SOME REMARKS ON KÄHLER MANIFOLDS WITH $c_1 = 0$

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These notes consist mainly of comments and applications of the results of [B]. We first recall the structure theorem for compact Kähler manifolds with $c_1 = 0$. Up to a finite covering, such a manifold splits as a product of irreducible factors of 3 possible types : complex tori, special unitary projective manifolds and Kähler symplectic manifolds. After showing a list of examples we give some applications, mainly to the study of automorphisms. We extend to our manifolds some results of Nikulin on automorphisms of K3 surfaces. We then consider automorphisms of symplectic manifolds which induce the identity in cohomology. Finally we conclude with an example of a birational automorphism (of a projective symplectic manifold) which is not biregular, contrary to a conjecture of Bogomolov.

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1. The Structure Theorem

Let me first set up some terminology. A manifold is always assumed to be connected. By a Kähler manifold I mean a complex manifold which admits at least one Kähler metric.

The structure theorem for manifolds with $c_1 = 0$ goes back, in a weak form, to Calabi [C]. A stronger version was proved by Bogomolov in 1974 [Bo]. Finally the proof by S-T. Yau of the Calabi conjecture made possible to give an easy proof of the strongest possible statement. This fact seems to have been noticed independently by various mathematicians, in particular S. Kobayashi and M.L. Michelsohn [M].

Theorem

Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$.

1) The universal covering of X is isomorphic to a product $\mathbb{C}^k \times \prod_i V_i \times \prod_j X_j$, where

a) V_i is a simply connected projective manifold, of dimension ≥ 3, with trivial canonical bundle, such that $H^0(V_i, \Omega^p_{V_i}) = 0$ for $0 < p < \dim(V_i)$.

b) X_j is a simply connected compact Kähler manifold, admitting a holomorphic 2-form φ_j which is everywhere non-degenerate (as an alternate form on the holomorphic tangent bundle). Any holomorphic form on X_j is (up to a scalar) a power of φ_j.

This decomposition is unique, up to the order of the V_i's and of the X_j's.

2) There exists a finite étale cover $\tilde{X}$ of X which is isomorphic to a product $T \times \prod_i V_i \times \prod_j X_j$, where T is a complex torus.

Let us give a sketch of the proof, referring to [B]for the details. According to Yau's theorem, X carries a Ricci-flat Kähler metric. The De Rham theorem ([K-N], IX.8) implies that the universal covering of X is isomorphic (as a Kähler manifold) to a product $\mathbb{C}^k \times \prod_i M_i$, where for each i the manifold M_i has irreducible holonomy. Moreover M_i is compact by the Cheeger-Gromoll theorem [C-G]. Since M_i is Ricci-flat, its holonomy group H_i is contained in $SU(m_i)$. The list of holonomy groups given by Berger [Be] leaves only two possibilities for H_i, namely $H_i = SU(m_i)$ and $H_i = Sp\,(m_i/2)$ (if m_i is even).

We now consider holomorphic forms on M_i. The Bochner principle implies that on a compact Kähler Ricci-flat manifold any holomorphic form is parallel. Therefore the space of holomorphic p-forms on M_i is holomorphic to the space of those p-forms at a given point which are invariant under H_i. From the representation theory of the unitary and symplectic groups one deduces easily that M_i satisfies property a) of the theorem if $H_i = SU(m_i)$ and property b) if $H_i = Sp(m_i/2)$ (in case $H_i = SU(m_i)$ with $m_i \geqslant 3$, we observe that the vanishing of $H^{2,0}$ implies that M_i is projective).

This proves the existence of the decomposition 1). The unicity is deduced easily from the unicity of the De Rham decomposition and the unicity of a Ricci-flat metric in a given cohomology class. Finally 2) follows essentially from the classical Bieberbach theorem.

For obvious reasons, manifolds satisfying property a) will be called special unitary, while those satisfying b) will be called (irreducible) symplectic.

Let us mention some obvious consequences of the theorem. The fundamental group of X is an extension of a finite group by a group $\mathbb{Z}^{2\tilde{q}}$ where $\tilde{q}$ is the maximum irregularity of the finite coverings of X. If $\chi(\mathcal{O}_X)$ is nonzero, then $\tilde{q} = 0$ and $\pi_1(X)$ is finite. In any case the canonical bundle is a torsion element of Pic(X).

The following consequences are perhaps less obvious :

<u>Corollary</u> :

Let X be a compact Kähler manifold with $c_1^{\mathbb{R}}(X) = 0$, of dimension n.

(i) *If n is odd, one has $\chi(\mathcal{O}_X) = 0$.*

(ii) *If $n = 2r$, one has $0 \leqslant \chi(\mathcal{O}_X) \leqslant 2^r$. The equality $\chi(\mathcal{O}_X) = 2^r$ holds if and only if X is a product of K3 surfaces.*

(iii) *One has $h^{p,0}(X) \leqslant \binom{n}{p}$ for all p. If equality holds for one value of p with $0 < p < n$, then X is a complex torus.*

The assertion (i) follows at once from Serre duality. Let $\tilde{X}$ be a finite covering of X which is isomorphic to a product of manifolds M_i

of dimension m_i, which either are complex tori, or satisfy property a) or b). Then

$$0 \leqslant \chi(\mathcal{O}_{M_i}) \quad \frac{m_i}{2} + 1 \leqslant 2^{m_i/2} \quad ,$$

with equality if and only if M_i is a K3 surface. Since

$$\chi(\mathcal{O}_X) \leqslant \chi(\mathcal{O}_X) = \prod_i \chi(\mathcal{O}_{M_i}) \quad , \text{ this implies (ii).}$$

Let us prove (iii). Let T_i be a complex torus of dimension m_i .

One has $$h^{p,0}(M_i) \leqslant h^{p,0}(T_i) \quad ,$$

and equality holds (for $0 < p < m_i$) if and only if M_i is a complex torus. We conclude that

$$h^{p,0}(\tilde{X}) \leqslant h^{p,0}(\prod T_i) = \binom{n}{p} \quad ,$$

with equality (for $0 < p < m_i$) if and only if $\tilde{X}$ is a complex torus. It remains to show that in this last case X also is a complex torus. We can assume that the covering $\tilde{X} \to X$ is Galois ; its Galois group G must act trivially on $H^{p,0}(\tilde{X})$. This implies that any element g of G acts on $H^{1,0}(\tilde{X})$ by multiplication by a p-th root of unity $\lambda(g)$. But then the holomorphic Lefschetz fixed-point formula, applied to g, gives

$$1 - \binom{n}{1} \lambda(g) + \binom{n}{2} \lambda^2(g) + \ldots + (-1)^n = (1-\lambda(g))^n = 0 \quad ,$$

hence $\lambda(g) = 1$, which means that G acts on $\tilde{X}$ by translations, so that X is a torus.

2. Examples

a) Special unitary manifolds.

Except K3 surfaces and their products, all usual examples of Kähler manifolds with trivial canonical bundle are special unitary : hypersurfaces of degree $(m+2)$ in $\mathbb{P}^{m+1}$, complete intersections of degrees $(d_1, \ldots, d_r)$ in $\mathbb{P}^n$, with $\Sigma d_i = n+1$; more generally, weighted complete intersections of degrees $(d_1, \ldots, d_r)$ in the twisted projective space

$\mathbb{P}(e_1,\dots,e_n)$ with $\Sigma d_i = \Sigma e_i$. If V is a projective manifold with ample anticanonical bundle (for instance a complex homogeneous space G/P, where G is a semi-simple complex Lie group and P a parabolic subgroup), then any smooth hypersurface $X \in |-K_V|$ is special unitary. More generally if $X_1,\dots,X_r$ are ample divisors in V meeting transversally, such that $\Sigma X_i \equiv -K_V$, then $X = \bigcap_i X_i$ is special unitary, etc ...

Let me give another example which is of a somewhat different nature. For m=3, 4 or 6, let E_m denote the elliptic curve which admits an automorphism of order m. Put $A_m = (E_m)^m$. The group μ_m of m-th roots of unity acts diagonally on A_m, with a finite number of fixed points. By blowing-up these points, we obtain a manifold $\hat{A}_m$ on which the group μ_m acts in such a way that the locus of fixed points of a generator is a smooth divisor. Therefore the manifold $X_m = \hat{A}_m/\mu_m$ is smooth. One checks easily that X_m is simply connected and that its canonical bundle is trivial. For $p \neq q$ one has

$$H^{p,q}(X_m) = H^{p,q}(\hat{A}_m)^{inv} = H^{p,q}(A_m)^{inv}$$

(here the sign = means "canonically isomorphic"). Let $V = H^{1,0}(A_m)$; the group μ_m acts on V by multiplication. Then we have $H^{p,q}(X_m) = (\Lambda^p V \otimes \Lambda^q \bar{V})^{inv}$, so $H^{p,q}(X_m) = 0$ for $p \neq 0,m$ and $p \neq q$. This implies that the manifolds X_m are special unitary. They have some interesting properties : in particular they are rigid, since

$$H^1(X_m,T_{X_m}) = H^1(X_m,\Omega^{m-1}_{X_m}) = H^{m-1,1} = 0 .$$

b) Symplectic manifolds

A symplectic structure on a complex manifold X is a holomorphic 2-form on X which is everywhere non-degenerate. The existence of such a structure implies that X is even-dimensional and has trivial canonical bundle. It follows from the structure theorem that a compact Kähler manifold is symplectic irreducible (in the sense of the theorem) iff it is simply connected and admits a unique symplectic structure (up to a scalar).

Let S be a compact complex surface. We denote by $S^{(r)}$ the r-th symmetric product of S (quotient of S^r by the symmetric group $\mathfrak{S}_r$) and by $\pi: S^r \to S^{(r)}$ the quotient map. The (singular) variety $S^{(r)}$ parametrizes effective 0-cycles of degree r on S. Let $S^{[r]}$ be the <u>Douady Space</u> of 0-dimensional subspaces $Z \subset S$ with $\ell g(\mathcal{O}_Z) = r$.

Consider the natural map $\varepsilon: S^{[r]} \to S^{(r)}$ which associates to a finite subspace the corresponding 0-cycle. Let D be the diagonal of $S^{(r)}$ (locus of cycles $2p_1 + \ldots + p_{r-1}$), and put $E = \varepsilon^{-1}(D)$. It is clear that $\varepsilon: S^{[r]} - E \to S^{(r)} - D$ is an isomorphism, so ε is a bimeromorphic morphism. Fogarty has proved that $S^{[r]}$ is smooth, so that ε is a resolution of the singularities of $S^{(r)}$. Note that the exceptional divisor E is irreducible (Iarrobino).

<u>Proposition 1</u>:

<u>Let</u> S <u>be a generic K3 surface</u>. <u>Then</u> $S^{[r]}$ <u>is a Kähler symplectic manifold, irreducible, of dimension 2r</u>.

Here the word "generic" means that S is allowed to vary in an open dense subset of the coarse moduli space of Kähler K3 surfaces, containing the projective ones (I have to make this rather unpleasant restriction only because I don't know how to prove that $S^{[r]}$ is Kähler for every S). For $r = 2$, this example has been first noticed by A. Fujiki (see [F2]).

Again I refer to [B] for a complete proof ; I just want to sketch how one gets the symplectic structure on $S^{[r]}$. Let S^r_* denote the set of r-uples $(x_1, \ldots, x_r)$ with at most two x_i's equal. Put $S^{(r)}_* = \pi(S^r_*)$ and $S^{[r]}_* = \varepsilon^{-1}(S^{(r)})$. Then the map $\varepsilon : S^{[r]}_* \to S^{(r)}_*$ is easy to understand. Since a subspace with associated cycle $2p$ is given by a point of $\mathbb{P}(T_p(S))$, it is easily checked that ε is just the blowing-up of $D \cap S^{(r)}_*$ in $S^{(r)}_*$. More precisely, let $\Delta = \pi^{-1}(D)$ be the diagonal of S^r ; note that $\Delta \cap S^r_*$ is smooth of codimension 2 in S^r_*. If $\eta: B_\Delta(S^r_*) \to S^r_*$ denotes the blowing-up of S^r_* along Δ, then we get a commutative diagram

$$\begin{array}{ccc} B_\Delta(S^r_*) & \xrightarrow{\ \eta\ } & S^r_* \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\pi} \\ S^{[r]}_* & \xrightarrow{\ \varepsilon\ } & S^{(r)}_* \end{array} \quad ,$$

where ρ is a Galois covering with group $\mathfrak{S}_r$, ramified simply along the exceptional divisor E' of η.

From a nonzero 2-form on S we deduce a symplectic structure ω on S^r. The form $\eta^*\omega$ is invariant under $\mathfrak{S}_r$, thus descends to a holomorphic 2-form φ on $S^{[r]}_*$ with $\rho^*\varphi = \eta^*\omega$. We have

$$\rho^*\mathrm{div}(\varphi^r) = \mathrm{div}(\rho^*\varphi^r)-E'=\mathrm{div}(\eta^*\omega^r)-E' = 0 \ ,$$

hence $\mathrm{div}(\varphi^r)=0$, which implies that φ is a symplectic structure on $S^{[r]}_*$. Now since E is irreducible, $S^{[r]}- S^{[r]}_*$ is of codimension $\geqslant 2$ in $S^{[r]}$; so by Hartogs'theorem φ extends to a holomorphic 2-form $\tilde{\varphi}$ on $S^{[r]}$. The divisor of $\tilde{\varphi}^r$, which should be contained in $S^{[r]}-S^{[r]}_*$ is zero, which means that $\tilde{\varphi}$ is a symplectic structure on $S^{[r]}$.

Now let A be a 2-dimensional complex torus. The manifold $A^{[r]}$ is again symplectic, but not simply connected. Let $s : A^{(r)} \to A$ be the sum map (defined by $s([a_1]+\ldots+[a_r]) = \sum\limits_i a_i$). By composition with ε, we obtain a morphism $S : A^{[r]} \to A$.

The group A acts on $A^{[r]}$ by translations. Let us also consider its action on A given by $(\alpha,a) \mapsto a + r\alpha$. Then the map S is equivariant with respect to these actions, so it is smooth and has isomorphic fibres. We put $K_{r-1} = S^{-1}(0)$. In the same way as prop. 1, we prove in [B] :

Proposition 2 :

For A *a generic 2-dimensional complex torus, the manifold* K_r *is Kähler symplectic, irreducible, of dimension* $2r$.

The manifold K_1 is simply the Kummer surface associated to A. So the manifolds $S^{[r]}$ appear as natural generalizations of K3 surfaces, while K_r seems to generalize Kummer surfaces. Note however that for $r \geqslant 2$ the manifolds $S^{[r]}$ and K_r are not isomorphic.

It turns out that the manifolds $S^{[r]}$ (resp. K_r) have more deformations than those coming from deformations of S (resp. A) : these deformations furnish new (although not very explicit) examples of Kähler symplectic manifolds. At the moment I know no other types of such manifolds.

3. Split coverings

In this section we want to state more precisely the assertion 2) of the structure theorem, and in particular give a corresponding assertion of unicity. This will follow from general remarks about Kähler manifolds which are covered by a product of a complex torus and a compact simply connected manifold.

Lemma :

Let T *be a complex torus and* S *be a compact Kähler manifold with* $b_1(S) = 0$. *Then any automorphism* u *of* $T \times S$ *is of the form* (v,w), *with* $v \in \mathrm{Aut}(T)$ *and* $w \in \mathrm{Aut}(S)$.

Since the projection $T \times S \to T$ is the Albanese map of $T \times S$, there is a commutative diagram

$$\begin{array}{ccc} T \times S & \xrightarrow{\quad u \quad} & T \times S \\ \downarrow & & \downarrow \\ T & \xrightarrow{\quad v \quad} & T \end{array}$$

This implies the existence of a map w of T into the complex Lie group Aut(S) such that

$$u(t,s) = (v(t),\ w_t(s)) \quad \text{for} \quad t \in T \quad \text{and} \quad s \in S.$$

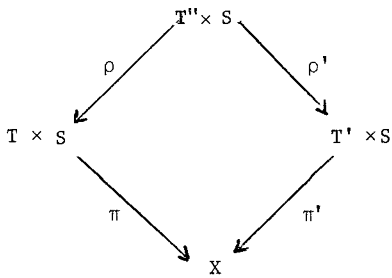
Now the map $t \mapsto w_t\, w_o^{-1}$ gives an action of T on S. Since S is Kähler with $b_1(S) = 0$, it is known that such an action is necessary trivial (see e.g. [F1]),which implies the lemma.

In what follows, a covering is always assumed to be étale. We'll say for short that a compact manifold $\tilde{X}$ is <u>split</u> if it is isomorphic to the product of a torus and a simply connected (compact)manifold. Let X be a compact manifold ; we'll say that a finite covering $\tilde{X} \to X$ is split if the manifold X is split. Finally we'll say that a split covering $T \times S \to X$ is <u>minimal</u> if it is Galois and if its Galois group does not contain any element of the form $(\tau, 1_S)$, where τ is a translation of the torus T.

<u>Proposition 3</u> :

<u>Let X be a compact complex manifold which admits a finite split covering. Then there exists a minimal split covering</u> $\tau : T \times S \to X$, <u>unique (up to a non-unique automorphism). Any split covering of</u> X <u>factors through</u> π .

We first observe that every finite covering of a split manifold is split ; therefore there exists a split covering $\pi : T \times S \to X$ which is Galois. Let G be its Galois group, and let K be the subgroup of G consisting of automorphisms $(\tau, 1_S)$, where τ is a translation. Put $T = \tilde{T}/K$. Then K is a normal subgroup of G (by the lemma) and the covering $\pi : T \times S \to X$ deduced from $\tilde{\pi}$ is Galois with Galois group G/K, hence is minimal. Let $\pi' : T' \times S' \to X$ be another split covering. Then there exists a Galois covering π'' of X, with Galois group G'', which factors through both π and π'. This implies that S' is isomorphic to S, and that there is a commutative diagram



Let us denote by H and H' the Galois groups of ρ and ρ' respectively. They are subgroups of G'', and their elements are of the form $(\tau, 1_S)$, where τ is a translation of T''. Then the minimality of π implies $K' \subseteq K$, hence π' factors through π .

In particular, any compact Kähler manifold has a unique minimal split covering, of the form $T \times \Pi\, V_i \times \Pi\, X_j$, where the V_i's are special unitary, the X_j's are irreducible symplectic and T is a complex torus.

This fact allows to describe the automorphism group of X, or more precisely to reduce its study to the case in which X is irreducible.

a) Let us write $X = (T \times S)/G$, where the covering is minimal. Then any automorphism of X extends to $T \times S$, so that the group $\mathrm{Aut}(X)$ is identified with the normalizer of G in $\mathrm{Aut}(T) \times \mathrm{Aut}(S)$.

b) Write $S = \prod_i S_i^{n_i}$, where the S_i's are non-isomorphic irreducible manifolds. Then by the unicity property in the structure theorem, $\mathrm{Aut}(S)$ is isomorphic to $\prod_i \mathrm{Aut}(S_i^{n_i})$.

c) Let Y be an irreducible manifold. Then again by the unicity property, $\mathrm{Aut}(Y^n)$ is the semi-direct product of the symmetric group $\mathfrak{S}_n$ by $\mathrm{Aut}(Y)^n$: every automorphism u of Y^n satisfies

$u(y_1,\dots,y_n) = (u_1(y_{\sigma 1}),\dots, u_n(y_{\sigma n}))$ for some $u_1,\dots,u_n \in \mathrm{Aut}(Y)$ and some $\sigma \in \mathfrak{S}_n$.

In the next sections we will investigate some properties of the groups Aut(Y).

4. Automorphisms and torsion

In this section we extend to manifolds of higher dimension some results of Nikulin [N] on automorphisms of K3 surfaces.

Let X be a compact complex manifold of dimension n, with trivial canonical bundle, and let G be a finite group of automorphisms of X. There exists a character $\alpha : G \to \mathbb{C}^*$ such that

$$g^*\omega = \alpha(g)\omega \quad \text{for} \quad g \in G,\ \omega \in H^o(X, K_X) \ .$$

The image of α is a cyclic subgroup μ_m of order m of $\mathbb{C}$, so that we get an exact sequence

$$1 \longrightarrow G_o \longrightarrow G \xrightarrow{\ \alpha\ } \mu_m \longrightarrow 1 \ .$$

The number m will be called the index of G. If $g \in G$, the index of g is defined as the index of the subgroup generated by g. Obviously it divides the index of G, and the index of G is the largest index of its elements. The following fact is equally obvious.

Proposition 4 :

Assume G acts freely on X. Then the index of G equals the order of the canonical bundle in Pic(X/G).

We now assume that X is Kähler. Let S_X be the subgroup of $H^n(X, \mathbf{Z})$ consisting of elements orthogonal to $H^{n,o}$, and let T_X denote the orthogonal of S_X in $H^n(X, \mathbf{Z})$. We denote by ρ_n the rank of the subgroup of $H^n(X, \mathbf{Z})$ spanned by analytic cycles (so that $\rho_n = 0$ when

n is odd, and $\rho_n \leqslant \mathrm{rk}\,(S_X)$).

Proposition 5 :

The index m of G satisfies $\varphi(m) \leqslant \mathrm{rk}\,(T_X)$, and in particular $\varphi(m) \leqslant b_n - \rho_n$.

Let $g \in G$ be an element of index m. Since $H^{n,o}$ is contained in $(T_X)_{\mathbb{C}}$, $\alpha(g)$ is an eigenvalue of g^* acting on T_X. Therefore its minimal polynomial Φ_m divides the characteristic polynomial of g^*, hence

$$\varphi\,(m) \leqslant \mathrm{rk}(T_X) = b_n - \mathrm{rk}\,(S_X) \leqslant b_n - \rho_n .$$

Remark : Assume that for some integer p, the group G acts faithfully on $H^p(X,\mathbb{C})$ or more generally that the elements of G acting trivially on $H^p(X,\mathbb{C})$ acts also trivially on $H^{n,o}$. Then the argument of the proof shows that the index m of G satisfies $\varphi(m) \leqslant b_p$. This applies in particular to complex tori, for which we get the bound $\varphi(m) \leqslant b_1 = 2n$.

In case X is symplectic irreducible, we can get better results. There is a character $\beta : G \to \mathbb{C}$ such that

$$g^* \varphi = \beta(g)\varphi \quad \text{for } g \in G \ , \ \varphi \in H^o(X,\Omega_X^2) .$$

The image of β is a cyclic subgroup μ_s of $\mathbb{C}$, and we call s the symplectic index of G. If $\dim(X) = 2r$ one has $\alpha = \beta^r$, hence $m = s/d$, where d is the g.c.d. of s and r. In particular m divides s.

Proposition 6 :

(i) If X is not projective then $s = 1$.

(ii) $\varphi(s)$ (hence also $\varphi(m)$) divides $b_2 - \rho_2$.

To prove (i), we assume $s > 1$. Since $H^2(X,\mathbb{Q})$ is dense in $H^2(X,\mathbb{R})$ and the cone of Kähler classes is open in $H^{1,1}_{\mathbb{R}}(X)$, there exists an element $c \in H^2(X,\mathbb{Q})$ whose $(1,1)$-component is a Kähler class. Then the class $h = \sum_{g \in G} g^* c$ has the same property, and is moreover invariant under G. Since $H^{2,0}$ does not contain any nonzero invariant element, we conclude that h is of type (1,1) and therefore a Kähler class. Since h is rational, this implies that X is projective.

To prove (ii), we can assume therefore that X is projective. We recall that we have defined in [B], §8 a canonical integral symmetric bilinear form q on $H^2(X, \mathbb{Z})$ with the following properties :

(i) $H^{1,1}$ is orthogonal to $H^{2,0}$ with respect to $q_{\mathbb{C}}$.

(ii) The restriction of $q_{\mathbb{R}}$ to $H^{1,1} \cap H^2(X,\mathbb{R})$ has signature $(1, b_2-3)$. If ω is a Kähler class, then $q_{\mathbb{R}}(\omega) > 0$.

It follows that the subgroup $S_X \subset H^2(X, \mathbb{Z})$ of algebraic cycles is orthogonal to $H^{2,0}$, and that the form q restricted to S_X is non-degenerate (since X is assumed to be projective, S_X contains an element h with $q(h) > 0$). Let T_X denote the orthogonal of S_X in $H^2(X, \mathbb{Z})$, and let $g \in G$. I claim that g^* acts trivially on T_X in the following two cases :

a) $\beta(g) = 1$.

b) There exists a nonzero element t of T_X with $g^*t = t$.

To prove this claim, observe that for any t in T_X

$$q(g^*\omega \, , \, g^*t) = \beta(g) \, q(\omega, g^*t) = q(\omega,t).$$

In case a) we get

$$g^*t - t \in S_X \cap T_X = (0) \, ,$$

while in case b) we get $(\beta(g)-1) \, q(\omega,t) = 0$. Since t does not belong to S_X this implies $\beta(g) = 1$, hence the result by case a).

Now let g be an element of G with symplectic index s . Applying a) to g^s we get $(g^*)^s = 1$, while b) applied to g^d, for all d dividing s, shows that the eigenvalues of g^* on T_X are primitive s-th roots of unity. Therefore the characteristic polynomial of g^* is a power of Φ_s, which implies (ii).

It is interesting to observe that prop. 6(ii) extends to automorphisms of infinite order :

Proposition 7 :

Let X *be a projective symplectic irreducible manifold, and let* $G = \mathrm{Aut}(X)$. *There exists an integer* s *and a surjective character* $\beta : G \to \mu_s$ *such that*

$$g^*\varphi = \beta(g)\varphi \quad \text{for all} \quad g \in G,\ \varphi \in H^o(X,\Omega_X^2).$$

Moreover $\varphi(s)$ *divides* $b_2 - \rho_2$.

We first notice that the only assumption on g which is needed in the proof of prop. 6(ii) is that the eigenvalues of g^* acting on T_X are roots of unity. By Kronecker's theorem, it is enough to prove that all these eigenvalues have modulus 1. Put $E = (H^{2,o} \oplus H^{o,2}) \cap H^2(X,\mathbb{R})$. Since $(T_X)_{\mathbb{R}}$ contains E, it admits an orthogonal decomposition

$$(T_X)_{\mathbb{R}} = E \oplus (T_X)_{\mathbb{R}} \cap H^{1,1}.$$

The form q is positive on the first space and negative on the second one. Since g^* preserves this decomposition and is unitary on each space, the assertion follows.

We will now try to apply these results to get a bound for the order of the canonical bundle. Recall that for a surface S with $c_1^{\mathbb{R}}(S) = 0$, one has $K_S^{\otimes m} = \mathcal{O}_S$ with $m = 1,2,3,4$ or 6.

Proposition 8 :

Let X *be a compact Kähler threefold with* $c_1^{\mathbb{R}}(X) = 0$. *Then there exists an integer* $m \leqslant 66$, *with* $\varphi(m) \leqslant 20$, *such that the bundle* $K_X^{\otimes m}$ *is trivial.*

According to the structure theorem, one can write $X = \tilde{X}/G$, where $\tilde{X}$ is one of the following :

a) A special unitary threefold.

b) A product $E \times S$, where E is an elliptic curve and S a K3 surface.

c) A complex torus.

In case a), one has $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{\tilde{X}}) = 0$, thus $h^o(K_X) = 1$ and K_X is trivial. In case c), the index m of G satisfies $\varphi(m) \leqslant b_1 = 6$ (remark following prop. 5), hence $m \leqslant 18$.

In case b), let G_t denote the subgroup of those elements in G which act by translation on E. Suppose first $G_t \neq G$, and let g be an element of $G - G_t$. Then g must act freely on S, therefore its image in $\mathrm{Aut}(S)$ is a fixed point free involution of index 2. The same must be true for gh, for any $h \in G_t$; this implies that h has index 1. We conclude in this case that the index of G is 2.

Suppose now that $G=G_t$, i.e. G acts on E by translations. Then the index of G equals its index as a subgroup of $\mathrm{Aut}(S)$. Nikulin's result (prop. 6) then gives $\varphi(m) \leqslant 21$, which implies $\varphi(m) \leqslant 20$ and $m \leqslant 66$ (exercise!).

Remarks. 1) I must honestly admit that prop. 8 does not use the full force of the structure theorem. In fact the weak version of [C] (proved without appealing to Calabi conjecture in [Fl]) would suffice.

2) According to [N], Dolgachev has given an example of a K3 surface with an automorphism of index 66. This implies that there exists a projective threefold whose canonical bundle has order 66.

3) Let us try to extend the proof, say in dimension 4. We can write $X = \tilde{X}/G$, where $\tilde{X}$ is one of the following :

a) A special unitary or irreducible symplectic fourfold.

b) A product $E \times V$, where E is an elliptic curve and V a special unitary threefold.

c) A product $T \times S$, where T is a 2-dimensional torus and S a K3 surface.

d) A complex torus.

In case a) one has $\chi(\mathcal{O}_X) = 2$ or 3, thus G has order $\leqslant 3$. In case d) we find as before $\varphi(m) \leqslant 8$. Case c) can be treated in the same way as case b) of prop. 8; one finds $m \leqslant 150$. The difficulty comes from case b) : we need a bound for the index of an arbitrary automorphism (of finite order) of the threefold V. The only way I can imagine to get such a bound would be to get a uniform bound for $b_3(V)$, for all threefolds V with trivial canonical bundle. I have not even a feeling whether one should expect such a bound to hold or not.

5. Automorphisms inducing the identity in cohomology

An important point in the construction of the universal family of marked K3 surfaces is the following result [B-R].

Let S be a K3 surface. Then any automorphism of S inducing the identity on $H^2(S,\mathbb{C})$ is the identity.

In this section we will investigate the analogous question for the symplectic manifolds K_{r-1} and $S^{[r]}$ (§2). Let us consider the first one. Recall that K_{r-1} is the fibre of the sum map $S : A^{[r]} \to A$. Let A_r denote the group of points of order r in A. Then the action of A_r on $A^{[r]}$ by translations restricts to an action of A_r on K_{r-1}.

Proposition 9 :

For $r \geqslant 3$, the group A_r acts trivially on $H^2(K_{r-1},\mathbb{C})$.

According to [B], §7 the restriction map

$$H^2(A^{[r]},\mathbb{C}) \to H^2(K_{r-1},\mathbb{C})$$

is surjective. But in $A^{[r]}$ the operations of A are homotopic to the identity, thus induce the identity on cohomology, hence the result.

We now turn to the manifolds $S^{[r]}$.

Lemma 1 :

Let S be a (Kähler) K3 surface with Pic(S) = 0. Let f be a meromorphic map from S to S. Then either f maps S to a point, or f is an automorphism.

Since S contains no curves, the map $f : S \to S$ is actually holomorphic (otherwise its indeterminacy points would give rise to curves on S). The image of f cannot be a curve, so f is either trivial or surjective. In this last case the morphism f is finite (because its fibres contain no curves). Its discriminant curve being empty, f is an étale covering, hence an automorphism since S is simply connected.

Lemma 2 :

Under the hypotheses of lemma 1, assume moreover that S has no non-trivial automorphisms. Then the group of bimeromorphic transformations of S^r is equal to the symmetric group $\mathfrak{S}_r$ (acting by permutation of the factors).

Let u be a bimeromorphic transformation of S^r. There exists a Zariski open set U in S^r such that the restriction of u to U is a holomorphic embedding. Let $p_1,\ldots,p_r$ denote the projections from S^r onto S. We will prove by induction on ℓ that, after a permutation of the coordinates,

$$p_i \circ u(s) = p_i(s) \quad \text{for } s \in U \text{ and } 1 \leq i \leq \ell .$$

Assume the result is proved for some integer $\ell (0 \leq \ell < r)$. Fix $(r-1)$ points $s_1,\ldots,s_\ell, s_{\ell+2},\ldots,s_r$ general enough so that

$$(\{s_1\}\times \ldots \times\{s_\ell\}\times S\times \ldots \times\{s_r\}) \cap U \neq \emptyset ,$$

and consider the meromorphic map from S to S defined by

$$s \mapsto p_j \circ u(s_1, \ldots, s_\ell, s, s_{\ell+2}, \ldots, s_r).$$

By construction, for some $j \geq \ell+1$, this map is nontrivial (and will remain so for general s_i' s), hence by lemma 1 is the identity. Permuting the last $(r-\ell)$ coordinates so that $j = \ell+1$ we get by continuity $p_{\ell+1} \circ u(s) = p_{\ell+1}(s)$ for all s in U. This proves the lemma.

Note that a generic K3 surface fulfills the hypotheses of lemma 2.

Lemma 3 : Under the hypotheses of lemma 2, the group $Aut(S^{[r]})$ is reduced to the identity.

Recall that there is an irreducible divisor E in $S^{[r]}$ such that the natural map $\varepsilon : S^{[r]} - E \to S^{[r]} - D$ is an isomorphism. Let $\Delta = \cup_{i,j} \Delta_{ij}$ denote the diagonal of S^r. Since Δ is of codimension ≥ 2, $S^r - \Delta$ is simply connected and the map $\pi : S^r - \Delta \to S^r - E$ is the universal covering of $S^{[r]} - E$.

Let u be an automorphism of $S^{[r]}$. It follows from [Fo] (see also [B], §6) that $Pic(S^{[r]})$ is the infinite cyclic group generated by a line bundle L such that $L^{\otimes 2} = \mathcal{O}(E)$. Therefore u preserves E, and induces an automorphism of $S^r - \Delta$, that is a bimeromorphic transformation of S^r. We conclude with the help of lemma 2.

Proposition 10 :

Let S be a K3 surface and r an integer. Then any automorphism u of $S^{[r]}$ inducing the identity on $H^2(S^{[r]}, \mathbb{C})$ is the identity.

Let $f : \mathcal{X} \longrightarrow \mathcal{M}$ be a Kuranishi family for $X = S^{[r]}$, such that X is the fibre at a point $o \in \mathcal{M}$. Then there exists an open neighborhood $\mathcal{M}'$ of o in $\mathcal{M}$ and a diagram

$$\begin{array}{ccc} \mathcal{X}' & \xrightarrow{\bar{u}} & \mathcal{X} \\ \downarrow f & & \downarrow f \\ \mathcal{M}' & \xrightarrow{v} & \mathcal{M} \end{array}$$

so that $v(o) = o$ and $\bar{u}|_X = u$.

We can assume that the local system $R^2 f_*(\mathbb{C})$ on $\mathcal{M}$ is trivial. The choice of a trivialisation $R^2 f_*(\mathbb{C}) \to L_{\mathcal{M}}$ (with $L = H^2(X,\mathbb{C})$) allows to define a period map $p : \mathcal{M} \to \mathbb{P}(L)$, which is an embedding when $\mathcal{M}$ is small enough ([B], §8). Since u induces the identity on $H^2(X,\mathbb{C})$ we get $p \circ v = p$, thus v is the identity, and we conclude that u extends to an automorphism $\bar{u}$ of $\mathcal{X}$ above $\mathcal{M}$.

Now let $g : \mathcal{S} \to \mathcal{N}$ be a Kuranishi family for S, and let $g^{[r]} : \mathcal{S}^{[r]} \to \mathcal{N}$ be the corresponding family of Douady spaces. According to [B], §9, we can identify $\mathcal{N}$ to a smooth hypersurface in $\mathcal{M}$, passing through o, such that $g^{[r]}$ is obtained by pulling-back to $\mathcal{N}$ the family f. In particular $\bar{u}$ restricts to an automorphism of $\mathcal{S}^{[r]}$ above $\mathcal{N}$. But there exists a dense subset T of $\mathcal{N}$ such that for t in T, the K3 surface $S=\mathcal{S}_t$ has $\mathrm{Pic}(S) = 0$ and $\mathrm{Aut}(S) = \{1\}$. Then lemma 3 implies $\bar{u}_t = \mathrm{Id}$ for $t \in T$. Since T is dense this gives $u = \mathrm{Id}$.

<u>Remark</u> : It is also easy to give an example of a special unitary manifold X of dimension m and of a non-trivial automorphism of X inducing the identity on $H^m(X,\mathbb{C})$. Take for instance $X_3 = \hat{A}_3/\mu_3$ (§2). Then one has $H^2(X_3,\mathbb{C}) = H^{3,o} \oplus H^{o,3}$, so that any automorphism of X_3 of index 1 acts trivially on $H^3(X_3,\mathbb{C})$. There are many such automorphisms, for instance the translations by the points of A_3 fixed under μ_3.

6. Birational Transformations

In this section we will answer negatively the following conjecture of Bogomolov [Bo] : any birational automorphism of a projective manifold with trivial canonical bundle is biregular. The counter-examples we give are symplectic manifolds. The question is still open for special unitary manifolds.

Let us first recall a particular case of a construction which is due to Mukai [Mu]. Let X be a complex symplectic manifold, of dimension $2r$, and let P be a submanifold of X isomorphic to $\mathbb{P}^r$. Then P is totally isotropic with respect to the symplectic form φ on X (because $\varphi|_P = 0$). So φ induces an isomorphism from the normal bundle of P in X onto its cotangent bundle. Let $\varepsilon: \hat{X} \to X$ be the blowing-up of X along P. The exceptional divisor E of $\hat{X}$ is then isomorphic to the projective cotangent bundle of P, which can be described as the incidence correspondence in $P \times P^*$(locus of pairs (p,h) such that the point p belongs to the hyperplane h). From this one deduces easily that E can be blown-down to P^*, giving rise to a symplectic manifold X' and to a diagram

$$\begin{array}{ccc} & \hat{X} & \\ {\scriptstyle\varepsilon}\swarrow & & \searrow{\scriptstyle\varepsilon'} \\ X & & X' \end{array} .$$

The bimeromorphic map $\varepsilon' \circ \varepsilon^{-1}$ (which obviously is not biregular at the points of P) is called by Mukai the elementary transformation along P. We will give an example where X' is isomorphic to X (I don't know whether this holds in general).

Let us consider the manifold $S^{[2]}$ associated with a smooth quartic surface $S \subset \mathbb{P}^3$. Given $Z \in S^{[2]}$, there exists a unique line $\ell(Z)$ in $\mathbb{P}^3$ such that the subscheme Z of S is contained in $\ell(Z)$. We define in this way a morphism $\ell : S^{[2]} \to G$, where G denotes the Grassmann manifold of lines in $\mathbb{P}^3$. If Z is general enough, the line $\ell(Z)$ meets S along 4 distinct points, so that

$$\ell(Z).S = Z \cup Z' ,$$

where $Z' \in S^{[2]}$ is disjoint from Z. By associating to Z the residual subscheme Z', we define a birational involution σ of $S^{[2]}$, such that the diagram

$$\begin{array}{ccc} S^{[2]} & \overset{\sigma}{\dashrightarrow} & S^{[2]} \\ & {\scriptstyle \ell}\searrow \quad \swarrow{\scriptstyle \ell} & \\ & G & \end{array}$$

is commutative.

Proposition 11 :

(i) The birational map σ is defined at a point $Z \in S^{[2]}$ if and only if the line $\ell(Z)$ is not contained in S.

(ii) If S contains p lines $d_1,\ldots,d_p$ then σ is the composition of the Mukai elementary transforms along the planes $d_1^{[2]},\ldots,d_p^{[2]}$.

Suppose first that $\ell(Z)$ is not contained in S. Then the morphism $\ell : S^{[2]} \to G$ is finite in a neighborhood of $\ell(Z)$. This is enough to imply that σ is defined at Z. Consider indeed the closure Γ in $S^{[2]}\times_G S^{[2]}$ of the graph of σ : the first projection $\Gamma \to S^{[2]}$, which is finite and birational, must be an isomorphism.

Suppose now that S contains a line d. We observe that $S^{[2]}\times_G{}^{[2]}$ is locally a complete intersection in $S^{[2]}\times S^{[2]}$: if $(x_1,\ldots,x_4)$ is a local coordinate system on G, the 4-dimensional variety $S^{[2]}\times_G S^{[2]}$ is defined by the 4 equations $pr_1^*\,\ell^*x_i - pr_2^*\,\ell^*x_i = 0$ $(1 \leqslant i \leqslant 4)$. In particular it is Cohen-Macaulay, therefore its two components Γ and $d^{[2]}\times d^{[2]}$

intersect along a 3-dimensional variety. This shows that σ is not defined at the points of $d^{[2]}$, thus proving (i).

Put $P = d^{[2]} = \ell^{-1}(d)$. It is a submanifold of $S^{[2]}$, isomorphic to the projective plane. Let us consider the diagram

$$\begin{array}{ccc} S_P^{[2]} & \longrightarrow & S^{[2]} \\ {\scriptstyle \hat{\ell}}\downarrow & & \downarrow{\scriptstyle \ell} \\ G_d & \longrightarrow & G \end{array} ,$$

where the horizontal maps are obtained by blowing-up the plane P in $S^{[2]}$ and the point d in G. We'll denote by E and T the exceptional divisors of $S_P^{[2]}$ and G_d respectively. The technical part of the proof is contained in the following lemma.

<u>Lemma</u> :

<u>The induced map</u> $\hat{\ell}: E \to T$ <u>is finite</u>.

<u>Proof</u> : Let Z_o be a point of $d^{[2]}$. We choose a hyperplane at infinity in $\mathbb{P}^3$ away from Z_o, and affine coordinates (x,y,z) such that Z has equations $x^2 + ax = y = z = 0$ for some $a \in \mathbb{C}$ (so that d is the line $y=z=0$). A local coordinate system in $(\mathbb{P}^3)^{[2]}$ at Z_o is then given by (r,s,t,u,v,w), where a subspace Z close to Z_o is defined by $f_1 = f_2 = f_3 = 0$, with $f_1 = x^2 + (a+r)x + s$, $f_2 = y - tx - u$, $f_3 = z - vx - w$.

Let $F(x,y,z) = 0$ be the equation of S. Then Z belongs to $S^{[2]}$ if and only if F belongs to the ideal (f_1, f_2, f_3) of $\mathbb{C}[x,y,z]$, which can be expressed by the condition

(*) $\quad f_1(x)$ divides $F(x, tx+u, vx+w)$ (in $\mathbb{C}[x]$).

The line $\ell(Z)$ is given by $f_2 = f_3 = 0$. We can take (t,u,v,w) as local coordinates on G near d, so that ℓ is simply the projection on the last four coordinates.

Let $\hat{Z}_o$ be a point of E above Z_o. We can assume that $\hat{\ell}(\hat{Z}_o)$ is the point of T corresponding to the direction $u=v=w=0$ at d, so that we can take local coordinates (t,u',v',w') on G_d near $\hat{\ell}(\hat{Z}_o)$ such that $u = tu', v = tv', w = tw'$. Then $(r,s;t,u',v',w')$ are local coordinates at $\hat{Z}_o$ on $(\mathbb{P}^3)^{[2]}$ blown-up along $d^{[2]}$. In this local chart the submanifold $S_P^{[2]}$ is defined as follows : since d is contained in S, one can write

$$F(x,y,z) = y\, G(x,y,z) + z\, H(x,y,z) \;;$$

then the condition (*) becomes

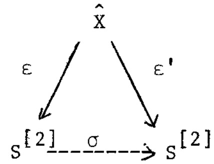
$$f_1(x) \text{ divides } (x+u')\, G(x,t(x+w),t(v'x+w))+(vx+w)H(x,t(x+u'), t(v'x+w')).$$

The exceptional divisor E is defined by $t=0$, thus a point of E has coordinates $(r,s;u',v'w')$ with the condition

$$f_1(x) \text{ divides } (x+u')\, G(x,0,0)+(v'x+w')\, H(x,0,0).$$

Now we claim that this last polynomial cannot vanish identically, for any given value of (u',v',w') : for this would imply that the line in $\mathbb{P}^3 \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$ given by $y = \varepsilon(x+u')$, $z = \varepsilon(v'x+w')$ (with $\varepsilon^2 = 0$) is in $\mathbb{P}^3 \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]$, which is impossible since $H^0(d,N_{d/S})=0$. Therefore given (u',v',w') there are at most 6 possibilities for (r,s). This shows that the map $\hat{\ell} : E \to T$ is quasi-finite at every point of E, hence the lemma.

The lemma implies that the map $\ell : S_P^{[2]} \to G_d$ is finite. Then the argument used in the proof of (i) shows that σ extends to an involution of $S_P^{[2]}$ which is defined at every point of E. Let $\varepsilon : \hat{X} \to S^{[2]}$ denote the blowing-up of $S^{[2]}$ along the union of the planes $P_1,\ldots,P_p$, and $E_i = \varepsilon^{-1}(P_i)$. Then σ extends to a biregular involution τ of $\hat{X}$, with $\tau(E_i) = E_i$. Recall that E_i is identified with the incidence correspondence in $P_i \times P_i^*$. Now since $\mathrm{Pic}(E_i) = \mathbf{Z} \oplus \mathbf{Z}$, one checks easily that E_i has only two rulings (i.e two $\mathbb{P}^1$-fibrations onto $\mathbb{P}^2$), given by the projections on P_i and P_i^*. The involution τ must exchange these rulings, because otherwise σ would be defined on P_i. Therefore the morphism $\varepsilon' = \varepsilon \circ \tau$ blows down each divisor E_i onto P_i^*, and we have a commutative digram

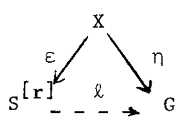


This achieves the proof of the proposition.

The preceding construction extends to higher-dimensional projective space. In fact, for any K3 surface S of degree $2r$ in $\mathbb{P}^{r+1}$, the manifold $S^{[r]}$ admits a (non-trivial) birational automorphism. A generic subspace $Z \in S^{[r]}$ is made up of r distinct points, which span a codimension 2 linear subspace $\ell(Z)$ in $\mathbb{P}^{r+1}$, intersecting S in $2r$ distinct points. We define a birational automorphism σ of $S^{[r]}$ by associating to Z the residual intersection $(\ell(Z).S) - Z$.

Contrary to the case $r = 2$, the map σ is never biregular for $r \geqslant 3$. I will just sketch the idea of the proof and insist on two examples. I'll assume for simplicity that $\mathrm{Pic}(S) = \mathbf{Z}$.

Let G denote the Grassmann manifold of codimension 2 linear subspaces of $\mathbb{P}^{r+1}$, and let X be the closure in $S^{[r]} \times G$ of the graph of the rational map $Z \mapsto \ell(Z)$, that is the locus of pairs (Z,π) with $Z \subset \pi$. If ε,η denote the projections of X on $S^{[r]}$ and G, we get a commutative diagram



The map ε is an isomorphism outside the subvariety B of $S^{[r]}$ which parametrizes those subspaces $Z \in S^{[r]}$ which are contained in a $\mathbb{P}^{r-2}$. Assume that B is smooth and that no $Z \in B$ is contained in $\mathbb{P}^{r-3}$ (this will be the case in our two examples). Then X is smooth and ε is the blowing up of $S^{[r]}$ along B. Because of the hypothesis on $\mathrm{Pic}(S)$, the map η is finite, so as in prop. 11 σ extends to a biregular involution τ of X, and we find a diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\varepsilon}\swarrow & & \searrow{\scriptstyle\varepsilon\circ\tau} \\ S^{[r]} & \overset{\sigma}{\dashrightarrow} & S^{[r]} \end{array} .$$

So again σ is obtained by blowing up B and blowing down the exceptional divisor onto B along another ruling. Let us consider now the low-dimensional cases.

a) $r = 3$.

Then S is a complete intersection of a smooth quadric Q and a cubic in $\mathbb{P}^4$. A point Z of $S^{[3]}$ is in B when it is contained in a line d. This happens exactly when d is contained in Q; thus B is isomorphic to the variety of lines contained in Q, that is to $\mathbb{P}^3$. So σ is the elementary transformation of $S^{[3]}$ along B.

b) $r = 4$.

Then S is the base locus of a net N of quadrics in $\mathbb{P}^5$. The points Z of B are in one-to-one correspondence with the 2-planes lying in one quadric of the net. We get in this way a morphism $B \to N$, which factors as $B \xrightarrow{p} \tilde{S} \xrightarrow{\pi} N$, where p is a $\mathbb{P}^3$-bundle and π a two-sheeted covering ramified along a sextic (so that $\tilde{S}$ is a K3 surface). In this case σ is the elementary transformation along the $\mathbb{P}^3$-bundle P in the sense of Mukai [Mu].

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