}_\varepsilon) + \varepsilon \partial_{xx}^2 \eta(\mathbf{U}_\varepsilon) \right] ds \\
& + \int_0^t \eta'(\mathbf{U}_\varepsilon) \mathbf{\Psi}^\varepsilon(\mathbf{U}_\varepsilon) \, dW(s) + \frac{1}{2} \int_0^t \mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)^2 \partial_{qq}^2 \eta(\mathbf{U}_\varepsilon) ds.
\end{aligned}
\tag{3.110}
$$

Then, for all $\alpha \in (0, 1/4)$, the $W^{-2,2}(\mathbb{T})$-valued process $(B_\varepsilon(t))$ has a modification which has almost surely α-Hölder trajectories and satisfies

$$\mathbb{E} \| B_\varepsilon \|_{C^\alpha([0,T]; W^{-2,2}(\mathbb{T}))}^2 = \mathcal{O}(1), \tag{3.111}$$

where $\mathcal{O}(1)$ depends on γ, T, p, on the constant A_0 in (3.2) and on the bound on (3.109) only.

Proof. Let $\varphi \in W^{2,2}(\mathbb{T})$ such that $\|\varphi\|_{W^{2,2}(\mathbb{T})} \leq 1$. For $0 \leq s \leq t \leq T$, the increment $\langle B_\varepsilon(t) - B_\varepsilon(s), \varphi \rangle_{W^{-2,2}(\mathbb{T}), W^{2,2}(\mathbb{T})}$ is the sum of various terms, which we denote by $D_\varepsilon^j(s,t)$, $j = 1, \ldots, 4$. The first term is

$$D_\varepsilon^1(s,t) = \int_s^t \langle H(\mathbf{U}_\varepsilon(\sigma)), \partial_x \varphi \rangle_{L^2(\mathbb{T})} d\sigma.$$

By (3.108) and (3.10), we have

$$\mathbb{E} \sup_{\sigma \in [0,T]} \| H(\mathbf{U}_\varepsilon(\sigma)) \|_{L^2(\mathbb{T})}^2 = \mathcal{O}(1).$$

It is easy to deduce from this estimate the bound

$$\mathbb{E} |D_\varepsilon^1(s,t)|^4 = \mathcal{O}(1)(t-s)^4.$$

We obtain the same bounds for $D_\varepsilon^j(s,t)$, $j = 2, 4$, where

$$D_\varepsilon^2(s,t) = \int_s^t \langle \varepsilon \eta(\mathbf{U}_\varepsilon(\sigma)), \partial_{xx}^2 \varphi \rangle_{L^2(\mathbb{T})} d\sigma, \quad D_\varepsilon^4(s,t) = \frac{1}{2} \int_s^t \langle \mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)^2 \partial_{qq}^2 \eta(\mathbf{U}_\varepsilon), \varphi \rangle_{L^2(\mathbb{T})} d\sigma.$$

To treat the term $D_\varepsilon^4(s,t)$, we use in particular the estimates (3.68) (with $m = 1$), (3.108) and (3.10), which give

$$\mathbb{E} \sup_{\sigma \in [0,T]} \| \mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)^2 \partial_{qq}^2 \eta(\mathbf{U}_\varepsilon) \|_{L^2(\mathbb{T})}^2 (\sigma) = \mathcal{O}(1).$$

Eventually, by (3.67) (with $m = 1$), (3.108) and (3.10) and the Burkholder-Davis-Gundy Inequality, we obtain

$$\mathbb{E} |D_\varepsilon^3(s,t)|^4 = \mathcal{O}(1)(t-s)^2,$$

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where

$$D_\varepsilon^3(s,t) = \int_s^t \langle \eta'(\mathbf{U}_\varepsilon)\mathbf{\Psi}^\varepsilon(\mathbf{U}_\varepsilon), \varphi \rangle_{L^2(\mathbb{T})} \, dW(\sigma).$$

We conclude by the Kolmogorov Theorem. ∎

Let us apply Proposition 3.21 with $g(\xi) = 1$ and $g(\xi) = \xi$, respectively. In that case $\eta(\mathbf{U}) = \rho$, respectively $\eta(\mathbf{U}) = q$. Therefore, we obtain the following statement.

Corollary 3.22 (Additional continuity estimate, corollary). *Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Let $p \in \mathbb{N}$ satisfy $p \geq 4 + \frac{1}{2\theta}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} \left(\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2p}(\mathbf{U}_{\varepsilon 0}) \right) dx \tag{3.112}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1). Then, for all $\alpha \in (0, 1/4)$,

$$\mathbb{E}\|\mathbf{U}_\varepsilon\|^2_{C^\alpha([0,T];W^{-2,2}(\mathbb{T}))} = \mathcal{O}(1), \tag{3.113}$$

where $\mathcal{O}(1)$ depends on γ, T, p, on the constant A_0 in (3.2) and on the bound on (3.109) only.

3.3.3 Bound on the viscosity term

Let $\mathcal{M}_b(\overline{Q_T})$ denote the set of bounded Borel measures on $\overline{Q_T}$ and $\mathcal{M}_b^+(\overline{Q_T})$ denote the subset of non-negative bounded measures.

Proposition 3.23. *Under the hypotheses of Proposition 3.21, the random measure e^ε on $\overline{Q_T}$ defined by*

$$\langle e^\varepsilon, \varphi \rangle_{\mathcal{M}_b(\overline{Q_T}), C(\overline{Q_T})} = \iint_{Q_T} \varepsilon \eta''(\mathbf{U}_\varepsilon) \cdot (\partial_x \mathbf{U}_\varepsilon, \partial_x \mathbf{U}_\varepsilon) \varphi(x,t) dx dt \tag{3.114}$$

is uniformly bounded in $L^2(\Omega; \mathcal{M}_b^+(\overline{Q_T}))$.

Proof. We apply the entropy balance equation (3.13) with $\varphi \equiv 1$ and $t = T$. We obtain then, with the notations of Proposition 3.21,

$$\|\eta(\mathbf{U}_\varepsilon)(T)\|_{L^1(\mathbb{T})} + \|e^\varepsilon\|_{\mathcal{M}_b(Q_T)} = \langle B_\varepsilon(T), \varphi \rangle_{W^{-2,2}(\mathbb{T}), W^{2,2}(\mathbb{T})}. \tag{3.115}$$

By (3.108) and (3.10), we have $\mathbb{E}\|\eta(\mathbf{U}_\varepsilon)(T)\|^2_{L^1(\mathbb{T})} = \mathcal{O}(1)$. By (3.111), we deduce from (3.115) that $\mathbb{E}\|e^\varepsilon\|^2_{\mathcal{M}_b(Q_T)} = \mathcal{O}(1)$. ∎

4 Probabilistic Young measures

Let $\mathbf{U}_\varepsilon$ be the solution to (3.1) given in Theorem 3.1. Our aim is to prove the convergence of $(\mathbf{U}_\varepsilon)$. The standard tool for this is the notion of measure-valued solution introduced by Di Perna, [DiP83a]. In this section we give some precisions about it in our context of random solutions. More precisely, we know that, almost surely, $(\mathbf{U}_\varepsilon)$ defines a Young measure ν_ε on $\mathbb{R}_+ \times \mathbb{R}$ by the formula

$$\langle \nu_{x,t}^\varepsilon, \varphi \rangle := \langle \delta_{\mathbf{U}_\varepsilon(x,t)}, \varphi \rangle = \varphi(\mathbf{U}_\varepsilon(x,t)), \quad \forall \varphi \in C_b(\mathbb{R}_+ \times \mathbb{R}). \tag{4.1}$$

Our aim is to show that $\nu_\varepsilon \rightharpoonup \nu$ (in a sense to be specified), where ν has some specific properties. To that purpose, we will use the probabilistic compensated compactness method developed in

the Appendix of [FN08] and some results on the convergence of probabilistic Young measures that we introduce here. Note that the notion of random Young measure has also been introduced and developed by Brzeźniak and Serrano in [BS13], compare in particular [BS13, Lemma 2.18] and Proposition 4.3 below. Brzeźniak and Serrano, [BS13], use the theory of Young measure developed by Castaing, Raynaud de Fitte, Valadier, [CRdFV04].

4.1   Young measures embedded in a space of Probability measures

Let $(Q, \mathcal{A}, \lambda)$ be a finite measure space. Without loss of generality, we will assume $\lambda(Q) = 1$. A Young measure on Q (with state space E) is a measurable map $Q \to \mathcal{P}_1(E)$, where E is a topological space endowed with the σ-algebra of Borel sets, $\mathcal{P}_1(E)$ is the set of probability measures on E, itself endowed with the σ-algebra of Borel sets corresponding to the topology defined by the weak[4] convergence of measures, *i.e.* $\mu_n \to \mu$ in $\mathcal{P}_1(E)$ if

$$\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle, \quad \forall \varphi \in C_b(E).$$

As in (4.1), any measurable map $w \colon Q \to E$ can be viewed as a Young measure ν defined by

$$\langle \nu_z, \varphi \rangle = \langle \delta_{w(z)}, \varphi \rangle = \varphi(w(z)), \quad \forall \varphi \in C_b(E), \quad \text{for } \lambda - \text{almost all } z \in Q.$$

A Young measure ν on Q can itself be seen as a probability measure on $Q \times E$ defined by

$$\langle \nu, \psi \rangle = \int_Q \int_E \psi(z, p) d\nu_z(p) d\lambda(z), \quad \forall \psi \in C_b(Q \times E).$$

We then have, for all $\psi \in C_b(Q)$ (ψ independent on $p \in E$), $\langle \nu, \psi \rangle = \langle \lambda, \psi \rangle$, that is to say

$$\pi_* \nu = \lambda, \tag{4.2}$$

where π is the projection $Q \times E \to Q$ and the push-forward of ν by π is defined by $\pi_* \nu(A) = \nu(\pi^{-1}(A))$, for all Borel subset A of Q. Assume now that Q is a compact subset of $\mathbb{R}^s$ and E is a closed subset of $\mathbb{R}^m$, $m, s \in \mathbb{N}^*$, and, conversely, let μ is a probability measure on $Q \times E$ such that $\pi_* \mu = \lambda$. Then, by the Slicing Theorem (*cf.* Attouch, Buttazzo, Michaille [ABM06, Theorem 4.2.4]), we have: for λ-a.e. $z \in Q$, there exists $\mu_z \in \mathcal{P}_1(E)$ such that,

$$z \mapsto \langle \mu_z, \varphi \rangle$$

is measurable from Q to $\mathbb{R}$ for every $\varphi \in C_b(E)$, and

$$\langle \mu, \psi \rangle = \int_Q \int_E \psi(z, p) d\mu_z(p) d\lambda(z),$$

for all $\psi \in C_b(Q \times E)$. This means precisely that μ is a Young measure on Q. We therefore denote by

$$\mathcal{Y} = \{\nu \in \mathcal{P}_1(Q \times E); \pi_* \nu = \lambda\}$$

the set of Young measures on Q.

We use now the Prohorov's Theorem, *cf.* Billingsley [Bil99, Theorem 5.1], to give a compactness criteria in $\mathcal{Y}$. We assume that Q is a compact subset of $\mathbb{R}^s$ and E is a closed subset of $\mathbb{R}^m$. We also assume that the σ-algebra $\mathcal{A}$ of Q is the σ-algebra of Borel sets of Q.

---

[4]actually, weak convergence *of probability measures*, also corresponding to the tight convergence of finite measures

Proposition 4.1 (Bound against a Lyapunov functional)**.** *Let $\eta \in C(E; \mathbb{R}_+)$ satisfy the growth condition*

$$\lim_{p \in E, |p| \to +\infty} \eta(p) = +\infty.$$

Let $C > 0$ be a positive constant. Then the set

$$K_C = \left\{ \nu \in \mathcal{Y}; \int_{Q \times E} \eta(p) d\nu(z, p) \leq C \right\} \tag{4.3}$$

is a compact subset of $\mathcal{Y}$.

Proof. The condition $\pi_* \nu = \lambda$ being stable by weak convergence, $\mathcal{Y}$ is closed in $\mathcal{P}_1(Q \times E)$. By Prohorov's Theorem, [Bil99, Theorem 5.1], K_C is relatively compact in $\mathcal{Y}$ if, and only if it is tight. Besides, K_C is closed since

$$\int_{Q \times E} \eta(p) d\nu(z, p) \leq \liminf_{n \to +\infty} \int_{Q \times E} \eta(p) d\nu_n(z, p)$$

if (ν_n) converges weakly to ν. It is therefore sufficient to prove that K_C is tight, which is classical: let $\varepsilon > 0$. For $R \geq 0$, let

$$V(R) = \inf_{|p| > R} \eta(p).$$

Then $V(R) \to +\infty$ as $R \to +\infty$ by hypothesis and, setting $M_R = Q \times [\overline{B}(0, R) \cap E]$, we have

$$V(R)\nu(M_R^c) \leq \int_{Q \times E} \eta(p) d\nu(z, p) \leq C,$$

for all $\nu \in K_C$, whence $\sup_{\nu \in K_C} \nu(M_R^c) < \varepsilon$ for R large enough. ∎

The following result is a generalization of [CRdFV04, Proposition 2.4.1] which considers $\eta(p) = d(p, p_0)^q$ with d a distance (or semi-distance) p_0 a given basis point, and $q \geq 1$.

Proposition 4.2 (Momentum-convergence)**.** *Let (ν_n) be a sequence of Young measures in $\mathcal{Y}$ satisfying the bound*

$$C(\eta) := \sup_n \int_{Q \times E} \eta(p)^s d\nu_n(z, p) < +\infty, \tag{4.4}$$

where η is a non-negative continuous function on E and $s > 1$. Assume that $\nu_n \to \nu$ in $\mathcal{Y}$. Then

$$\int_{Q \times E} \eta(p)^s d\nu(z, p) < +\infty, \tag{4.5}$$

and

$$\int_{Q \times E} \varphi(z)\eta(p)^r d\nu_n(p, z) \to \int_{Q \times E} \varphi(z)\eta(p)^r d\nu(p, z), \tag{4.6}$$

for all $r \in [1, s)$ and all $\varphi \in L^{\frac{s}{s-r}}(Q, \lambda)$.

Proof. Let $\chi \in C(\mathbb{R}_+)$ be a non-negative non-increasing function supported in $[0, 2]$ with value 1 on $[0, 1]$ and define the truncature function $\chi_R(a) = \chi\left(\frac{a}{R}\right)$, $R > 1$. By the monotone convergence theorem,

$$\int_{Q \times E} \eta(p)^s d\nu(z, p) = \lim_{R \to +\infty} \int_{Q \times E} \eta(p)^s \chi_R(\eta(p)) d\nu(z, p).$$

Since $p \mapsto \eta(p)^s \chi_R(\eta(p))$ is continuous and bounded, we also have

$$\int_{Q \times E} \eta(p)^s \chi_R(\eta(p)) d\nu(z, p) = \lim_{n \to +\infty} \int_{Q \times E} \eta(p)^s \chi_R(\eta(p)) d\nu_n(z, p) \leq C(\eta).$$

In particular,

$$\int_{Q \times E} \eta(p)^s d\nu(z, p) \leq C(\eta), \tag{4.7}$$

which gives (4.5). Similarly, we have $J_R(\nu_n, \varphi) \to J_R(\nu, \varphi)$, where

$$J_R(\nu, \varphi) = \int_{Q \times E} \varphi(z) \eta(p)^r \chi_R(\eta(p)) d\nu(p, z),$$

for a fixed $\varphi \in C_b(Q)$. Let us also set

$$J(\nu, \varphi) = \int_{Q \times E} \varphi(z) \eta(p)^r d\nu(p, z).$$

The bounds (4.4) and (4.7), a kind of equi-integrability conditions, give the following estimate:

$$|J_R(\mu, \varphi) - J(\mu, \varphi)| \leq \|\varphi\|_{C_b(Q)} \frac{C(\eta)}{R^{s-r}}, \tag{4.8}$$

for $\mu \in \{\nu_n; n \geq 0\} \cup \{\nu\}$. Indeed, we have $1 - \chi_R(a) \leq \mathbf{1}_{a > R} \leq \frac{a^{s-r}}{R^{s-r}}$, and therefore

$$|J_R(\mu, \varphi) - J(\mu, \varphi)| \leq \frac{1}{R^{s-r}} \int_{Q \times E} |\varphi(z)| \eta(p)^s d\mu(p, z) \leq \|\varphi\|_{C_b(Q)} \frac{C(\eta)}{R^{s-r}}.$$

Consequently,

$$|J(\nu, \varphi) - J(\nu_n, \varphi)| \leq 2\|\varphi\|_{C_b(Q)} \frac{C(\eta)}{R^{s-r}} + |J_R(\nu, \varphi) - J(\nu_n, \varphi)|,$$

and thus $J(\nu_n, \varphi) \to J(\nu, \varphi)$. This gives (4.6) in the case where φ is continuous and bounded on Q. Let $\mu \in \{\nu_n; n \geq 0\} \cup \{\nu\}$. Since $\varphi \mapsto J(\mu, \varphi)$ is linear and

$$|J(\mu, \varphi)| \leq \left[\int_{Q \times E} |\varphi(z)|^{\frac{s}{s-r}} d\mu(p, z) \right]^{\frac{s-r}{s}} \left[\int_{Q \times E} |\eta(p)|^s d\mu(p, z) \right]^{\frac{r}{s}} \leq C(\eta)^{\frac{r}{s}} \|\varphi\|_{L^{\frac{s}{s-r}}(Q, \lambda)},$$

the general case follows. $\blacksquare$

4.2 A compactness criterion for probabilistic Young measures

As above, we assume that Q is a compact subset of $\mathbb{R}^s$ and E is a closed subset of $\mathbb{R}^m$. We endow $\mathcal{P}_1(Q \times E)$ (and thus $\mathcal{Y}$ also) with the Prohorov's metric d. Then $(\mathcal{P}_1(Q \times E), d)$ is a complete, separable metric space, weak convergence coincides with d-convergence, and a subset A is relatively compact if, and only if it is tight, [Bil99, p.72].

Definition 4.1. A random Young measure is a $\mathcal{Y}$-valued random variable.

Proposition 4.3. *Let $\eta \in C(E; \mathbb{R}_+)$ satisfy the growth condition*

$$\lim_{p \in E, |p| \to +\infty} \eta(p) = +\infty.$$

Let $M > 0$ be a positive constant. If (ν_n) is a sequence of random Young measures on Q satisfying the bound

$$\mathbb{E} \int_{Q \times E} \eta(p) d\nu_n(z, p) \leq M,$$

then, up to a subsequence, (ν_n) is converging in law.

Proof. Let $\mathcal{L}(\nu_n) \in \mathcal{P}_1(\mathcal{Y})$ denote the law of ν_n. To prove that it is tight, we use the Prohorov's Theorem. Let $\varepsilon > 0$. For $C > 0$, let K_C be the compact set defined by (4.3). If ν is a random Young measure, then we have

$$\mathbb{P}(\nu \notin K_C) = \mathbb{P}\left(1 < \frac{1}{C} \int_{Q \times E} \eta(p) d\nu(z, p)\right) \leq \frac{1}{C} \mathbb{E} \int_{Q \times E} \eta(p) d\nu(z, p),$$

hence

$$\sup_{n \in \mathbb{N}} \mathcal{L}(\nu_n)(\mathcal{Y} \setminus K_C) = \sup_{n \in \mathbb{N}} \mathbb{P}(\nu_n \notin K_C) \leq \frac{M}{C} < \varepsilon,$$

for C large enough, which proves the result. ■

The following proposition is a version of Proposition 4.2 for random Young measures.

Proposition 4.4 (Momentum-convergence, random Young measures). *Let (ν_n) be a sequence of random Young measures satisfying the bound*

$$C(\eta) := \sup_n \mathbb{E} \int_{Q \times E} \eta(p)^s d\nu_n(z, p) < +\infty, \tag{4.9}$$

where η is a non-negative continuous function on E and $s > 1$. Assume that, almost-surely, $\nu_n \to \nu$ in $\mathcal{Y}$. Then

$$\mathbb{E} \int_{Q \times E} \eta(p)^s d\nu(z, p) < +\infty, \tag{4.10}$$

and

$$\lim_{n \to +\infty} \mathbb{E} \left| \int_{Q \times E} \varphi(z) \eta(p)^r d\nu_n(p, z) - \int_{Q \times E} \varphi(z) \eta(p)^r d\nu(p, z) \right| = 0, \tag{4.11}$$

for all $r \in [1, s)$ and all $\varphi \in L^{\frac{s}{s-r}}(Q, \lambda)$.

Proof. As in the proof of Proposition 4.2, an argument of truncature and of monotone convergence gives

$$\mathbb{E} \int_{Q \times E} \eta(p)^s d\nu(z, p) \leq C(\eta), \tag{4.12}$$

hence (4.10). Let $\varphi \in C_b(Q)$. Set

$$X_n^R = \int_{Q \times E} \varphi(z) \eta(p)^r \chi_R(\eta(p)) d\nu_n(z, p), \quad X^R = \int_{Q \times E} \varphi(z) \eta(p)^r \chi_R(\eta(p)) d\nu(z, p),$$

where χ_R is the truncature function introduced in the proof of Proposition 4.2. We know by hypothesis that $X_n^R \to X^R$ almost surely. We have also

$$\mathbb{E}|X_n^R|^{s/r} \leq \|\varphi\|_{C_b(Q)}^{s/r} C(\eta), \tag{4.13}$$

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hence $(X_n^R)_n$ is equi-integrable. By the Vitali Theorem, we deduce that $X_n^R \to X^R$ in $L^1(\Omega)$. Then one can show, as in the proof of Proposition 4.2, that

$$\mathbb{E}|X_n - X| \leq 2\|\varphi\|_{C_b(Q)} \frac{C(\eta)}{R^{s-r}} + \mathbb{E}|X_n^R - X^R|,$$

where

$$X_n = \int_{Q \times E} \varphi(z)\eta(p)^r d\nu_n(z,p), \quad X = \int_{Q \times E} \varphi(z)\eta(p)^r d\nu(z,p).$$

The convergence (4.11) then follows (again, we refer to the proof of Proposition 4.2 for the details). ∎

Remark 4.1. Under the hypotheses of Proposition 4.4, we have a result of convergence a little bit stronger than (4.11). Indeed, if $\varphi \in L^{\frac{s}{s-r}}(Q, \lambda)$, we have a bound on X_n and X in $L^{s/r}(\Omega)$ (*cf.* (4.13)). Consequently

$$\lim_{n \to +\infty} \mathbb{E}\left| \int_{Q \times E} \varphi(z)\eta(p)^r d\nu_n(p,z) - \int_{Q \times E} \varphi(z)\eta(p)^r d\nu(p,z) \right|^\delta = 0, \tag{4.14}$$

for all $\delta \in \left[1, \frac{s}{r}\right)$

We end this section with a result about random Young measure being almost surely Dirac masses.

Definition 4.2 (Random Dirac mass). Let $r \geq 1$ and let ν be a random Young measure. We say that ν is an L^r-random Dirac mass if there exists $u \in L^r(\Omega \times Q; E)$ such that, almost-surely, $\nu = \delta_u \rtimes \lambda$, *i.e.* (indicating by the superscript ω the dependence on ω): for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$\int_{Q \times E} \varphi(p,z) d\nu_z^\omega(p) d\lambda(z) = \int_Q \varphi(u^\omega(z), z) d\lambda(z), \tag{4.15}$$

for all $\varphi \in C_b(Q \times E)$.

Proposition 4.5. *Let $r \geq 1$, let ν be a random Young measure on the probability space $(\Omega, \mathbb{P})$ and let $\tilde{\nu}$ be a random Young measure on a probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$ such that ν and $\tilde{\nu}$ have same laws. Then ν is an L^r-random Dirac mass if, and only if, $\tilde{\nu}$ is an L^r-random Dirac mass, i.e. the fact that ν is an L^r-random Dirac mass depends on the distribution of ν uniquely.*

Proof. We denote by $\tilde{\mathbb{E}}$ the expectancy with respect to $\tilde{\mathbb{P}}$. Let $\psi \colon \mathbb{R}^m \to \mathbb{R}$ be a strictly convex function satisfying the growth condition

$$C_1|p|^r \leq |\psi(p)| \leq C_2(1 + |p|^r).$$

If ν is an L^r-random Dirac mass, then

$$\mathbb{E} \int_{Q \times E} \psi(p) d\nu_z(p) d\lambda(z) = \mathbb{E} \int_Q \psi\left(\int_E p d\nu_z(p) \right) d\lambda(z), \tag{4.16}$$

and both sides of the equation (equal to $\mathbb{E}\|\psi(u)\|_{L^1(Q)}$) are finite. Equation (4.16) can be rewritten

$$\mathbb{E}\varphi(\nu) = \mathbb{E}\theta(\nu), \tag{4.17}$$

where the functions φ and θ are defined on $\mathcal{Y}$ as the applications

$$\varphi \colon \mu \mapsto \int_{Q \times E} \psi(p) d\mu_z(p) d\lambda(z), \quad \theta \colon \mu \mapsto \int_Q \psi\left(\int_E p d\mu_z(p) \right) d\lambda(z).$$

The function φ is continuous on $\mathcal{Y}$ and, by the Lebesgue dominated convergence theorem, θ is continuous on the subset

$$\mathcal{Y}_r := \left\{ \mu \in \mathcal{Y}; \int_{Q \times E} |p|^r d\mu_z(p) d\lambda(z) < +\infty \right\}.$$

If $\tilde{\nu}$ has same law as ν, then (4.17) shows that $\tilde{\mathbb{P}}$-almost surely $\tilde{\nu} \in \mathcal{Y}_r$, that

$$\mathbb{E} \int_{Q \times E} \psi(p) d\tilde{\nu}_z(p) d\lambda(z) = \mathbb{E} \int_Q \psi\left(\int_E p d\tilde{\nu}_z(p) \right) d\lambda(z), \qquad (4.18)$$

and that both sides of the equation (4.18) are finite. Note that, $\tilde{\mathbb{P}}$-almost surely, for λ-almost all $z \in Q$,

$$\int_E \psi(p) d\tilde{\nu}_z(p) \geq \psi\left(\int_E p d\tilde{\nu}_z(p) \right), \qquad (4.19)$$

by the Jensen Inequality. By strict convexity of ψ, there is equality in (4.19) if, and only if, $\tilde{\nu}_z$ is the Dirac mass $\delta_{\tilde{u}(z)}$, where

$$\tilde{u}(z) := \int_E p d\tilde{\nu}_z(p). \qquad (4.20)$$

Therefore (4.19) shows that $\tilde{\mathbb{P}}$-almost surely, for λ-almost all $z \in Q$, $\nu_z = \delta_{\tilde{u}(z)}$. In particular, (4.15) is satisfied by $\tilde{\nu}$ and $\tilde{u}$. By (4.20), $\tilde{u}$ is measurable from $\Omega \times Q$ to E. Since

$$\mathbb{E} \int_Q \psi(u) d\lambda = \mathbb{E} \int_{Q \times E} \psi(p) d\tilde{\nu}_z(p) d\lambda(z) < +\infty$$

in (4.18), we have $u \in L^r(\Omega \times Q; E)$. ∎

4.3   Convergence to a random Young measure

Let $\mathbf{U}_\varepsilon$ be a bounded solution to (3.1). We will apply the results of paragraphs 4.1-4.2 to the case $Q = Q_T$, λ is the 2-dimensional Lebesgue measure on Q_T, $E = \mathbb{R}_+ \times \mathbb{R}$ and $\nu^\varepsilon = \delta_{(\rho_\varepsilon, u_\varepsilon)} \rtimes \lambda$, that is to say

$$\int_{Q_T \times \mathbb{R}_+ \times \mathbb{R}} \varphi(x,t,\rho,u) d\nu_{x,t}^\varepsilon(\rho,u) dx dt = \int_{Q_T} \varphi(x,t,\rho_\varepsilon(x,t),u_\varepsilon(x,t)) dx dt, \qquad (4.21)$$

for all $\varphi \in C_b(Q_T \times \mathbb{R}_+ \times \mathbb{R})$. Hence our aim is to show the convergence, in an appropriate sense, of (ν^ε). We will also need the convergence of (e^ε), where e^ε is defined by (3.114). To that purpose, we introduce the path space

$$\mathcal{X}_e = \mathcal{M}_b^+(\overline{Q_T})$$

Recall that $\mathcal{M}_b^+(\overline{Q_T})$ is the set of non-negative bounded Borel measures on $\overline{Q_T}$. Then $\mathcal{X}_e$ is a subset of $\mathcal{M}_b(\overline{Q_T})$, the set of bounded Borel measures on $\overline{Q_T}$, which is the dual to $C(\overline{Q_T})$. The topology that we consider on $\mathcal{X}_e$ is the weak-star topology induced by $C(\overline{Q_T})$. We also use the convergence of $(\mathbf{U}_\varepsilon)$ in the path-space

$$\mathcal{X}_\mathbf{U} := C^\beta([0,T]; W^{-3,2}(\mathbb{T})).$$

Here $\beta \in (0, 1/4)$ is a given exponent and the negative Sobolev space $W^{-3,2}(\mathbb{T})$ is the dual to $W^{3,2}(\mathbb{T})$.

Proposition 4.6. *Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} \frac{1}{2} \rho_{\varepsilon 0} u_{\varepsilon 0}^2 + \frac{\kappa}{\gamma - 1} \rho_{\varepsilon 0}^{\gamma} \ dx \tag{4.22}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1) and let ν^ε be the Random Young measure associated to $\mathbf{U}_\varepsilon$ defined by (4.21). Let (ε_n) be a sequence of real numbers decreasing to zero and let $\mathcal{X}_W$ be the path space defined by (3.99). Then, up to a subsequence, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, some random variables $(\tilde{\nu}^{\varepsilon_n}, \tilde{W}, \tilde{e}^{\varepsilon_n}, \tilde{\mathbf{U}}_{\varepsilon_n})$ and $(\tilde{\nu}, \tilde{W}, \tilde{e}, \tilde{\mathbf{U}})$ with values in $\mathcal{Y} \times \mathcal{X}_W \times \mathcal{X}_e \times \mathcal{X}_\mathbf{U}$ such that

1. *the law of $(\tilde{\nu}^{\varepsilon_n}, \tilde{W}, \tilde{e}^{\varepsilon_n}, \tilde{\mathbf{U}}_{\varepsilon_n})$ under $\tilde{\mathbb{P}}$ coincide with the law of $(\nu^{\varepsilon_n}, W, e^{\varepsilon_n}, \mathbf{U}_{\varepsilon_n})$,*

2. *$(\tilde{\nu}^{\varepsilon_n}, \tilde{e}^{\varepsilon_n}, \tilde{\mathbf{U}}_{\varepsilon_n})$ converges $\tilde{\mathbb{P}}$-almost surely to $(\tilde{\nu}, \tilde{e}, \tilde{\mathbf{U}})$ in the topology of $\mathcal{Y} \times \mathcal{X}_e \times \mathcal{X}_\mathbf{U}$.*

Furthermore, $\tilde{\mathbf{U}} \in C([0,T]; W^{-2,2}(\mathbb{T}))$ $\tilde{\mathbb{P}}$-almost-surely.

Proof. let η be the entropy (energy in that case) defined by (2.8) with $g(\xi) = |\xi|^2$. Then η is coercive by (3.63). For such an η, (4.22) and the uniform estimate (3.10) show with Proposition 4.3 that the sequence of random Young measures (ν^{ε_n}) is tight. We deduce from (3.113) (taking $\alpha \in (\beta, 1/4)$) that $(\mathbf{U}_{\varepsilon_n})$ is tight in $\mathcal{X}_\mathbf{U}$. The tightness of (e^{ε_n}) in $\mathcal{X}_e$ follows from Proposition 3.23. Since the single random variable W is tight on the Polish space $\mathcal{X}_W$, the 4-uple $(\nu^{\varepsilon_n}, W, e^{\varepsilon_n}, \mathbf{U}_{\varepsilon_n})$ is tight on $\mathcal{Y} \times \mathcal{X}_W \times \mathcal{X}_e \times \mathcal{X}_\mathbf{U}$. We can apply then the Skorohod Theorem [Bil99, p. 70] to conclude. Since the closed balls of $C^\beta([0,T]; W^{-2,2}(\mathbb{T})$ are closed in $\mathcal{X}_\mathbf{U}$, it follows from (3.113) that $\tilde{\mathbf{U}} \in C([0,T]; W^{-2,2}(\mathbb{T}))$ $\tilde{\mathbb{P}}$-almost-surely. ∎

Remark 4.2. We may take $\tilde{\Omega} = [0,1]$, with $\tilde{\mathcal{F}}$ the σ-algebra of the Borelians on $[0,1]$ and $\tilde{\mathbb{P}}$ the Lebesgue measure on $[0,1]$, see [Sko56].

5 Reduction of the Young measure

Proposition 4.6 above gives the existence of a random young measure $\tilde{\nu}$ such that $\tilde{\nu}_\varepsilon$ converges in law and almost surely in the sense of Young measures to $\tilde{\nu}$. We will now apply the compensated compactness method to prove that a.s., for a.e. $(x,t) \in Q_T$, either $\tilde{\nu}_{x,t}$ is a Dirac mass or $\tilde{\nu}_{x,t}$ is concentrated on the vacuum region $\{\rho = 0\}$. To do this, we will use the probabilistic compensated compactness method of [FN08] to obtain a set of functional equations satisfied by $\tilde{\nu}$. Then we conclude by adapting the arguments of [LPS96].

5.1 Compensated compactness

Let $\mathcal{G}$ denote the set of functions $g \in C^2(\mathbb{R})$, convex, with g sub-quadratic and g' sub-linear:

$$|g(\xi)| \leq C(g)(1 + |\xi|^2), \quad |g'(\xi)| \leq C(g)(1 + |\xi|), \tag{5.1}$$

for all $\xi \in \mathbb{R}$, for a given constant $C(g) \geq 0$.

5.1.1 Preparation to Murat's Lemma

For $p \in [1, +\infty]$, we denote by $W_0^{1,p}(Q_T)$ the set of functions u in the Sobolev space $W^{1,p}(Q_T)$ such that $u = 0$ on $\mathbb{T} \times \{0\}$ and $\mathbb{T} \times \{T\}$. We denote by $W^{-1,p}(Q_T)$ the dual of $W_0^{1,p'}(Q_T)$, where p' is the conjugate exponent to p. First we prove the tightness of the sequence $(\varepsilon \partial_{xx}^2 \eta(\mathbf{U}_\varepsilon))_{\varepsilon > 0}$.

Proposition 5.1 (Case $\gamma \leq 2$). *We assume $\gamma \leq 2$. Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_{\varepsilon}}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} \frac{1}{2} \rho_{\varepsilon 0} u_{\varepsilon 0}^2 + \frac{\kappa}{\gamma - 1} \rho_{\varepsilon 0}^{\gamma} \, dx \tag{5.2}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_{\varepsilon}$ be the bounded solution to (3.1). Let $r \in (1, 2)$ and let η be an entropy of the form (2.8) with $g \in \mathcal{G}$ (cf. (5.1)). Then the sequence of random variables $(\varepsilon \partial_{xx}^2 \eta(\mathbf{U}_{\varepsilon}))_{\varepsilon > 0}$ is tight in $W^{-1,r}(Q_T)$.

Proof. We suppose first that $\gamma < 2$ and we set $m = \dfrac{r}{2 - r}(2 - \gamma)$. We can assume that $r \in \left(\dfrac{2}{3 - \gamma}, 2\right)$. Then $m > 1$. We will show that $(\varepsilon \partial_{xx}^2 \eta(\mathbf{U}_{\varepsilon}))$ converges to zero in probability on $W^{-1,r}(Q_T)$ by proving that

$$\lim_{\varepsilon \to 0} \varepsilon \partial_x \eta(\mathbf{U}_{\varepsilon}) = 0 \text{ in probability in } L^r(Q_T). \tag{5.3}$$

To obtain (5.3), it is sufficient to prove the convergence

$$\lim_{\varepsilon \to 0} \varepsilon \partial_x \eta(\mathbf{U}_{\varepsilon}) = 0 \text{ in } L^r(Q_T), \tag{5.4}$$

conditionally to the bounds

$$\|\rho_{\varepsilon}\|_{L^m(Q_T)}^m \leq R, \tag{5.5}$$

and

$$\varepsilon \iint_{Q_T} \left\{ \left[\rho_{\varepsilon}^{\gamma} + |u_{\varepsilon}|^4 \right] \rho_{\varepsilon}^{\gamma - 2} |\partial_x \rho_{\varepsilon}|^2 + \left[\rho_{\varepsilon}(1 + \rho_{\varepsilon}^{2\theta} + |u_{\varepsilon}|^2) \right] \rho_{\varepsilon} |\partial_x u_{\varepsilon}|^2 \right\} dx dt \leq R, \tag{5.6}$$

where $R > 1$ is fixed. Indeed, by the estimates (3.10), (3.11), (3.12) and the Markov Inequality, the probabilities of the events (5.5) and (5.6) are arbitrary large for large R, uniformly with respect to ε. The proof of (5.4) is similar to the analysis in [LPS96, pp.627-629], with the difference that we do not use L^{∞} estimates here. We note first that, by (5.1), we have

$$|\partial_{\rho} \eta(\mathbf{U})| \leq C \left(1 + |u|^2 + \rho^{2\theta} \right),$$

and

$$|\partial_u \eta(\mathbf{U})| \leq C \rho \left(1 + |u| + \rho^{\theta} \right),$$

for a given non-negative constant that we still denote by C. By the Young Inequality, we obtain the bounds

$$
\begin{aligned}
|\partial_x \eta(\mathbf{U}_{\varepsilon})|^r \leq & C \left\{ \left(1 + |u_{\varepsilon}|^{2r} + |\rho_{\varepsilon}|^{2r\theta} \right) |\partial_x \rho_{\varepsilon}|^r + \left(1 + |u_{\varepsilon}|^r + |\rho_{\varepsilon}|^{r\theta} \right) \rho_{\varepsilon}^r |\partial_x u_{\varepsilon}|^r \right\} \\
\leq & C \left\{ 1 + \left(1 + |u_{\varepsilon}|^{2r} \right) |\partial_x \rho_{\varepsilon}|^r + |\rho_{\varepsilon}|^{4\theta} |\partial_x \rho_{\varepsilon}|^2 + \rho_{\varepsilon} \left[1 + |\rho_{\varepsilon}|^{2\theta} + |u_{\varepsilon}|^2 \right] \rho_{\varepsilon} |\partial_x u_{\varepsilon}|^2 \right\},
\end{aligned} \tag{5.7}
$$

where C denotes some constant possibly varying from places to places that depends only on r. By (5.5), (5.6) therefore,

$$\varepsilon^r \iint_{Q_T} |\partial_x \eta(\mathbf{U}_{\varepsilon})|^r \, dx dt \leq C_R \varepsilon^{r-1} + C \varepsilon^r \iint_{Q_T} \left(1 + |u_{\varepsilon}|^{2r} \right) |\partial_x \rho_{\varepsilon}|^r \, dx dt, \tag{5.8}$$

where the constant C_R depends on R. Since $\gamma \leq 2$, we have furthermore

$$
\begin{aligned}
\left(1 + |u_{\varepsilon}|^{2r} \right) |\partial_x \rho_{\varepsilon}|^r &= \rho_{\varepsilon}^{\frac{r}{2}(2 - \gamma)} \left(1 + |u_{\varepsilon}|^{2r} \right) \rho_{\varepsilon}^{\frac{r}{2}(\gamma - 2)} |\partial_x \rho_{\varepsilon}|^r \\
&\leq C \rho_{\varepsilon}^m + C \left(1 + |u_{\varepsilon}|^4 \right) \rho_{\varepsilon}^{\gamma - 2} |\partial_x \rho_{\varepsilon}|^2.
\end{aligned}
$$

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By (5.5), (5.6) and (5.8), we conclude to

$$\varepsilon^r \iint_{Q_T} |\partial_x \eta(\mathbf{U}_\varepsilon)|^r \, dx dt \leq C_R \varepsilon^{r-1}, \tag{5.9}$$

for all $\varepsilon \in (0,1)$. This gives the convergence (5.4). If $\gamma = 2$, the arguments used above remain valid, taking $r = 2$. ∎

Proposition 5.2 (Case $\gamma > 2$). *We assume $\gamma > 2$. Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} \frac{1}{2} \rho_{\varepsilon 0} u_{\varepsilon 0}^2 + \frac{\kappa}{\gamma - 1} \rho_{\varepsilon 0}^\gamma \, dx \tag{5.10}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1). Assume that there exists $m > 4$ such that the sequence $(\varepsilon^{\frac{1}{\gamma-2}} \|u^\varepsilon\|_{L^m(Q_T)})$ is stochastically bounded: for all $\alpha > 0$, there exists $M > 0$ such that, for all $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(\varepsilon^{\frac{1}{\gamma-2}} \|u^\varepsilon\|_{L^m(Q_T)} > M\right) < \alpha. \tag{5.11}$$

Let $r \in (1,2)$ and let η be an entropy of the form (2.8) with $g \in \mathcal{G}$ (cf. (5.1)). Then the sequence of random variables $(\varepsilon \partial_{xx}^2 \eta(\mathbf{U}_\varepsilon))_{\varepsilon > 0}$ is tight in $W^{-1,r}(Q_T)$.

Proof. We begin as in the proof of Proposition 5.1. Without loss of generality, we assume $\frac{4r}{2-r} \geq m$. We will obtain (5.3) here by proving that, given $\eta > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}(A_{\varepsilon,\eta}) = 0, \quad A_{\varepsilon,\eta} := \left\{ \|\varepsilon \partial_x \eta(\mathbf{U}_\varepsilon)\|_{L^r(Q_T)} > \eta \right\}. \tag{5.12}$$

For $R > 1$, we consider the events (5.6) and

$$\|u_\varepsilon\|_{L^m(Q_T)} \leq R, \quad \|u_\varepsilon \rho_\varepsilon^{\frac{1}{2}}\|_{L^2(Q_T)} \leq R. \tag{5.13}$$

By (3.10), (3.11), (3.12) and (5.11), the probability of the event

$$B_{\varepsilon,R} := \left\{ (5.6) \text{ \& } (5.13) \text{ \& } \varepsilon^{\frac{1}{\gamma-2}} \|u_\varepsilon\|_{L^m(Q_T)} \leq M \right\} \tag{5.14}$$

is arbitrarily close to 1 for large R, uniformly with respect to ε. To obtain (5.12), it is therefore sufficient to prove

$$\lim_{\varepsilon \to 0} \|\varepsilon \partial_x \eta(\mathbf{U}_\varepsilon)\|_{L^r(Q_T)} = 0 \text{ a.e. on } B_{\varepsilon,R}, \tag{5.15}$$

for every $R > 1$. To get (5.15), we use the estimate (5.7), which gives (5.8). The remaining term in the right-hand side of (5.8) is estimated as follows: let $\delta > 0$. First, we have $1 \leq \delta^{r(2-\gamma)/2} \rho_\varepsilon^{r(\gamma-2)/2}$ on the set $\{\rho_\varepsilon \geq \delta\}$ and, by the Hölder Inequality and (5.6),

$$\varepsilon^r \iint_{Q_T} \left(1 + |u_\varepsilon|^{2r}\right) |\partial_x \rho_\varepsilon|^r \, dx dt$$

$$\leq C \delta^{\frac{r(2-\gamma)}{2}} \left(\varepsilon^2 \iint_{Q_T} \left(1 + |u_\varepsilon|^4\right) \rho_\varepsilon^{\gamma-2} |\partial_x \rho_\varepsilon|^2 \, dx dt\right)^{r/2} + \varepsilon^r \iint_{Q_T} \left(1 + |u_\varepsilon|^{2r}\right) \mathbf{1}_{\rho_\varepsilon < \delta} |\partial_x \rho_\varepsilon|^r \, dx dt$$

$$\leq C_R \varepsilon^{r/2} \delta^{\frac{r(2-\gamma)}{2}} + \varepsilon^r \iint_{Q_T} \left(1 + |u_\varepsilon|^{2r}\right) \mathbf{1}_{\rho_\varepsilon < \delta} |\partial_x \rho_\varepsilon|^r \, dx dt.$$

To estimate the part corresponding to $\{\rho_\varepsilon < \delta\}$, we first use the Hölder Inequality to obtain

$$\varepsilon^r \iint_{Q_T} \left(1 + |u_\varepsilon|^{2r}\right) |\partial_x \rho_\varepsilon|^r \mathbf{1}_{\rho_\varepsilon < \delta} \le \varepsilon^{r/2} \left(\iint_{Q_T} \left(1 + |u_\varepsilon|^{2r}\right)^{\frac{2}{2-r}} \right)^{\frac{2-r}{2}} \left(\varepsilon \iint_{Q_T} |\partial_x \rho_\varepsilon|^2 \mathbf{1}_{\rho_\varepsilon < \delta} \right)^{\frac{r}{2}}$$

$$\lesssim \varepsilon^{r/2} (1 + \|u_\varepsilon\|_{L^m(Q_T)})^{2r} \left(\varepsilon \iint_{Q_T} |\partial_x \rho_\varepsilon|^2 \mathbf{1}_{\rho_\varepsilon < \delta} \right)^{\frac{r}{2}}. \qquad (5.16)$$

Then, we multiply the first Equation of the system (3.1a), *i.e.* Equation

$$\partial_t \rho_\varepsilon + \partial_x(\rho_\varepsilon u_\varepsilon) = \varepsilon \partial_{xx}^2 \rho_\varepsilon,$$

by $\min(\rho_\varepsilon, \delta)$, and then sum the result over Q_T. This gives, by (5.13) and for some constants varying from lines to lines

$$\varepsilon \iint_{Q_T} |\partial_x \rho_\varepsilon|^2 \mathbf{1}_{\rho_\varepsilon < \delta} \le C\delta + C\left(\iint_{Q_T} \rho_\varepsilon |u_\varepsilon| |\partial_x \rho_\varepsilon| \mathbf{1}_{\rho_\varepsilon < \delta} \right)$$

$$\le C\delta + C\delta^{1/2} \left[\iint_{Q_T} |u_\varepsilon|^2 \rho_\varepsilon \right]^{\frac{1}{2}} \left[\iint_{Q_T} |\partial_x \rho_\varepsilon|^2 \mathbf{1}_{\rho_\varepsilon < \delta} \right]^{\frac{1}{2}}$$

$$\le C_R \left(\delta + \frac{\delta}{\varepsilon} \right) + \frac{\varepsilon}{2} \iint_{Q_T} |\partial_x \rho_\varepsilon|^2 \mathbf{1}_{\rho_\varepsilon < \delta},$$

from which we deduce

$$\varepsilon \iint_{Q_T} |\partial_x \rho_\varepsilon|^2 \mathbf{1}_{\rho_\varepsilon < \delta} \le C_R \left(\delta + \frac{\delta}{\varepsilon} \right).$$

Reporting this result in (5.8) and (5.16), we get

$$\varepsilon^r \iint_{Q_T} |\partial_x \eta(\mathbf{U}_\varepsilon)|^r \, dx dt \le C_R \left(\varepsilon^{r-1} + \delta^{\frac{r}{2}(2-\gamma)} \varepsilon^{r/2} + \delta^{r/2} (1 + \|u_\varepsilon\|_{L^m(Q_T)})^{2r} \right). \qquad (5.17)$$

We take $\delta = o(\varepsilon^{\frac{1}{\gamma-2}})$. On the event $B_{\varepsilon,R}$ (*cf.* (5.14)), (5.17) reads then

$$\varepsilon^r \iint_{Q_T} |\partial_x \eta(\mathbf{U}_\varepsilon)|^r \, dx dt = o(1).$$

This concludes the proof of (5.15) and of Proposition 5.2. ∎

Remark 5.1 (Growth of $\|u^\varepsilon\|_{L^{4+}(Q_T)}$). Since $\Lambda_{\varkappa_\varepsilon}$ is an invariant region for $\mathbf{U}_\varepsilon$, a sufficient condition to (5.11) is that $\varepsilon^{\frac{1}{\gamma-2}} \varkappa_\varepsilon$ is bounded:

$$\varepsilon^{\frac{1}{\gamma-2}} \varkappa_\varepsilon \lesssim 1. \qquad (5.18)$$

In that case we have even $\varepsilon^{\frac{1}{\gamma-2}} \|u^\varepsilon\|_{L^\infty(Q_T)} \lesssim 1$ almost surely.

The next Proposition is similar to Lemma 4.20 in [FN08].

Proposition 5.3. *Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \ge c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Let $p \in \mathbb{N}$ satisfy $p \ge 4 + \frac{1}{2\theta}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} (\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2p}(\mathbf{U}_{\varepsilon 0})) \, dx \qquad (5.19)$$

is bounded uniformly with respect to ε (recall that η_m denotes the entropy associated by (2.8) to the convex function $\xi \mapsto \xi^{2m}$). Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1). Let η be an entropy of the form (2.8) with $g \in \mathcal{G}$ (cf. (5.1)). Let

$$M^\varepsilon(t) = \int_0^t \partial_q \eta(\mathbf{U}_\varepsilon)(s)\Phi^\varepsilon(\mathbf{U}_\varepsilon)(s)dW(s). \qquad (5.20)$$

Then $\partial_t M^\varepsilon$ is tight in $W^{-1,2}(Q_T)$.

Proof. The proof is in essential the proof of Lemma 4.19 in [FN08]. However, we will proceed slightly differently (instead of using Marchaud fractional derivative we work directly with fractional Sobolev spaces and an Aubin-Simon compactness lemma). We begin by giving some precisions on the sense of $\partial_t M^\varepsilon$: this is the random element of $W^{-1,2}(Q_T)$ defined $\mathbb{P}$-almost surely by

$$\langle \partial_t M^\varepsilon, z \rangle_{W^{-1,2}(Q_T),W_0^{1,2}(Q_T)} = -\langle M^\varepsilon, \partial_t z \rangle_{L^2(Q_T),L^2(Q_T)}.$$

Let $0 \leq s \leq t \leq T$. In what follows we denote by C any constant, that may vary from line to line, which depends on the data only and is independent on ε. By the Burkholder-Davis-Gundy Inequality, we have

$$\mathbb{E}\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^4(\mathbb{T})}^4 \leq C \int_\mathbb{T} \mathbb{E}\left|\int_s^t |\partial_q \eta(\mathbf{U}_\varepsilon)|^2 |\mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)|^2 d\sigma\right|^2 dx,$$

and, using the Hölder Inequality,

$$\mathbb{E}\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^4(\mathbb{T})}^4 \leq C|t - s| \int_s^t \mathbb{E}\int_\mathbb{T} \left[|\partial_q \eta(\mathbf{U}_\varepsilon)|^2 |\mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)|^2\right]^2 d\sigma dx.$$

By (5.1), and (3.67) with $m = 1$, we have

$$|\partial_q \eta(\mathbf{U})|^2 \mathbf{G}^2(\mathbf{U}) \leq C(\eta_0(\mathbf{U}) + \eta_2(\mathbf{U})).$$

Taking the square of both sides, we obtain

$$\left[|\partial_q \eta(\mathbf{U})|^2 \mathbf{G}^2(\mathbf{U})\right]^2 \leq C(\eta_0(\mathbf{U}) + \eta_p(\mathbf{U})) \qquad (5.21)$$

by Lemma 3.8. The uniform estimate (3.10) and (5.19) give

$$\mathbb{E}\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^4(\mathbb{T})}^4 \leq C|t - s|^2, \qquad (5.22)$$

and, by integration with respect to t and s,

$$\mathbb{E}\int_0^T\int_0^T \frac{\|M^\varepsilon(t) - M^\varepsilon(s)\|_{L^4(\mathbb{T})}^4}{|t - s|^{1+2\nu}}dtds \leq C, \qquad (5.23)$$

as soon as $\nu < 1/2$. The left-hand side in this inequality (5.23) is the norm of M^ε in the space $L^4(\Omega; W^{\nu,4}(0,T; L^4(\mathbb{T})))$. Since $L^4(\mathbb{T}) \hookrightarrow H^{-1}(\mathbb{T})$, it follows that

$$\mathbb{E}\|M^\varepsilon\|_{W^{\nu,4}(0,T;H^{-1}(\mathbb{T}))}^4 \leq C.$$

We use the continuous injection

$$W^{\nu,4}(0,T; H^{-1}(\mathbb{T})) \hookrightarrow C^{0,\mu}([0,T]; H^{-1}(\mathbb{T}))$$

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for every $0 < \mu < \nu - \frac{1}{4}$ to obtain

$$\mathbb{E}\|M^\varepsilon\|^4_{C^{0,\mu}([0,T];H^{-1}(\mathbb{T}))} \leq C. \tag{5.24}$$

Besides, taking $s = 0$ in (5.22) and summing with respect to $t \in (0, T)$ gives also

$$\mathbb{E}\|M^\varepsilon\|^4_{L^4(Q_T)} \leq C. \tag{5.25}$$

By the Aubin-Simon compactness Lemma, [Sim87], the set

$$A_R := \left\{ M \in L^2(Q_T); \|M\|_{C^{0,\mu}([0,T];H^{-1}(\mathbb{T}))} \leq R, \ \|M\|_{L^4(Q_T)} \leq R \right\}$$

is compact in $C([0,T];H^{-1}(\mathbb{T}))$, hence compact in $L^2(0,T;H^{-1}(\mathbb{T}))$. Consequently (5.24) and (5.25) show that (M^ε) is tight as a $L^2(0,T;H^{-1}(\mathbb{T}))$-random variable, and we conclude that $(\partial_t M^\varepsilon)$ is tight as a $W^{-1,2}(Q_T)$-random variable. ∎

5.1.2 Functional equation

Proposition 5.4. *Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Let $p \in \mathbb{N}$ satisfy $p \geq 4 + \frac{1}{2\theta}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} (\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2p}(\mathbf{U}_{\varepsilon 0}))\, dx \tag{5.26}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1). If $\gamma > 2$, we suppose that (5.11) is satisfied. Let (η, H) be an entropy-entropy flux of the form (2.8)-(2.10) with $g \in \mathcal{G}$ (cf. (5.1)). Then the family

$$\{\mathrm{div}_{t,x}(\eta(\mathbf{U}_\varepsilon), H(\mathbf{U}_\varepsilon)); \varepsilon \in (0,1)\}$$

is tight in $W^{-1,2}(Q_T)$.

Proof. **Step 1.** By Proposition 3.20, $\eta(\mathbf{U}_\varepsilon)$ and $H(\mathbf{U}_\varepsilon)$ are uniformly bounded in $L^s(\Omega; L^s(Q_T))$. As a consequence, $\mathrm{div}_{t,x}(\eta(\mathbf{U}_\varepsilon), H(\mathbf{U}_\varepsilon))$ is stochastically bounded in $W^{-1,s}(Q_T)$.

Step 2. We consider the entropy balance equation (3.13), which we rewrite as the following distributional equation on Q_T:

$$\mathrm{div}_{t,x}(\eta(\mathbf{U}_\varepsilon), H(\mathbf{U}_\varepsilon)) = -\varepsilon\eta''(\mathbf{U}_\varepsilon) \cdot (\partial_x \mathbf{U}_\varepsilon, \partial_x \mathbf{U}_\varepsilon) + \varepsilon\partial^2_{xx}\eta(\mathbf{U}_\varepsilon) + \partial_t M^\varepsilon + \frac{1}{2}\mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)^2 \partial^2_{qq}\eta(\mathbf{U}_\varepsilon),$$

where M^ε is defined by (5.20). Let $r \in (1,2)$. By Proposition 5.1, Proposition 5.2 and Proposition 5.3, the families $\{\varepsilon\partial^2_{xx}\eta(\mathbf{U}_\varepsilon)\}_{\varepsilon \in (0,1)}$ and $\{\partial_t M^\varepsilon\}_{\varepsilon \in (0,1)}$ are tight in $W^{-1,r}(Q_T)$ and $W^{-1,2}(Q_T)$ respectively. The two remaining terms

$$\varepsilon\eta''(\mathbf{U}_\varepsilon) \cdot (\partial_x \mathbf{U}_\varepsilon, \partial_x \mathbf{U}_\varepsilon) \quad \text{and} \quad \frac{1}{2}|\mathbf{G}(\mathbf{U}_\varepsilon)|^2 \partial^2_{qq}\eta(\mathbf{U}_\varepsilon)$$

are stochastically bounded in measure on Q_T by (3.11)-(3.12) and (2.2)-(3.10) respectively (we use (3.68) with $m = 1$ to estimate this latter term).

Step 3. We want now to apply the stochastic version of the Murat's Lemma, Lemma A.3 in [FN08]. If we refer strictly to the statement of Lemma A.3 in [FN08], there is an obstacle here, due to the fact that $\varepsilon\partial^2_{xx}\eta(\mathbf{U}_\varepsilon)$ is neither tight in $W^{-1,2}(Q_T)$, neither stochastically bounded in measure on Q_T. However, in the proof of Lemma A.3 in [FN08], the property which is used

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regarding the term that is stochastically bounded in measure on Q_T is only the fact that it is tight in $W^{-1,r}(Q_T)$ for $1 < r < 2$, due to the compact injection $W_0^{1,\sigma}(Q_T) \hookrightarrow C(\overline{Q_T})$ for $\sigma > 2$. The argument of interpolation theory which combines this compactness result with the stochastic bound in $W^{-1,r}(Q_T)$ can therefore be directly applied here: we deduce that the sequence of $W^{-1,2}(Q_T)$ random variables

$$\text{div}_{t,x}(\eta(\mathbf{U}_\varepsilon), H(\mathbf{U}_\varepsilon)) = \partial_t \eta(\mathbf{U}_\varepsilon) + \partial_x H(\mathbf{U}_\varepsilon)$$

is tight. ∎

We apply now the div-curl lemma to obtain the functional equation (5.28) below.

Proposition 5.5 (Functional Equation). *Let $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ satisfy $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ a.e. in $\mathbb{T}$, for a positive constant $c_{\varepsilon 0}$. Let $p \in \mathbb{N}$ satisfy $p \geq 4 + \frac{1}{2\theta}$. Assume that hypotheses (3.2), (3.3), (3.5) are satisfied, that $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ and that*

$$\mathbb{E} \int_{\mathbb{T}} \left(\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2p}(\mathbf{U}_{\varepsilon 0}) \right) dx \tag{5.27}$$

is bounded uniformly with respect to ε. Let $\mathbf{U}_\varepsilon$ be the bounded solution to (3.1). If $\gamma > 2$, we furthermore suppose that the possible growth of $\varkappa_\varepsilon$ with ε is limited according to (5.18). Let (η, H), $(\hat{\eta}, \hat{H})$ be some entropy-entropy flux pairs of the form (2.8)-(2.10) associated to some convex functions $g, \hat{g} \in \mathcal{G}$ respectively (cf. (5.1)). Let $\tilde{\nu}$ be the random Young measure given by Proposition 4.6. Then, almost surely, for a.e. $(x,t) \in Q_T$,

$$\langle \hat{\eta}, \tilde{\nu}_{x,t} \rangle \langle H, \tilde{\nu}_{x,t} \rangle - \langle \eta, \tilde{\nu}_{x,t} \rangle \langle \hat{H}, \tilde{\nu}_{x,t} \rangle = \langle \hat{\eta} H - \eta \hat{H}, \tilde{\nu}_{x,t} \rangle. \tag{5.28}$$

Besides, if (5.28) is realized, then, for all $v, v' \in \mathbb{R}$,

$$2\lambda \Big(\langle \chi(v) u \rangle \langle \chi(v') \rangle - \langle \chi(v) \rangle \langle \chi(v') u \rangle \Big) = (v - v') \Big(\langle \chi(v) \chi(v') \rangle - \langle \chi(v) \rangle \langle \chi(v') \rangle \Big), \tag{5.29}$$

where $\chi(\mathbf{U}, v) = (v - z)_+^\lambda (w - v)_+^\lambda$, $z := u - \rho^\theta$, $w := u + \rho^\theta$, and

$$\langle \chi(v) \rangle = \int \chi(\mathbf{U}, v) \, d\tilde{\nu}_{x,t}(\mathbf{U}).$$

Proof. Let (ε_n) be the sequence considered in Proposition 4.6 (to be exact, this is a subsequence of (ε_n) that we are considering). By Proposition 4.5, $\tilde{\nu}_{x,t}^{\varepsilon_n}$ is an L^r-random Dirac mass for every n. In particular, it satisfies almost surely, for a.e. $(x,t) \in Q_T$, the identity

$$\langle \hat{\eta}, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle \langle H, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle - \langle \eta, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle \langle \hat{H}, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle = \langle \hat{\eta} H - \eta \hat{H}, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle. \tag{5.30}$$

Let

$$X_n(x,t) = (\langle \eta, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle, \langle H, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle), \quad \hat{X}_n(x,t) = (\langle \hat{\eta}, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle, \langle \hat{H}, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle).$$

By Proposition 3.20 and Proposition 4.4, we have, for all $\varphi \in L^2(Q_T; \mathbb{R}^2)$,

$$\lim_{n \to +\infty} \mathbb{E} \left| \langle X_n - X, \varphi \rangle_{L^2(Q_T; \mathbb{R}^2)} \right| = 0,$$

where $X(x,t) = (\langle \eta, \tilde{\nu}_{x,t} \rangle, \langle H, \tilde{\nu}_{x,t} \rangle)$. Up to a subsequence, the event $\{\langle X_n - X, \varphi \rangle_{L^2(Q_T; \mathbb{R}^2)} \to 0$ as $n \to +\infty\}$ is of probability one. The subsequence may depend on φ at this stage. Since $L^2(Q_T; \mathbb{R}^2)$ is separable, however, there exists a subsequence of (ε_n) such that, for all $\varphi \in$

$L^2(Q_T; \mathbb{R}^2)$, $\langle X_n - X, \varphi \rangle_{L^2(Q_T; \mathbb{R}^2)} \to 0$ as $n \to +\infty$ a.s., that is to say $X_n \to X$ in weak-$L^2(Q_T)$ almost surely. Similarly (by taking a further subsequence of (ε_n) if necessary), we obtain $\hat{X}_n \to \hat{X}$ in weak-$L^2(Q_T)$ almost surely, where $\hat{X}(x,t) = (\langle \hat{\eta}, \tilde{\nu}_{x,t} \rangle, \langle \hat{H}, \tilde{\nu}_{x,t} \rangle)$. Let

$$\hat{X}_n^\perp = (-\langle \hat{H}, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle, \langle \hat{\eta}, \tilde{\nu}_{x,t}^{\varepsilon_n} \rangle)$$

and let $\alpha > 0$. Note that

$$\mathrm{curl}_{t,x} \hat{X}_n^\perp = \mathrm{div}_{t,x} \hat{X}_n.$$

By Proposition 5.4 (we use Remark 5.1 to ensure that (5.11) is satisfied if $\gamma > 2$), there exists a compact subset K_α of $W^{-1,2}(Q_T)$ such that the event

$$\mathrm{div}_{t,x} X_n \in K_\alpha \ \& \ \mathrm{curl}_{t,x} \hat{X}_n^\perp \in K_\alpha \tag{5.31}$$

has probability greater than $1 - \alpha$. If (5.31) is realized, then the div-curl lemma, [Mur78] ensures that the product $X_n \cdot \hat{X}_n^\perp$ is converging in weak-$L^1(Q_T)$ to the product $X \cdot \hat{X}^\perp$. The product $X_n \cdot \hat{X}_n^\perp$ is the left-hand side of (5.30). Therefore, we can pass to the limit in (5.30) to obtain (5.28) almost surely conditionally to (5.31), for a.e. $(x,t) \in Q_T$, that is to say for almost all $(\omega, x, t) \in A_\alpha$ with $\tilde{\mathbb{P}} \times \mathcal{L}^2(A_\alpha) \geq (1 - \alpha)\mathcal{L}^2(Q_T)$ (we denote by $\mathcal{L}^2$ the Lebesgue measure on Q_T). We consider a sequence (α_n) converging to 0. We can choose the sets K_{α_n} as an increasing sequence, in which case (A_{α_n}) is also increasing. We set

$$A = \bigcup_{n \in \mathbb{N}} A_{\alpha_n}.$$

Then A is of full $\tilde{\mathbb{P}} \times \mathcal{L}^2$-measure and (5.28) is satisfied on A. The identity (5.29) follows from the formulas (2.8), (2.10) and (5.28). $\blacksquare$

5.2 Reduction of the Young measure

We now follow [LPS96] to conclude. We switch from the variables (ρ, u) or (ρ, q) to (w, z), where w and z are the Riemann invariants

$$z = u - \rho^\theta, \quad w = u + \rho^\theta.$$

We write then $\chi(w, z, v)$ for $\chi(\mathbf{U}, v)$. Let us fix (ω, x, t) such that (5.29) is satisfied. Set

$$\mathcal{C} = \{v \in \mathbb{R} \, ; \langle \chi(v) \rangle > 0\} = \bigcup_{(w,z) \in \mathrm{supp}\, \tilde{\nu}_{x,t}} \{v \, ; \, z < v < w\}.$$

Let

$$V = \{(\rho, u) \in \mathbb{R}_+ \times \mathbb{R} | \rho = 0\} = \{(w, z) \in \mathbb{R}^2 | w = z\}$$

denote the vacuum region. If $\mathcal{C}$ is empty, then $\tilde{\nu}_{x,t}$ is concentrated on V. Assume $\mathcal{C}$ not empty. By Lemma I.2 in [LPS96] then, $\mathcal{C}$ is an open interval in $\mathbb{R}$, say $\mathcal{C} =]a, b[$, where $-\infty \leq a < b \leq +\infty$ (we use here the french notation for open intervals to avoid the confusion with the point (a, b) of $\mathbb{R}^2$). Furthermore all the computations of [LPS96] apply here, and thus, as in Section I.6 of [LPS96], we obtain

$$\langle \rho^{2\lambda\theta} \langle \chi \circ \pi_i \rangle \phi \circ \pi_i \rangle = 0, \tag{5.32}$$

for any continuous function ϕ with compact support in $\mathcal{C}$, where $\pi_i \colon \mathbb{R}^2 \to \mathbb{R}$ denote the projection on the first coordinate w if $i = 1$, and the projection on the second coordinate z if $i = 2$.

Note that, if $\text{supp}(\tilde{\nu}_{x,t}) \setminus V$ is reduced to a single point $\{Q\}$, then $\pi_i(Q) \in \overline{\mathcal{C}} \setminus \mathcal{C}$ for $i = 1$ and $i = 2$. Assume by contradiction that there exists $Q \in \mathbb{R}^2$ satisfying

$$Q \in \text{supp}(\tilde{\nu}_{x,t}) \setminus V, \quad \pi_i(Q) \in \mathcal{C}, \tag{5.33}$$

for a i in $\{1, 2\}$. Then there exists a neighbourhood K of Q such that $K \cap V = \emptyset$, $\nu_{x,t}(K) > 0$, $\pi_i(K) \subset \mathcal{C}$. But then $\langle \chi \circ \pi_i \rangle > 0$ on K, $\rho > 0$ on K and, choosing a continuous function ϕ compactly supported in $\mathcal{C}$ such that $\phi > 0$ on K we obtain a contradiction to (5.32). Consequently (5.33) cannot be satisfied. This shows that there cannot exists *two distinct* points P, Q in $\text{supp}(\tilde{\nu}_{x,t}) \setminus V$. Indeed, if two such points exists, then either $\pi_1(Q) < \pi_1(P)$, and then Q satisfies (5.33) with $i = 1$, or $\pi_1(Q) = \pi_1(P)$ and, say, $\pi_2(P) < \pi_2(Q)$ and then Q also satisfies (5.33). The other cases are similar by symmetry of P and Q.

Therefore if $\mathcal{C} \neq \emptyset$, then the support of the restriction of $\tilde{\nu}_{x,t}$ to $\mathcal{C}$ is reduced to a point. In particular, a and b are finite. Then, by Lemma I.2 in [LPS96], $P := (a, b) \in \text{supp}(\nu_{x,t})$ and $\tilde{\nu}_{x,t} = \tilde{\mu}_{x,t} + \alpha\delta_{(\tilde{\rho}(x,t),\tilde{u}(x,t))}$, where $\tilde{\mu}_{x,t} = \tilde{\nu}_{x,t}|_V$. Using (5.29), we obtain

$$0 = (v - v')\chi(b, a, v)\chi(b, a, v')(\alpha - \alpha^2),$$

for all $v, v' \in (a, b)$, and thus $\alpha = 0$ or 1. We have therefore proved the following result.

Proposition 5.6 (Reduction of the Young measure). *Let* $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ *satisfy* $\rho_{\varepsilon 0} \geq c_{\varepsilon 0}$ *a.e. in* $\mathbb{T}$, *for a positive constant* $c_{\varepsilon 0}$. *Let* $p \in \mathbb{N}$ *satisfy* $p \geq 4 + \frac{1}{2\theta}$. *Assume that hypotheses* (3.2), (3.3), (3.5) *are satisfied, that* $\mathbf{U}_{\varepsilon 0} \in \Lambda_{\varkappa_\varepsilon}$ *and that*

$$\mathbb{E} \int_{\mathbb{T}} (\eta_0(\mathbf{U}_{\varepsilon 0}) + \eta_{2p}(\mathbf{U}_{\varepsilon 0}))\, dx$$

is bounded uniformly with respect to ε. *Let* $\mathbf{U}_\varepsilon$ *be the bounded solution to* (3.1). *If* $\gamma > 2$, *we furthermore suppose that the possible growth of* $\varkappa_\varepsilon$ *with* ε *is limited according to* (5.18). *Let* $\tilde{\nu}$ *be the random Young measure given by Proposition* 4.6. *Then, almost surely, for a.e.* $(x, t) \in Q_T$, *either* $\tilde{\nu}_{x,t}$ *is concentrated on the vacuum region* V, *or* $\tilde{\nu}_{x,t}$ *is reduced to a Dirac mass* $\delta_{(\tilde{\rho}(x,t),\tilde{u}(x,t))}$.

Remark 5.2. The notation $\tilde{\rho}$ is already used in Proposition 4.6 since $\tilde{U}$ denotes the limit of $\tilde{\mathbf{U}}_{\varepsilon_n}$ in $\mathcal{X}_{\mathbf{U}} = C^\beta([0, T]; W^{-3,2}(\mathbb{T}))$. Our notation is consistent however. Indeed, let us fix $\omega \in \tilde{\Omega}$ such that the convergence of $\tilde{\nu}^{\varepsilon_n}$ and $\tilde{\mathbf{U}}_{\varepsilon_n}$ are satisfied. Let us set

$$\mathbf{U}_*(x, t) = \int_E \begin{pmatrix} \eta(p) \\ H(p) \end{pmatrix} d\tilde{\nu}_{(x,t)}(p), \quad \begin{pmatrix} \eta(p) \\ H(p) \end{pmatrix} = \begin{pmatrix} \rho \\ q \end{pmatrix}, \tag{5.34}$$

where $(\eta(p), H(p)) = (\rho, q)$ is the entropy-entropy flux pair obtained when taking $g(\xi) = 1$ in (2.8)-(2.10). Let $\varphi \in C^3(\overline{Q_T}; \mathbb{R}^2)$. By passing to the limit in the sense of Young measure, respectively in the sense of the convergence in $\mathcal{X}_{\mathbf{U}}$ in the term

$$\iint_{Q_T} \mathbf{U}_{\varepsilon_n}(x, t) \cdot \varphi(x, t) dx dt,$$

we obtain

$$\iint_{Q_T} \mathbf{U}_*(x, t) \cdot \varphi(x, t) dx dt = \iint_{Q_T} \tilde{\mathbf{U}}(x, t) \cdot \varphi(x, t) dx dt.$$

Consequently, $\mathbf{U}_* = \tilde{\mathbf{U}}$ a.e. since φ is arbitrary.

5.3 Martingale solution

In this section we will prove Theorem 2.1.

5.3.1 Convergence of non-linear functionals of $(\mathbf{U}_\varepsilon)$

Let $E = \mathbb{R}_+ \times \mathbb{R}$. By Proposition 5.6, we have: almost surely, for every continuous and *bounded* function S on E and every $\varphi \in C_b(Q_T)$,

$$\iint_{Q_T} S(\tilde{\mathbf{U}}_{\varepsilon_n}(x,t))\varphi(x,t)dxdt \to \iint_{Q_T} \int_E S(p)\varphi(x,t)d\tilde{\nu}_{x,t}(p)dxdt, \qquad (5.35)$$

and we know that

$$\text{supp}(\nu_{x,t}) \cap V = \emptyset \implies \int_E S(p)d\tilde{\nu}_{x,t}(p) = S(\tilde{\rho}(x,t), \tilde{u}(x,t)).$$

Proposition 5.7 (Limit in the vacuum). *Let $g \in \mathcal{G}$ (cf. (5.1)) and let (η, H) be the entropy-entropy flux pair defined by (2.8)-(2.10). Under the hypotheses of Proposition 5.6, the convergence (5.35) holds true, in probability, for every $\varphi \in L^\infty(Q_T)$ and $S \in \{\eta, H\}$. Besides, the limit is trivial in the vacuum region: almost surely, for a.e. $(x,t) \in Q_T$, for $S \in \{\eta, H\}$,*

$$\text{supp}(\tilde{\nu}_{x,t}) \subset V \implies \int_E S(p)d\tilde{\nu}_{x,t}(p) = 0. \qquad (5.36)$$

Proof. By Proposition 3.20 and Proposition 4.4, the convergence (5.35) holds true, in $L^1(\tilde{\Omega})$ for all $\varphi \in L^{\frac{s}{s-1}}(Q_T)$. In particular, it is satisfied for every $\varphi \in L^\infty(Q_T)$, with a convergence in probability.

To prove (5.36), we use the two last estimates in Lemma 3.8 with $m = 1$ and $s = 1$. This gives the equi-integrability estimates

$$\mathbb{E} \iint_{Q_T} (|\eta(\mathbf{U}_\varepsilon)| + |H(\mathbf{U}_\varepsilon)|) |u_\varepsilon|dxdt \leq C.$$

where C is a constant independent on ε. By Proposition 4.4 and (4.10) follows

$$\tilde{\mathbb{E}} \iint_{Q_T} \int_E (|\eta(p)| + |H(p)|) |u| d\tilde{\nu}_{x,t}dxdt \leq C.$$

In particular, for $\tilde{\mathbb{P}}$-almost all ω, there exists $C(\omega) < +\infty$ such that

$$\iint_{Q_T} \int_E (|\eta(p)| + |H(p)|) |u| d\tilde{\nu}_{x,t}^\omega dxdt \leq C(\omega).$$

Consequently, by making the distinction between the ranges $\{|u| > R\}$ and $\{|u| \leq R\}$, we see that

$$\iint_{Q_T} \int_E (|\eta(p)| + |H(p)|) d\tilde{\nu}_{x,t}^\omega dxdt \leq \iint_{Q_T} \int_E (|\eta(p)| + |H(p)|) \mathbf{1}_{|u| \leq R} d\tilde{\nu}_{x,t}^\omega dxdt + \frac{C(\omega)}{R}.$$

If $\text{supp}(\tilde{\nu}_{x,t}^\omega) \subset V$ then

$$\iint_{Q_T} \int_E (|\eta(p)| + |H(p)|) \mathbf{1}_{|u| \leq R} d\tilde{\nu}_{x,t}^\omega dxdt = 0$$

and

$$\iint_{Q_T} \int_E (|\eta(p)| + |H(p)|) \, d\tilde{\nu}_{x,t}^\omega dx dt \leq \frac{C(\omega)}{R}.$$

Letting $R \to +\infty$ in this last estimate, we obtain (5.36). ∎

Remark 5.3. In the case where a priori L^∞ bounds on $(\rho_\varepsilon, u_\varepsilon)$ are known, Proposition 5.7 is almost automatic. In the absence of such L^∞ bounds it requires some additional estimates to be established. In our context, we have some estimates on moments of arbitrary orders (see (3.10)). In some situations, like the isentropic Euler system with geometric effects, it is quite difficult to obtain enough equi-integrability to conclude. See in particular [LW07] where such estimates are proved for the isentropic Euler system with geometric effects.

Recall the definition (5.34) of $\mathbf{U}_* = \begin{pmatrix} \rho_* \\ q_* \end{pmatrix}$. Outside the vacuum, $\tilde{\nu}_{x,t}$ is the Dirac mass

$$\tilde{\nu}_{x,t} = \delta_{(\rho_*(x,t), u_*(x,t))}, \quad u_*(x,t) := \frac{q_*(x,t)}{\rho_*(x,t)}.$$

By Proposition 5.7, this is still true in the vacuum, by setting

$$u_*(x,t) = 0. \tag{5.37}$$

In particular, we have then

$$\int_E S(p) d\tilde{\nu}_{(x,t)}(p) = S(\mathbf{U}_*(x,t)),$$

for almost all $(\omega, x, t) \in \Omega \times Q_T$ if $S = \eta$ or $S = H$, where (η, H) is associated to a subquadratic function g (*i.e.* $g \in \mathcal{G}$, as defined in (5.1)). We have proved in Remark 5.2 that $\tilde{\mathbf{U}} = \mathbf{U}_\star$ a.e. Consequently, we have

$$\int_E S(p) d\tilde{\nu}_{(x,t)}(p) = S(\tilde{\mathbf{U}}(x,t)), \tag{5.38}$$

for almost all $(\omega, x, t) \in \Omega \times Q_T$.

We can prove now the following strong convergence result.

Proposition 5.8 (Strong convergence). *Let $g \in \mathcal{G}$ (cf. (5.1)) and let (η, H) be the entropy-entropy flux pair defined by (2.8)-(2.10). Under the hypotheses of Proposition 5.6, we have, up to subsequence,*

$$\eta(\tilde{\mathbf{U}}_{\varepsilon_n}) \to \eta(\tilde{\mathbf{U}}), \quad H(\tilde{\mathbf{U}}_{\varepsilon_n}) \to H(\tilde{\mathbf{U}}) \tag{5.39}$$

in $L^2(\tilde{\Omega} \times Q_T)$-strong and a.s. as $L^2(Q_T)$-valued random variables.

Proof. Let $S \in \{\eta, H\}$. Let us set

$$\Psi_n(x,t) = S(\tilde{\mathbf{U}}_{\varepsilon_n}(x,t)), \quad \Psi(x,t) = \int_E S(p)\varphi(x,t) d\tilde{\nu}_{x,t}(p).$$

By (5.38), we have $\Psi(x,t) = S(\tilde{U}(x,t))$. Then we use Proposition 3.20, Proposition 4.4, in particular (4.14) with $r = 1$, $\delta = 2$ to show that, for all $\varphi \in L^2(Q_T)$,

$$\tilde{\mathbb{E}} \left| \langle \Psi_n - \Psi, \varphi \rangle_{L^2(Q_T)} \right|^2 \to 0, \tag{5.40}$$

and (4.11) with $r = 2$ to obtain

$$\tilde{\mathbb{E}} \left| \|\Psi_n\|_{L^2(Q_T)}^2 - \|\Psi\|_{L^2(Q_T)}^2 \right| = 0, \tag{5.41}$$

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when $n \to +\infty$. Using the separability of $L^2(Q_T)$, we deduce that there is a subsequence of (Ψ_n), which we still denote by (Ψ_n), such that, $\tilde{\mathbb{P}}$-almost surely, $\Psi_n \to \Psi$ in weak-$L^2(Q_T)$ and $\|\Psi_n\|_{L^2(Q_T)} \to \|\Psi\|_{L^2(Q_T)}$. This implies $\Psi_n \to \Psi$ in $L^2(Q_T)$-strong. In particular, up to a subsequence, we have $\Psi_n \to \Psi$ for almost all $(\tilde{\omega}, x, t)$. We use the estimate $\tilde{\mathbb{E}}\|\Psi_n\|^s_{L^s(Q_T)} < +\infty$, $s > 2$, to conclude by the Vitali Theorem that $\Psi_n \to \Psi$ in $L^2(\tilde{\Omega} \times Q_T)$-strong. ∎

5.3.2 Martingale solution

Let us apply Proposition 5.8 to the entropy-entropy flux pair associated to the affine function $g\colon \xi \mapsto \alpha\xi + \beta$. Then $\eta(\mathbf{U}) = \alpha q + \beta\rho$. We deduce that

$$\tilde{\mathbf{U}}_{\varepsilon_n} \to \tilde{\mathbf{U}} \tag{5.42}$$

in $L^2(\tilde{\Omega} \times Q_T)$ strong and a.s. as $L^2(Q_T)$-valued random variables.

For the moment we have only supposed that $\mathbf{U}_{\varepsilon 0} \in W^{2,2}(\mathbb{T})$ with some uniform bounds. Assume furthermore

$$\lim_{\varepsilon \to 0} \mathbf{U}_{\varepsilon 0} = \mathbf{U}_0 \quad \text{in } L^2(\mathbb{T}) \tag{5.43}$$

and a.e. Since $\mathbf{U}_{\varepsilon 0}$ avoids the vacuum ($\rho_{\varepsilon 0} \geq c_{\varepsilon 0} > 0$ a.e.), the velocity $u_{\varepsilon 0} = \frac{q_{\varepsilon 0}}{\rho_{\varepsilon 0}}$ is well defined. We assume also the convergence

$$\lim_{\varepsilon \to 0} u_{\varepsilon 0} = u_0 \quad \text{in } L^2(\mathbb{T}) \tag{5.44}$$

and a.e. This means in particular that, for a.e. x in the set $\{\rho_0 = 0\}$, $q_0(x) = 0$. Let $g \in C^2(\mathbb{R})$ be a convex subquadratic function (*i.e.* $g \in \mathcal{G}$, as defined in (5.1)). If (3.109) is uniformly bounded, then we can apply the dominated convergence Theorem to obtain

$$\lim_{\varepsilon \to 0} \eta(\mathbf{U}_{\varepsilon 0}) = \eta(\mathbf{U}_0) \quad \text{in } L^2(\mathbb{T}), \tag{5.45}$$

for any η defined by (2.8).

Recall that $(\tilde{\Omega}, \tilde{\mathbb{P}}, \tilde{\mathcal{F}}, \tilde{W})$ is given by Proposition 4.6. Let $(\tilde{\mathcal{F}}_t)$ be the $\tilde{\mathbb{P}}$-augmented canonical filtration of the process $(\tilde{\mathbf{U}}, \tilde{W})$, *i.e.*

$$\tilde{\mathcal{F}}_t = \sigma\big(\sigma\big(\varrho_t\tilde{\mathbf{U}}, \varrho_t\tilde{W}\big) \cup \big\{N \in \tilde{\mathcal{F}};\ \tilde{\mathbb{P}}(N) = 0\big\}\big), \quad t \in [0, T],$$

where the restriction operator ϱ_t is defined in (3.101). We will show that the sextuplet

$$\big(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t), \tilde{\mathbb{P}}, \tilde{W}, \tilde{\mathbf{U}}\big)$$

is a weak martingale solution to (1.1).

Our aim is to pass to the limit in the balance entropy equation (3.13). Actually, given (5.39), it would be more natural to pass to the limit in the weak-in-time formulation of (3.13), which is the following one: almost surely, for all $\varphi \in C^2(\overline{Q_T})$ such that $\varphi \equiv 0$ on $\mathbb{T} \times \{t = T\}$,

$$\iint_{Q_T} \big[\eta(\mathbf{U}_\varepsilon)\partial_t\varphi + H(\mathbf{U}_\varepsilon)\partial_x\varphi + \varepsilon\eta(\mathbf{U}_\varepsilon)\partial^2_{xx}\varphi\big]\,dxdt + \int_{\mathbb{T}} \eta(\mathbf{U}_{\varepsilon 0})\varphi(0)dx$$

$$+ \int_0^T \int_{\mathbb{T}} \eta'(\mathbf{U}_\varepsilon)\mathbf{\Psi}^\varepsilon(\mathbf{U}_\varepsilon)\varphi\,dxdW(t) + \frac{1}{2}\iint_{Q_T} \mathbf{G}^\varepsilon(\mathbf{U}_\varepsilon)^2\partial^2_{qq}\eta(\mathbf{U}_\varepsilon)\varphi dxdt$$

$$= \iint_{Q_T} \varepsilon\eta''(\mathbf{U}_\varepsilon) \cdot (\partial_x\mathbf{U}_\varepsilon, \partial_x\mathbf{U}_\varepsilon)\varphi dxdt. \tag{5.46}$$

However, we need to work on the processes to pass to the limit in the stochastic integral with the martingale formulation of (3.13). Therefore, let $\varphi_0 \in C^2(\mathbb{T})$ be fixed. Since

$$t \mapsto \left\langle \eta(\tilde{\mathbf{U}}_{\varepsilon_n}(t)), \varphi_0 \right\rangle$$

converges to $t \mapsto \left\langle \eta(\tilde{\mathbf{U}}(t)), \varphi_0 \right\rangle$ in $L^1(\tilde{\Omega} \times (0, T))$, we can assume, up to a subsequence (and using the Fubini Theorem), that for a.e. $t \in [0, T]$, almost surely, $\left\langle \eta(\mathbf{U}_{\varepsilon_n}(t)), \varphi_0 \right\rangle$ converges to $\left\langle \eta(\tilde{\mathbf{U}}(t)), \varphi_0 \right\rangle$. Therefore there is a Borel subset $\mathcal{D}$ of $[0, T]$ of full measure such that, for every $t \in \mathcal{D}$, almost surely, we have the convergence

$$\left\langle \eta(\tilde{\mathbf{U}}_{\varepsilon_n})(t), \varphi_0 \right\rangle - \left\langle \eta(\mathbf{U}_{\varepsilon_n\,0}), \varphi_0 \right\rangle - \int_0^t \left\langle H(\tilde{\mathbf{U}}_{\varepsilon_n}), \partial_x \varphi_0 \right\rangle + \varepsilon_n \left\langle \eta(\tilde{\mathbf{U}}_{\varepsilon_n}), \partial_{xx}^2 \varphi_0 \right\rangle ds$$

$$\to \left\langle \eta(\tilde{\mathbf{U}})(t), \varphi_0 \right\rangle - \left\langle \eta(\mathbf{U}_0), \varphi_0 \right\rangle - \int_0^t \left\langle H(\tilde{\mathbf{U}}), \partial_x \varphi_0 \right\rangle ds$$

Note that, by (5.45), we have $0 \in \mathcal{D}$. Furthermore, by Proposition 4.6, we have, for every $\varphi \in C_b(\overline{Q_T})$, almost-surely,

$$\left\langle \tilde{e}^{\varepsilon_n}, \varphi \right\rangle_{\mathcal{M}_b(\overline{Q_T}), C_b(\overline{Q_T})} \to \left\langle \tilde{e}, \varphi \right\rangle_{\mathcal{M}_b(\overline{Q_T}), C_b(\overline{Q_T})}. \tag{5.47}$$

By the the bound on $(\tilde{e}^{\varepsilon_n})$ in $L^2(\tilde{\Omega}; \mathcal{M}_b(\overline{Q_T}))$ (see Proposition 3.23) and the Vitali Theorem, we have the convergence (5.47) in $L^\delta(\tilde{\Omega})$ for every $\delta \in [1, 2)$. It follows that, for every $Y \in L^2(\tilde{\Omega})$, for every $\varphi \in C_b(\overline{Q_T})$,

$$\tilde{\mathbb{E}}(\left\langle \tilde{e}^{\varepsilon_n}, \varphi \right\rangle_{\mathcal{M}_b(\overline{Q_T}), C_b(\overline{Q_T})} Y) \to \tilde{\mathbb{E}}(\left\langle \tilde{e}, \varphi \right\rangle_{\mathcal{M}_b(\overline{Q_T}), C_b(\overline{Q_T})} Y).$$

Let $\mathfrak{A}$ denote the countable set of the atoms of the non-negative measure $\mathbb{E}\tilde{e}$. Let $\mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$. Replace $\mathcal{D}$ by $\mathcal{D} \setminus \mathfrak{A}^*$. Then $\mathcal{D}$ remains a set of full measure in $[0, T]$ containing $t = 0$ and, for every $t \in \mathcal{D}$, for every $\varphi \in C(\mathbb{T})$, we have

$$\tilde{\mathbb{E}}\left(\iint_{\overline{Q_T}} \mathbf{1}_{[0,t)} \varphi \, d\tilde{e}^{\varepsilon_n}\, Y \right) \to \tilde{\mathbb{E}}\left(\iint_{\overline{Q_T}} \mathbf{1}_{[0,t)} \varphi \, d\tilde{e}\, Y \right). \tag{5.48}$$

Let

$$\tilde{M}^\varepsilon(t) = \left\langle \eta(\tilde{\mathbf{U}}_\varepsilon)(t), \varphi_0 \right\rangle - \left\langle \eta(\mathbf{U}_{\varepsilon 0}), \varphi_0 \right\rangle - \int_0^t \left\langle H(\tilde{\mathbf{U}}_\varepsilon), \partial_x \varphi_0 \right\rangle dx$$

$$- \int_0^t \varepsilon_n \left\langle \eta(\tilde{\mathbf{U}}_\varepsilon), \partial_{xx}^2 \varphi_0 \right\rangle ds + \iint_{\overline{Q_T}} \mathbf{1}_{[0,t)} \varphi_0 \, d\tilde{e}_\varepsilon,$$

and

$$\tilde{M}(t) = \left\langle \eta(\tilde{\mathbf{U}})(t), \varphi_0 \right\rangle - \left\langle \eta(\mathbf{U}_0), \varphi_0 \right\rangle - \int_0^t \left\langle H(\tilde{\mathbf{U}}), \partial_x \varphi_0 \right\rangle ds + \iint_{\overline{Q_T}} \mathbf{1}_{[0,t)} \varphi_0 \, d\tilde{e}.$$

For every $t \in \mathcal{D}$, for every $Y \in L^2(\tilde{\Omega})$, we have

$$\tilde{\mathbb{E}}\left(\tilde{M}^{\varepsilon_n}(t) Y \right) \to \tilde{\mathbb{E}}\left(\tilde{M}(t) Y \right). \tag{5.49}$$

With the result of convergence (5.49) at hand, we will prove now that $\tilde{M}(t)$ is a stochastic integral with respect to $\tilde{W}$. The argumentation is very similar to the argumentation in Section 3.2.8. First, there exists some independent $(\tilde{\mathcal{F}}_t)$-adapted Wiener processes $(\tilde{\beta}_k(t))$ such that

$$\tilde{W} = \sum_{k \geq 1} \tilde{\beta}_k(t) e_k$$

almost surely in $\mathcal{X}_W$: the proof is analogous to the proof of Lemma 3.18. In analogy with Lemma 3.19 then, we can show that the processes

$$\tilde{M}, \ \tilde{M}^2 - \sum_{k\geq 1}\int_0^\cdot \left\langle \sigma_k(\tilde{\mathbf{U}})\partial_q\eta(\tilde{\mathbf{U}}),\varphi\right\rangle^2 dr, \ \tilde{M}\tilde{\beta}_k - \int_0^\cdot \left\langle \sigma_k(\tilde{\mathbf{U}})\partial_q\eta(\tilde{\mathbf{U}}),\varphi\right\rangle dr \qquad (5.50)$$

are $(\tilde{\mathcal{F}}_t)$-martingales. There is however a notable difference between the result of Lemma 3.19 and the result (5.50) here, in the fact that the martingales in (5.50) are indexed by $\mathcal{D} \subset [0,T]$ since we have used the convergence (5.49). This means that

$$\tilde{\mathbb{E}}(\tilde{M}(t) - \tilde{M}(s)|\tilde{\mathcal{F}}_s) = 0$$

is satisfied only for $s \leq t$ and $s,t \in \mathcal{D}$, and similarly for the other martingales in (5.50). If all the processes in (5.50) were continuous martingales indexed by $[0,T]$, we would infer, as in the proof of Proposition 3.17, that

$$\left\langle \eta(\tilde{\mathbf{U}})(t),\varphi_0\right\rangle - \left\langle \eta(\mathbf{U}_0),\varphi_0\right\rangle - \int_0^t \left\langle H(\tilde{\mathbf{U}}),\partial_x\varphi_0\right\rangle ds$$
$$= -\iint_{\overline{Q_T}} \mathbf{1}_{[0,t)}\varphi_0 d\tilde{e} + \sum_{k\geq 1}\int_0^t \left\langle \sigma_k(\tilde{\mathbf{U}})\partial_q\eta(\tilde{\mathbf{U}}),\varphi_0\right\rangle d\tilde{\beta}_k(s), \qquad (5.51)$$

for all $t \in [0,T]$, $\tilde{\mathbb{P}}$-almost surely. Nevertheless, $\mathcal{D}$ contains 0 and is dense in $[0,T]$ since it is of full measure, and it turns out, by the Proposition A.1 in [Hof13] on densely defined martingales, that this is sufficient[5] to obtain (5.51) for all $t \in \mathcal{D}$, $\tilde{\mathbb{P}}$-almost surely. Then we conclude as in the *proof of Theorem 4.13* of [Hof13]: let $N(t)$ denote the continuous semi-martingale defined by

$$N(t) = \int_0^t \left\langle H(\tilde{\mathbf{U}}),\partial_x\varphi_0\right\rangle ds + \sum_{k\geq 1}\int_0^t \left\langle \sigma_k(\tilde{\mathbf{U}})\partial_q\eta(\tilde{\mathbf{U}}),\varphi_0\right\rangle d\tilde{\beta}_k(s).$$

Let $t \in (0,T]$ be fixed and let $\alpha \in C_c^1([0,t))$. By the Itō Formula we compute the stochastic differential of $N(s)\alpha(s)$ to get

$$0 = \int_0^t N(s)\alpha'(s)ds + \int_0^t \left\langle H(\tilde{\mathbf{U}}),\partial_x\varphi_0\right\rangle\alpha(s)\,ds$$
$$+ \sum_{k\geq 1}\int_0^t \left\langle \sigma_k(\tilde{\mathbf{U}})\partial_q\eta(\tilde{\mathbf{U}}),\varphi_0\right\rangle\alpha(s)\,d\tilde{\beta}_k(s). \qquad (5.52)$$

By (5.51), we have

$$N(t) = \left\langle \eta(\tilde{\mathbf{U}})(t),\varphi_0\right\rangle - \left\langle \eta(\mathbf{U}_0),\varphi_0\right\rangle + \iint_{\overline{Q_T}} \mathbf{1}_{[0,t)}\varphi_0 d\tilde{e},$$

for all $t \in \mathcal{D}$, $\tilde{\mathbb{P}}$-almost surely. In particular, by the Fubini Theorem,

$$\int_0^t N(s)\alpha'(s)ds = \int_0^t \left\langle \eta(\tilde{\mathbf{U}})(s),\varphi_0\right\rangle\alpha'(s)\,ds$$
$$+ \left\langle \eta(\mathbf{U}_0),\varphi_0\right\rangle\alpha(0) - \int_{[0,t]}\alpha(\sigma)d\tilde{\rho}(\sigma), \qquad (5.53)$$

---

[5]indeed, it is possible to prove the equivalent equations to (3.103)-(3.105) for all $s,t \in \mathcal{D}$

$\tilde{\mathbb{P}}$-almost surely, where we have defined the non-negative measure $\tilde{\rho}$ by

$$\tilde{\rho}(B) = \iint_{\overline{Q_T}} \mathbf{1}_B \varphi_0 d\tilde{e},$$

for B a Borel subset of $[0, T]$. If $\alpha, \varphi_0 \geq 0$, then

$$\int_{[0,t]} \alpha(\sigma) d\tilde{\rho}(\sigma) \geq 0, \quad \tilde{\mathbb{P}} - \text{almost surely},$$

and we deduce (2.17) from (5.52), (5.53). This concludes the proof of Theorem 2.1.

6 Conclusion

We want to discuss in this concluding section some open questions related to the long-time behaviour of solutions to (1.1). It is known that for *scalar* stochastic conservation laws with additive noise, and for non-degenerate fluxes, there is a unique ergodic invariant measure, *cf.* [EKMS00, DV14]. Since both fields of (1.1) are genuinely non-linear, a form of non-degeneracy condition is clearly satisfied in (1.1). Actually, in the deterministic case $\Phi \equiv 0$, the solution converges to the constant state determined by the conservation of the two invariants

$$\int_0^1 \rho(x)dx, \quad \int_0^1 q(x)dx. \tag{6.1}$$

see [CF99, Theorem 5.4]. This indicates that some kind of dissipation effects (*via* interaction of waves, *cf.* also [GL70]) occur in the Euler system for isentropic gas dynamics. However, in a system there is in a way more room for waves to evolve than in a scalar conservation law, and the long-time behaviour in (1.1) may be different from the one described in [EKMS00, DV14].

Specifically, consider the case $\gamma = 2$. For such a value the system of Euler equations for isentropic gas dynamics is equivalent to the following Shallow water system:

$$h_t + \partial_x(hu)dt = 0, \qquad \qquad \text{in } Q_T, \tag{6.2a}$$

$$(hu)_t + \partial_x(hu^2 + g\frac{h^2}{2}) + gh\partial_x Z = 0, \qquad \text{in } Q_T, \tag{6.2b}$$

with $Z(x, t) = \Phi^*(x)\frac{dW}{dt}$ and $Q_T = \mathbb{T} \times (0, T)$. For example, we may take

$$dZ(x, t) = \sum_{k \in \mathbb{N}} \sigma_k \left[\cos(2\pi kx)d\beta_k^\flat(t) + \sin(2\pi kx)d\beta_k^\sharp(t) \right], \tag{6.3}$$

with $\sigma \in l^2(\mathbb{N})$ and $\beta_k^\flat(t)$, $\beta_k^\sharp(t)$ some independent Brownian motions on $\mathbb{R}$ ((6.3) is an example of space-homogeneous noise).

When $Z = Z(x)$, (6.2) is a model for the one-dimensional flow of a fluid of height h and speed u over a ground described by the curve $z = Z(x)$ ($u(x)$ is the speed of the column of water over the abscissa x)[6]. For a random Z as in (6.2b), the system (6.2) describes the evolution of the

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[6]the fact that u is independent on the altitude z is admissible as long as h is small compared to the longitudinal length L of the channel, $L = 1$ here, *cf.* [GP01]

fluid in terms of (h, u) when its behaviour is forced by the moving topography. Note that, for *smooth* solutions to (6.2), with a noise given by (6.3), the balance of Energy writes

$$\frac{d}{dt}\mathbb{E}\int_{\mathbb{T}}\eta_E(\mathbf{U}(x,t))dx = \frac{1}{2}\|\sigma\|_{l^2(\mathbb{N})}^2\mathbb{E}\int_{\mathbb{T}}h(x,t)dx, \quad \eta_E(\mathbf{U}) := h\frac{u^2}{2} + g\frac{h^2}{2}. \tag{6.4}$$

Since the total height $\int_{\mathbb{T}}h(x,t)dx$ is conserved in the evolution, the input of energy by the noise is done at *constant* rate:

$$\frac{d}{dt}\mathbb{E}\int_{\mathbb{T}}\eta_E(\mathbf{U}(x,t))dx = \text{Cst} = \frac{1}{2}\|\sigma\|_{l^2(\mathbb{N})}^2\mathbb{E}\int_{\mathbb{T}}h_0(x)dx. \tag{6.5}$$

Of course, the equality is not satisfied (6.4). We have

$$\frac{d}{dt}\mathbb{E}\int_{\mathbb{T}}\eta_E(\mathbf{U}(x,t))dx \leq \frac{1}{2}\|\sigma\|_{l^2(\mathbb{N})}^2\mathbb{E}\int_{\mathbb{T}}h_0(x)dx, \tag{6.6}$$

as a consequence of entropy inequalities. In particular dissipation of energy occurs in shocks. Therefore, the question is to determine if an equilibrium in law (and which kind of equilibrium) for such a random process as the solution to (6.2) can be reached when time goes to $+\infty$ as a result of the balance between production of energy in the stochastic source term and dissipation of energy in shocks. An hint for the existence of a unique, ergodic, invariant measure is the "loss of memory in the system" given by the ergodic theorem: if f is a bounded, continuous functional of the solution $\mathbf{U}(t)$, then

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(\mathbf{U}(t))dt \to \langle f, \mu\rangle \text{ a.s.} \tag{6.7}$$

where μ is the invariant measure. Before testing the ergodic convergence (6.7), one has first to restrict the evolution to the right manifold. Indeed, in the scalar case [EKMS00, DV14], say for the equation

$$dv + \partial_x(A(v)) = \partial_x\phi(x)dW(t), \quad x \in \mathbb{T}, t > 0,$$

there is a unique invariant measure μ_λ indexed by the constant parameter

$$\lambda = \int_{\mathbb{T}}v(x)dx \in \mathbb{R}.$$

For (6.2), the entropy solution is evolving on the manifold

$$\int_{\mathbb{T}}h(x)dx = \text{cst}.$$

Since $\mathbb{E}\int_0^t h(s)d\beta_k^\flat(s) = \mathbb{E}\int_0^t h(s)d\beta_k^\sharp(s) = 0$ for all k (this is the expectancy of a stochastic integral), we have a second equation of conservation by (6.2b):

$$\mathbb{E}\int_{\mathbb{T}}q(x)dx = 0.$$

It seems therefore that the final equilibrium and the invariant measure, if they exist, should be determined uniquely by the initial value of the parameters (6.1). This is what we illustrate by numerical approximations on Figure 6. On Figure 6, time is the abscissa coordinate, the averaged

Energy evolution, Tfinal=10
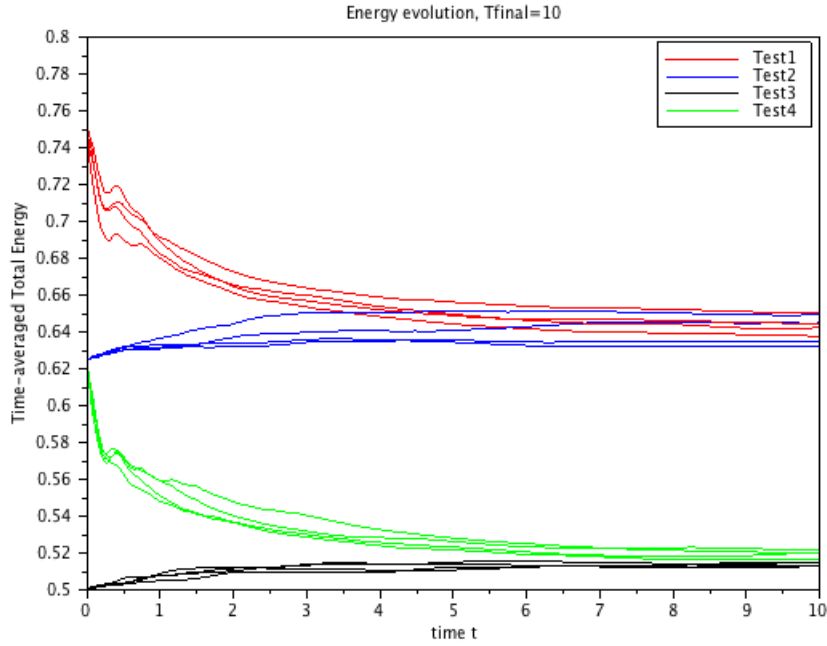
energy

$$\frac{1}{t} \int_0^t \int_{\mathbb{T}} \eta_E(\mathbf{U}(x,s))ds$$

is the ordinate coordinate. There are four different tests corresponding to four different initial conditions. The simulation on the time interval $[0, T]$, $T = 10$, has been done several times, for several realizations of the noise therefore. The numerical values corresponding to each test are the following ones: first, we have taken $g = 2$, Z as in (6.3) with $\sigma_k = \mathbf{1}_{1 \leq k \leq 5}$ and $h_0(x) \equiv 1$ in each four tests. The value of the initial velocity is then

$$u_0(x) = \mathbf{1}_{0 < x < 1/2} \ [\text{Test 1}], \quad u_0(x) = \frac{1}{2} \ [\text{Test 2}], \quad u_0(x) = 0 \ [\text{Test 3}],$$

and

$$u_0(x) = -\frac{1}{2}\mathbf{1}_{0 < x < 1/2} + \frac{1}{2}\mathbf{1}_{1/2 < x < 1} \ [\text{Test 4}].$$

For the four test cases considered, the quantity $\int_{\mathbb{T}} h dx$ is the same of course and $\int_{\mathbb{T}} q dx$ has a common value in Tests 1-2 and 3-4 respectively. Observe indeed the common convergence in Tests 1-2 and 3-4. The proof of the existence of an invariant measure will be addressed in a future work.

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A   A bound from below

Definition A.1. Let $\tau > 0$. Let $\mathbf{1}_{\det}$ be the step function defined by (3.49). Let $u \in L^3(Q_T)$ and $\rho_0 \in L^2(\mathbb{T})$. A function $\rho \in C([0,T]; L^2(\mathbb{T}))$ is said to be a generalized solution of the problem

$$\frac{1}{2}\partial_t\rho + \mathbf{1}_{\det}\left[\partial_x(\rho u) - \partial_x^2\rho\right] = 0 \text{ in } Q_T, \tag{A.1}$$

with initial condition

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{T}, \tag{A.2}$$

if

$$\rho \in C([0,T]; L^2(\mathbb{T})), \quad \mathbf{1}_{\det}\rho \in L^2(0,T; H^1(\mathbb{T})), \tag{A.3}$$

and, for all $\varphi \in L^2(\mathbb{T}; H^1(0,T))$ with $\varphi(\cdot, T) = 0$ such that

$$\mathbf{1}_{\det}\varphi \in L^2(0,T; H^1(\mathbb{T})),$$

one has

$$\iint_{Q_T} \frac{1}{2}\rho\partial_t\varphi + \mathbf{1}_{\det}\left[\rho u - \partial_x\rho\right]\partial_x\varphi \, dxdt + \frac{1}{2}\int_{\mathbb{T}} \rho_0(x)\varphi(x,0)dx = 0. \tag{A.4}$$

This definition of solution to (A.1)-(A.2) corresponds to the definition of generalized solutions in [LSU68, Eq. (1.16), Chap III]. The term

$$\iint_{Q_T} \mathbf{1}_{\det}\rho u\partial_x\varphi \, dxdt$$

in (A.4) is well defined as we can see by using the Hölder inequality, which gives

$$\iint_{Q_T} |\mathbf{1}_{\det}\rho u\partial_x\varphi| \, dxdt \leq \|\partial_x\varphi\|_{L^2(Q_T)}\|u\|_{L^3(Q_T)}\|\mathbf{1}_{\det}\rho\|_{L^6(Q_T)},$$

and then using the following estimate: for all function z satisfying the same condition (A.3) as ρ,

$$\|\mathbf{1}_{\det}z\|_{L^6(Q_T)} \leq C\left(\sup_{t\in[0,T]} \|z(t)\|_{L^2(\mathbb{T})}\right)^{2/3}\|\mathbf{1}_{\det}\partial_x z\|_{L^2(Q_T)}^{1/3} + C\sup_{t\in[0,T]} \|z(t)\|_{L^2(\mathbb{T})}. \tag{A.5}$$

Let us recall the proof of (A.5). We apply the Gagliardo-Nirenberg inequality, [Nir59, Formula (2.2)] with $j = 0$, $m = n = 1$, $a = \frac{1}{3}$, $r = q = 2$, using also [Nir59, Remark 5. p. 126] with $\tilde{q} = 2$, to $z(t)$, $t \in [0,T]$. This gives

$$\|z(t)\|_{L^6(\mathbb{T})} \leq C\|z(t)\|_{L^2(\mathbb{T})}^{2/3}\|\partial_x z(t)\|_{L^2(\mathbb{T})}^{1/3} + C\|z(t)\|_{L^2(\mathbb{T})}.$$

Then we multiply the result by $\mathbf{1}_{\det}(t)$ and we sum over $t \in [0,T]$.

Note also that, if $0 \leq t_{2n} < T - \tau$ (where $t_k = k\tau$) and if φ vanishes outside (t_{2n}, t_{2n+1}), then, by (A.4), we have

$$\iint_{Q_{t_{2n},t_{2n+1}}} \frac{1}{2}\rho\partial_t\varphi + [\rho u - \partial_x\rho]\partial_x\varphi \, dxdt = 0.$$

Let $\rho_{2n}(x) = \rho(x, t_{2n})$. Taking $\varphi(x,t) = \psi(x,t)\min\left(\frac{t-t_{2n}}{h}, 1\right)$ where $h \in (0, \tau)$ and where $\psi \in H^1(Q_{t_{2n}, t_{2n+1}})$ vanishes at $t = t_{2n+1}$, then letting $h \to 0$ (this is possible since ρ is continuous at $t = t_{2n}$ with values in $L^2(\mathbb{T})$), we obtain

$$\iint_{Q_{t_{2n}, t_{2n+1}}} \frac{1}{2}\rho\partial_t\psi + [\rho u - \partial_x\rho]\,\partial_x\psi \ dxdt + \frac{1}{2}\int_{\mathbb{T}} \rho_{2n}(x)\psi(x, t_{2n})dx = 0.$$

This means that, in restriction to $Q_{t_{2n}, t_{2n+1}}$, ρ is a generalized solution to the problem

$$\frac{1}{2}\partial_t\rho + \left[\partial_x(\rho u) - \partial_x^2\rho\right] = 0 \text{ in } Q_{t_{2n}, t_{2n+1}}, \tag{A.6}$$

with initial condition

$$\rho(x, t_{2n}) = \rho_{2n}(x), \quad x \in \mathbb{T}. \tag{A.7}$$

Similarly, we show that $\rho(x,t) = \rho(x, t_{t_{2n+1}})$ for a.e. $x \in \mathbb{T}$, for all $t \in [t_{2n+1}, t_{2n+2}]$. In particular, Problem (A.1)-(A.2) has a unique solution. Indeed, by [LSU68, Theorem 2.1, Chap III], we have

$$\|\rho(t_{2n+1})\|_{L^2(\mathbb{T})} \leq \sup_{t \in [t_{2n}, t_{2n+1}]} \|\rho(t)\|_{L^2(\mathbb{T})} + \|\partial_x\rho\|_{L^2(Q_{t_{2n}, t_{2n+1}})} \leq c\|\rho(t_{2n})\|_{L^2(\mathbb{T})},$$

where c depends on $\|u\|_{L^{3/2}(Q_T)}$ only. Since $\rho(t)$ is constant on intervals of the form $[t_{2n}, t_{2n+1}]$, it follows that

$$\sup_{t \in [0,T]} \|\rho(t)\|_{L^2(\mathbb{T})} \leq c^K\|\rho_0\|_{L^2(\mathbb{T})},$$

where K is such that $T \leq K\tau$. In particular, $\rho = 0$ if $\rho_0 = 0$. Introduce the notation

$$t_\sharp := \min(2t - t_{2n}, t_{2n+2}), \quad t_\flat := \frac{t + t_{2n}}{2}, \quad t_{2n} \leq t < t_{2n+2}.$$

Note that $(t_\sharp)_\flat = t$ if $t_{2n} < t < t_{2n+1}$ and that $(t_\flat)_\sharp = t$ for all t. Set $u_\flat(x,t) = u(x, t_\flat)$. By uniqueness, we have

$$\rho(x,t) = \zeta(x, t_\sharp) \text{ in } Q_T, \tag{A.8}$$

where $\zeta \in C([0, T_\flat]; L^2(\mathbb{T}))$ is the generalized solution of the problem

$$\partial_t\zeta + \partial_x(\zeta u_\flat) - \partial_x^2\zeta = 0 \text{ in } Q_{T_\flat}, \tag{A.9}$$

with initial condition

$$\zeta(x, 0) = \rho_0(x), \quad x \in \mathbb{T}. \tag{A.10}$$

Indeed, we start from

$$\iint_{Q_{T_\flat}} \zeta\partial_t\psi + [\zeta u_\flat - \partial_x\zeta]\,\partial_x\psi \ dxdt + \int_{\mathbb{T}} \rho_0(x)\psi(x, 0)dx = 0, \tag{A.11}$$

for all $\psi \in H^1(Q_{T_\flat})$ with $\psi(T_\flat) = 0$. Let $\varphi \in L^2(\mathbb{T}; H^1(0, T))$ with $\varphi(\cdot, T) = 0$ be such that

$$\mathbf{1}_{\det}\varphi \in L^2(0, T; H^1(\mathbb{T})).$$

Set $\psi(x,t) := \varphi(x, t_\flat)$. Then $\psi \in L^2(0, T_\flat; H^1(\mathbb{T}))$ and ψ vanishes at $t = T_\flat$. We do not have $\psi \in L^1(\mathbb{T}; H^1(0, T_\flat))$ since ψ has jumps at every points $t = t_{2n}$. However, an argument of

approximation of the discontinuous function $s \mapsto s_\flat$ allows us to deduce from (A.11) that

$$\sum_n \int_{\mathbb{T}} \zeta(x, t_{2n+2})(\varphi(x, t_{2n+2}) - \varphi(x, t_{2n+1}))dx$$

$$+ \iint_{Q_{T_\flat}} \frac{1}{2}\zeta(x,t)\partial_t\varphi(x,t_\flat) + [\zeta u_\flat - \partial_x\zeta](x,t)\partial_x\varphi(x,t_\flat) \; dxdt$$

$$+ \int_{\mathbb{T}} \rho_0(x)\varphi(x,0)dx = 0.$$

By a change of variable $t_\flat \mapsto t$ on every (t_{2n}, t_{2n+2}), we obtain

$$\sum_n \int_{\mathbb{T}} \zeta(x, t_{2n+2})(\varphi(x, t_{2n+2}) - \varphi(x, t_{2n+1}))dx$$

$$+ \sum_n \iint_{Q_{t_{2n},t_{2n+1}}} \zeta(x,t_\sharp)\partial_t\varphi(x,t) + 2[\zeta u_\flat - \partial_x\zeta](x,t_\sharp)\partial_x\varphi(x,t) \; dxdt$$

$$+ \int_{\mathbb{T}} \rho_0(x)\varphi(x,0)dx = 0.$$

Rewriting

$$\varphi(x, t_{2n}) - \varphi(x, t_{2n-1}) = \int_{t_{2n-1}}^{t_{2n}} \partial_t\varphi \, dt,$$

we have

$$\sum_n \int_{\mathbb{T}} \zeta(x, t_{2n+2})(\varphi(x, t_{2n+2}) - \varphi(x, t_{2n+1}))dx = \sum_n \iint_{Q_{t_{2n-1},t_{2n}}} \zeta(x,t_\sharp)\partial_t\varphi(x,t)dxdt.$$

Furthermore, we use $\sum_n \iint_{Q_{t_{2n},t_{2n+1}}} Z \; dxdt = \sum_n \iint_{Q_{t_{2n},t_{2n+2}}} \mathbf{1}_{\det}Z \; dxdt$ with the function $Z(t,x) = [\zeta u_\flat - \partial_x\zeta](x,t_\sharp)\partial_x\varphi(x,t)$. Using $u_\flat(t_\sharp) = u(t)$ shows that $(x,t) \mapsto \zeta(x,t_\sharp)$ satisfies (A.4). Consequently, $\rho(x,t) = \zeta(x,t_\sharp)$ as asserted.

Theorem A.1 (Positivity). *Let $\tau > 0$. Let $\mathbf{1}_{\det}$ be the step function defined by (3.49). Let $u \in L^3(Q_T)$ and $\rho_0 \in L^2(\mathbb{T})$. Let $\rho \in C([0,T]; L^2(\mathbb{T}))$ be the generalized solution of the problem (A.1)-(A.2). Assume $\rho_0 \geq c_0$ a.e. in $\mathbb{T}$ where c_0 is a positive constant and let $m > 6$. Then there exists a constant $c > 0$ depending continuously on c_0, T, m and*

$$\iint_{Q_T} \rho|\partial_x u|^2 dxdt \quad and \quad \|u\|_{L^m(Q_T)}, \tag{A.12}$$

such that

$$\rho \geq c \tag{A.13}$$

a.e. in Q_T.

Proof. By (A.8), it is sufficient to consider the equation (A.9) satisfied by ζ. Note that $\zeta \in L^6(Q_{T_\flat})$ by (A.5) since $\zeta \in C([0,T_\flat]; L^2(\mathbb{T}))$ and $\zeta \in L^2(0,T; H^1(\mathbb{T}))$. Since $u_\flat \in L^3(Q_{T_\flat})$, we have $\zeta u_\flat \in L^2(Q_{T_\flat})$. It follows from (A.9) that $\zeta_t \in L^2_t H^{-1}_x$. Let $h: \mathbb{R}_+ \to (0, +\infty)$ be a function of class $W^{2,\infty}$ and let $w = h(|\zeta|)$. Actually ζ is non-negative (see (A.19) below), so $w = h(\zeta)$. We will use the function

$$h(\zeta) = -\frac{\zeta}{\max(\zeta, r)^2} + \frac{2}{\max(\zeta, r)}, \tag{A.14}$$

where r is a positive parameter, $r \in (0,1)$. We will prove an L^∞-estimate on w that is uniform in $r > 0$. By passing to the limit $r \to 0$, this will give a bound from below on ζ and on ρ. By a chain-rule formula (*cf.* Lemma 1.4 in Carrillo, Wittbold [CW99] and Lemma 1.1 in Stampacchia [Sta65] for example) we derive the following equation for w:

$$\partial_t w - \partial_{xx}^2 w = -\frac{2}{w} \mathbf{1}_{rw \leq 1} |\partial_x w|^2 - \zeta h'(\zeta) \partial_x u_\flat - u_\flat \partial_x w. \tag{A.15}$$

Similarly, we have, for $p \geq 2$,

$$\partial_t \frac{z^2}{p} - \partial_{xx}^2 \frac{z^2}{p} = -\frac{4}{p^2} \left(2\mathbf{1}_{rw \leq 1} + p - 1 \right) |\partial_x z|^2 - \frac{\zeta h'(\zeta)}{w} z^2 \partial_x u_\flat - \frac{u_\flat}{p} \partial_x z^2, \tag{A.16}$$

where $z := w^{p/2}$. We will use (A.16) and an energy estimate to prove the bound

$$\sup_{t \in [0,T_\flat]} \|w(t)\|_{L^p(\mathbb{T})}^p + \|\partial_x w^{p/2}\|_{L^2(Q_{T_\flat})}^2 \leq C \|w(0)\|_{L^p(\mathbb{T})}^p, \tag{A.17}$$

where C is a constant depending on p, m, T, $\|u\|_{L^m(Q_T)}$. Let us sum (A.16) on $\mathbb{T}$ (*i.e.* we consider a test-function independent on x): we obtain

$$\frac{d}{dt} \int_\mathbb{T} z^2 dx + \frac{4(p-1)}{p} \int_\mathbb{T} |\partial_x z|^2 dx \leq -2 \int_\mathbb{T} u_\flat z \partial_x z dx - \int_\mathbb{T} G(z) \partial_x u_\flat dx,$$

where we have introduced the function $G(z)$ defined by the implicit identity

$$G(z) = p \frac{\zeta h'(\zeta)}{w} z^2.$$

By integration by parts, [Sta65, Lemma 1.1], we get

$$\frac{d}{dt} \int_\mathbb{T} z^2 dx + \frac{4(p-1)}{p} \int_\mathbb{T} |\partial_x z|^2 dx \leq \int_\mathbb{T} |u_\flat| |\partial_x z| \left[2|z| + |G'(z)| \right] dx.$$

It is easy to check that $|G'(z)| \leq (2p+2)|z|$. Consequently, we have

$$\frac{d}{dt} \int_\mathbb{T} z^2 dx + \frac{2(p-1)}{p} \int_\mathbb{T} |\partial_x z|^2 dx \leq \frac{2p(p+2)^2}{(p-1)} \int_\mathbb{T} u_\flat^2 z^2 dx.$$

Integrating then over $t \in [0, \sigma]$ where $\sigma \leq T_\flat$, we obtain

$$U_\sigma \leq \frac{2p(p+2)^2}{(p-1)} \iint_{Q_\sigma} u_\flat^2 z^2 dx dt + \|z(0)\|_{L^2(\mathbb{T})}^2,$$

where

$$U_\sigma := \sup_{t \in [0,\sigma]} \|z(t)\|_{L^2(\mathbb{T})}^2 + \frac{2(p-1)}{p} \|\partial_x z\|_{L^2(Q_\sigma)}^2.$$

By the Hölder Inequality, it follows that

$$U_\sigma \leq \frac{2p(p+2)^2}{(p-1)} \|u_\flat\|_{L^3(Q_\sigma)}^2 \|z\|_{L^6(Q_\sigma)}^2 + \|z(0)\|_{L^2(\mathbb{T})}^2. \tag{A.18}$$

To obtain an estimate on the right hand-side of (A.18), we apply (A.5) (without $\mathbf{1}_{det}$). This gives

$$U_\sigma \leq C^2 \, C(p) \|u_\flat\|_{L^3(Q_\sigma)}^2 \, U_\sigma + \|z(0)\|_{L^2(\mathbb{T})}^2,$$

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with $C(p) = \frac{4p(p+2)^2}{(p-1)} \left(\frac{2p}{(p-1)}\right)^{1/3}$, and then, since $m > 3$,

$$U_\sigma \le C^2\, C(p)\sigma^e \|u\|^2_{L^m(Q_T)} U_\sigma + \|z(0)\|^2_{L^2(\mathbb{T})}, \quad e := \frac{2}{3} - \frac{2}{m}.$$

Let $\sigma_0 > 0$ be defined by

$$C^2\, C(p)\sigma_0^e \|u\|^2_{L^m(Q_T)} = \frac{1}{2}.$$

For $\sigma < \sigma_0$, we obtain $U_\sigma \le 2\|w^{p/2}(0)\|^2_{L^2(\mathbb{T})}$, where $\sigma_0 > 0$ is some constant depending only on $\|u\|_{L^m(Q_T)}$ and p. Since an estimate on U_σ gives in turn an estimate on $\|z(\sigma)\|^2_{L^2(\mathbb{T})} = \|w^{p/2}(\sigma)\|^2_{L^2(\mathbb{T})}$, we can iterate our procedure from $[0, \sigma_0]$ to $[\sigma_0, 2\sigma_0]$ and so on to deduce the bound (A.17) with $C = 2^N$ with N an integer such that $N\sigma_0 \ge T_\flat$.

Note that the energy estimates argument we used can be adapted to show that

$$\zeta \ge 0 \quad \text{a.e. in } Q_T \tag{A.19}$$

(simply work on the equation satisfied by ζ_-, the negative part of ζ).

In the second step of the proof, we will derive the following L^∞ estimate on w:

$$\|w\|_{L^\infty(Q_{T_\flat})} \le C\left(T, c_0, \|u\|_{L^m(Q_T)}, \|\rho^{1/2}\partial_x u\|_{L^2(Q_T)}\right). \tag{A.20}$$

To prove (A.20), we use the equation (A.15). It is classical [Bal77] that the weak solution w is also a mild solution to (A.15):

$$w(t) = S(t)w(0) + \int_0^t S(t-s)f(s)ds,$$

where f is the right hand-side of (A.15). Since

$$f \le |\zeta h'(\zeta)||\partial_x u_\flat| - u_\flat \partial_x w,$$

we obtain

$$0 \le w(t) \le S(t)w(0) + W_1(t) + W_2(t),$$

with

$$W_1(t) = \int_0^t S(t-s)(|\zeta h'(\zeta)||\partial_x u_\flat|)(s)ds,$$

$$W_2(t) = -\int_0^t S(t-s)(u_\flat \partial_x w)(s)ds.$$

Let us set $g = \zeta^{1/2}|\partial_x u_\flat|$. We check on (A.14) that

$$\zeta^{1/2}|h'(\zeta)| \le h(\zeta)^{3/2} = w^{3/2} \tag{A.21}$$

for all $\zeta \ge 0$. Indeed, $\zeta^{1/2}|h'(\zeta)| = h(\zeta)^{3/2}$ if $\zeta \ge r$. If $0 \le \zeta \le r$, then $\zeta^{1/2}|h'(\zeta)| = \frac{\zeta^{1/2}}{r^2} \le \frac{1}{r^{3/2}}$ and $h(\zeta) \ge \frac{1}{r}$. By (A.21), we have

$$W_1(t) \le W_3(t) := \int_0^t S(t-s)(w^{3/2}g)(s)ds.$$

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Let $q \in [1, +\infty)$, $\nu \in [1, 2)$ be given. By (B.7a) with $j = 0$, we have

$$\|W_3\|_{L^\infty(Q_{T_\flat})} \le C \|w^{3/2} g\|_{L^\nu(Q_{T_\flat})}, \quad \frac{1}{\nu} < \frac{2}{3}, \tag{A.22}$$

$$\le C \|w\|_{L^q(Q_{T_\flat})}^{3/2} \|g\|_{L^2(Q_{T_\flat})}, \tag{A.23}$$

provided q and ν satisfy the relation

$$\frac{1}{2} + \frac{3}{2q} = \frac{1}{\nu}. \tag{A.24}$$

Using the energy estimate (A.17) with $p = 2$ and (B.7a), we have also

$$\|W_2\|_{L^\infty(Q_{T_\flat})} \le C \|u_\flat \partial_x w\|_{L^\nu(Q_{T_\flat})}, \quad \frac{1}{\nu} < \frac{2}{3}, \tag{A.25}$$

$$\le C \|w(0)\|_{L^2(\mathbb{T})} \|u\|_{L^m(Q_T)}$$

$$\le \frac{C}{c_0} \|u\|_{L^m(Q_T)}, \tag{A.26}$$

provided

$$\frac{1}{2} + \frac{1}{m} = \frac{1}{\nu}. \tag{A.27}$$

Finally, we have

$$\|S(\cdot) w(0)\|_{L^\infty(Q_{T_\flat})} \le \frac{1}{c_0}. \tag{A.28}$$

If $m > 6$, then there exists $\nu > 1$ satisfying (A.25)-(A.27). If $q > \frac{3}{2}$, then there exists $r > 1$ satisfying (A.22)-(A.24). Thus we use (A.17) with $p > \frac{3}{2}$. It follows from (A.23) that $\|W_3\|_{L^\infty(Q_{T_\flat})} \le \frac{C}{c_0^{3/2}} \|g\|_{L^2(Q_{T_\flat})}$. By (A.26)-(A.28), we conclude that $\|w\|_{L^\infty(Q_{T_\flat})}$ is bounded by a quantity depending only on c_0 and on the norms in (A.12). This concludes the proof of Theorem A.1.∎

Remark A.1. Note also that it is possible to give some precisions on the bound from below (I.57) in [LPS96], regarding the positivity of the density ρ in the deterministic parabolic approximation of the isentropic Euler system. Since, for such a system, the terms in (A.12) are bounded, respectively, by the initial entropy

$$\int_{\mathbb{T}} \eta_E(\mathbf{U}_0(x)) dx \le C(\|\rho_0\|_{L^\infty(\mathbb{T})}, \|u_0\|_{L^\infty(\mathbb{T})})$$

and the L^∞ norm

$$\|u\|_{L^\infty(Q_T)} \le TC(\|\rho_0\|_{L^\infty(\mathbb{T})}, \|u_0\|_{L^\infty(\mathbb{T})}),$$

where here C is a continuous function of its arguments, we obtain $\rho \ge c_1$ a.e. in Q_T, where c_1 depends continuously on T, $\|\rho_0\|_{L^\infty(\mathbb{T})}$, $\|u_0\|_{L^\infty(\mathbb{T})}$, c_0, where $c_0 = \inf_{x \in \mathbb{T}} \rho_0(x)$.

B Regularizing effects of the one-dimensional heat equation

In this section, we collect some results on the regularizing effects of the one-dimensional periodic heat equation: all the estimates below are very classical, but given for the heat equation on the whole line $\mathbb{R}$ in general. Since the proofs for the case of the circle are simple and short, we give them in full detail, see [CH98] for further results.

B.1 Heat semi-group

Let us denote by $(S(t))$ the one-dimensional Heat semi-group associated to the Heat Equation

$$(\partial_t - \partial_{xx}^2)z = f, \tag{B.1}$$

on $Q_T = \mathbb{T} \times (0, T)$. For some given data z_0 and f (*e.g.* integrable on $\mathbb{T}$ and Q_T respectively), the mild solution in $C([0, T]; L^1(\mathbb{T}))$ to (B.1) satisfying $z(0) = z_0$ is given by the formula

$$z(t) = S(t)z_0 + \int_0^t S(t - s)f(s)ds. \tag{B.2}$$

Using either a spectral decomposition or working on $\mathbb{R}$ with periodic functions, we obtain

$$S(t)u(x) = K_t * u(x) = \int_{\mathbb{T}} K_t(y)u(x - y)dy,$$

where the Kernel $K_t(x)$ is defined by

$$K_t(x) = \sum_{n \in \mathbb{Z}} e^{-4\pi^2 n^2 t} e_n(x) = \sum_{n \in \mathbb{Z}} G_t(x + n). \tag{B.3}$$

Here e_n is the n-th Fourier basis element on $\mathbb{T}$ and G_t the heat kernel on $\mathbb{R}$:

$$e_n(x) := e^{2\pi i n x}, \quad G_t(x) := \frac{1}{(4\pi t)^{1/2}} e^{-\frac{|x|^2}{4t}}.$$

By the second identity in (B.3), we easily obtain for $p = 1$ or $p = +\infty$ the estimate

$$\|\partial_t^k \partial_x^j K_t\|_{L^p(\mathbb{T})} \le C(k, j, p) t^{-k - j/2 - 1/(2p')}, \tag{B.4}$$

for all $k, j \in \mathbb{N}$, $t > 0$, where p' is the conjugate exponent to p and $C(k, j, p)$ is a constant depending on k, j, p only. By interpolation between the cases $p = 1$ and $p = +\infty$, we obtain (B.4) for all $p \in [1, +\infty]$. By the Young Inequality, we have, for $1 \le p \le q$,

$$\|S(t)\|_{L_x^p \to L_x^q} \le \|K_t\|_{L^m(\mathbb{T})},$$

where $\frac{1}{p} + \frac{1}{m} = 1 + \frac{1}{q}$. It follows from (B.4) that

$$\|S(t)\|_{L_x^p \to L_x^q} \le C(p, q) t^{-\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right)}, \tag{B.5}$$

for $1 \le p \le q \le +\infty$, for a given constant $C(p, q)$ and, more generally,

$$\|\partial_t^k \partial_x^j S(t)\|_{L_x^p \to L_x^q} \le C(p, q, k, j) t^{-\frac{1}{2}\left(\frac{1}{p} - \frac{1}{q}\right) - \frac{j}{2} - k}, \tag{B.6}$$

for $k, j \in \mathbb{N}$, $1 \le p \le q \le +\infty$. We deduce from (B.6) the following result.

Lemma B.1. *Let* $1 \le p \le +\infty$, $j \in \mathbb{N}$, $f \in L^p(Q_T)$, $z_0 \in L^p(\mathbb{T})$ *then*

$$\left\|\partial_x^j \int_0^t S(t - s)f(s)ds\right\|_{L^q(Q_T)} \le C\|f\|_{L^p(Q_T)} \quad if \quad \frac{1}{q} \le \frac{1}{p} < \frac{1}{q} + \frac{2 - j}{3}, \tag{B.7a}$$

$$\left\|\partial_x^j S(t)z_0\right\|_{L^q(Q_T)} \le C\|z_0\|_{L^p(\mathbb{T})} \quad if \quad \frac{1}{q} \le \frac{1}{p} < \frac{3}{q} - j, \tag{B.7b}$$

where the constant C depends on p, q, j, T.

Proof. we have

$$\left\|\partial_x^j S(t)z_0\right\|_{L^q(\mathbb{T})} \leq C(p,q,j)\|z_0\|_{L^p(\mathbb{T})}t^{-\mu}, \quad \mu := \frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right) + \frac{j}{2},$$

by (B.6) if $p \leq q$. The right-hand side is in L_t^q if $\mu < \frac{1}{q}$, *i.e.* $\frac{1}{p} < \frac{3}{q} - j$. Similarly,

$$\left\|\partial_x^j S(t-s)f(s)\right\|_{L^q(\mathbb{T})} \leq C(p,q,j)\|f(s)\|_{L^p(\mathbb{T})}(t-s)^{-\mu},$$

if $p \leq q$. Let $g(t) = t^{-\mu}\mathbf{1}_{t\in(0,T)}$, $h(t) = \|f(t)\|_{L^p(\mathbb{T})}\mathbf{1}_{t\in(0,T)}$. By the Young Inequality for the convolution of functions, we have

$$\left\|\partial_x^j \int_0^t S(t-s)f(s)ds\right\|_{L^q(Q_T)} \leq C(p,q,j)\|h\|_{L^p(0,T)}\|g\|_{L^m(0,T)}, \quad \frac{1}{p} + \frac{1}{m} = 1 + \frac{1}{q},$$

and $\|g\|_{L^m(0,T)} < +\infty$ if, and only if, $m\mu < 1$. This last condition is equivalent to

$$\frac{1}{2}\left(\frac{1}{p}-\frac{1}{q}\right) + \frac{j}{2} < 1 + \frac{1}{q} - \frac{1}{p},$$

i.e. $\frac{1}{p} < \frac{1}{q} + \frac{2-j}{3}$. ∎

Let $J = (\mathrm{Id} - \partial_x^2)^{-1/2}$ and $s \in (0,1)$. Using the first identity in (B.3) (spectral decomposition), we have

$$\|J^s S(t)u\|_{L^2(\mathbb{T})}^2 = \sum_{n\in\mathbb{Z}}(1 + 4\pi^2|n|^2)^{s/2}e^{-8\pi^2|n|^2 t}|\langle u, e_n\rangle|^2,$$

which gives

$$\|J^s S(t)\|_{L_x^2 \to L_x^2} \leq Ct^{-\frac{s}{2}}, \tag{B.8}$$

where C is a constant depending on s only.

We finish this part with the proof of the following estimate (B.9), that we will need in (3.27). Let $u \in H^1(\mathbb{T})$. Using the first identity in (B.3) (spectral decomposition), we have

$$\begin{aligned}
\|S(t)u - u\|_{L^2(\mathbb{T})}^2 &= \sum_{n\in\mathbb{Z}}|\langle u, e_n\rangle|^2\left|1 - e^{-4\pi n^2 t}\right|^2 \\
&\leq 2\sum_{n\in\mathbb{Z}}|\langle u, e_n\rangle|^2\left|1 - e^{-4\pi n^2 t}\right| \\
&\leq 8\pi\sum_{n\in\mathbb{Z}}|\langle u, e_n\rangle|^2 n^2 t = \frac{2}{\pi}\|\nabla u\|_{L^2(\mathbb{T})}^2 t.
\end{aligned} \tag{B.9}$$

B.2 Fractional Sobolev space

For $0 < s < 1$, $1 < p < +\infty$, $1 \leq q \leq +\infty$ we recall that we can define the Besov Space $B_{p,q}^s(\mathbb{T})$ as a space of functions $u \in L^p(\mathbb{T})$ such that

$$[u]_{B_{p,q}^s(\mathbb{T})} = \left(\int_{-1/2}^{1/2}\|\Delta_h u\|_{L^p(\mathbb{T})}^q|h|^{-1-sp}dh\right)^{1/q} < +\infty, \quad \Delta_h u(x) := u(x+h) - u(x). \tag{B.10}$$

Then we set $\|u\|_{B_{p,q}^s(\mathbb{T})} = \|u\|_{L^p(\mathbb{T})} + [u]_{B_{p,q}^s(\mathbb{T})}$, see Theorem 1.2.5 in Triebel, [Tri92].

It is easy to show the algebra property

$$\|uv\|_{B^s_{p,q}(\mathbb{T})} \lesssim \|u\|_{L^\infty(\mathbb{T})}\|v\|_{B^s_{p,q}(\mathbb{T})} + \|v\|_{L^\infty(\mathbb{T})}\|u\|_{B^s_{p,q}(\mathbb{T})}, \tag{B.11}$$

for $u, v \in B^s_{p,q}(\mathbb{T}) \cap L^\infty(\mathbb{T})$. Similarly, any $F\colon \mathbb{R} \to \mathbb{R}$ locally Lipschitz satisfying $F(0) = 0$ operates on $B^s_{p,q}(\mathbb{T}) \cap L^\infty(\mathbb{T})$:

$$\|F(u)\|_{B^s_{p,q}(\mathbb{T})} \leq \mathrm{Lip}_R(F)\|u\|_{B^s_{p,q}(\mathbb{T})}, \quad R := \|u\|_{L^\infty(\mathbb{T})}, \tag{B.12}$$

where $\mathrm{Lip}_R(F)$ is the Lipschitz constant of F in restriction to $[-R, R]$.

For $0 < s < 1$ and $1 \leq p < +\infty$, we denote by $H^s_p(\mathbb{T})$ the Bessel potential space of functions $u \in L^p(\mathbb{T})$ such that $J^s u \in L^p(\mathbb{T})$, where $J = (\mathrm{Id} - \partial^2_x)^{-1/2}$, with the norm

$$\|u\|_{H^s_p(\mathbb{T})} = \|u\|_{L^p(\mathbb{T})} + \|J^s u\|_{L^p(\mathbb{T})}.$$

We then have

$$B^s_{22}(\mathbb{T}) = H^s_2(\mathbb{T}), \tag{B.13}$$

see Equation (7) in Theorem 1.3.2 of [Tri92]. Actually the references we give in [Tri92] are for spaces of functions on $\mathbb{R}$, but the results are valid on $\mathbb{T}$, see Remark 4, paragraph 1.5.4 of [Tri92]. We denote by $W^{s,2}(\mathbb{T})$ the space in (B.13), used in Proposition 3.3 for example.

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