IMRN International Mathematics Research Notices
2001, No. 1

Endomorphisms of Hypersurfaces and Other Manifolds

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0 Introduction

In this note we prove the following result.

Theorem. A smooth complex projective hypersurface of dimension greater than 1 and degree greater than 2 admits no endomorphism of degree greater than 1. □

Since the case of quadrics is treated in [PS], this settles the question of endomorphisms of hypersurfaces. We prove the theorem in Section 1, using a simple but efficient trick devised by E. Amerik, M. Rovinsky, and A. Van de Ven [ARV]. In Section 2 we collect some general results on endomorphisms of projective manifolds; we classify in particular the Del Pezzo surfaces that admit an endomorphism of degree greater than 1.

1 Hypersurfaces

We consider in this note a compact complex manifold X that admits an endomorphism $f : X \to X$ which is generically finite (or equivalently surjective), of degree greater than 1. If X is projective (or more generally Kähler), f is actually *finite*; otherwise it contracts some curve C to a point, so that the class of $[C]$ in $H^*(X, \mathbf{Q})$ is mapped to zero by f_*; this contradicts the following lemma.

Lemma 1. Let $d = \deg f$. The endomorphisms f^* and $d^{-1} f_*$ of $H^*(X, \mathbf{Q})$ are inverse to each other.

This follows from the formula $f_* f^* = d\,\mathrm{Id}$. □

Received 5 August 2000. Revision received 8 September 2000.
Communicated by Dan Abramovich.

The proof of the theorem stated in the Introduction is based on the following result, which appears essentially in [ARV].

Proposition 1. Let X be a submanifold of $\mathbf{P}^N$, of dimension n, and let f be an endomorphism of X such that $f^*\mathcal{O}_X(1) = \mathcal{O}_X(m)$ for some integer $m \geq 2$. Then

$$c_n\big(\Omega^1_X(2)\big) \leq 2^n \deg(X). \qquad \square$$

Proof. Let us sketch the proof following [ARV]. We first observe that the sheaf $\Omega^1_{\mathbf{P}^N}(2)$ is spanned by its global sections; therefore $\Omega^1_X(2)$, which is a quotient of $\Omega^1_{\mathbf{P}^N}(2)_{|X}$, is also spanned by its global sections. Let σ be a general section of $\Omega^1_X(2)$; then σ and its pullback $f^*\sigma \in H^0(X, \Omega^1_X(2m))$ have isolated zeros (see [ARV, Lemma 1.1]). By counting these zeros, we get

$$c_n\big(\Omega^1_X(2m)\big) \geq \deg(f)\, c_n\big(\Omega^1_X(2)\big).$$

Since $\deg(f) = m^n$, we get $c_n(\Omega^1_X(2)) \leq m^{-n} c_n(\Omega^1_X(2m))$. By the splitting principle, $c_n(\Omega^1_X(2m))$ is a polynomial in m with leading term $(2m)^n \deg(X)$. Replacing f by f^k, we obtain the above inequality for m arbitrarily large; therefore

$$c_n\big(\Omega^1_X(2)\big) \leq \lim_{m\to\infty} m^{-n} c_n\big(\Omega^1_X(2m)\big) = 2^n \deg(X). \qquad \blacksquare$$

Proof of the theorem. Let X be a smooth hypersurface of dimension n and of degree d. We first discuss the case $n \geq 3$. Then $b_2(X) = 1$, so that the condition $f^*\mathcal{O}_X(1) = \mathcal{O}_X(m)$ is automatic. In view of the proposition, we just have to prove that $c_n(\Omega^1_X(2)) > 2^n\, d$. Twisting the Euler exact sequence and the conormal sequence by $\mathcal{O}_X(2)$, we get exact sequences

$$0 \longrightarrow \Omega^1_{\mathbf{P}^{n+1}}(2)_{|X} \longrightarrow \mathcal{O}_X(1)^{n+2} \longrightarrow \mathcal{O}_X(2) \longrightarrow 0,$$
$$0 \longrightarrow \mathcal{O}_X(2-d) \longrightarrow \Omega^1_{\mathbf{P}^{n+1}}(2)_{|X} \longrightarrow \Omega^1_X(2) \longrightarrow 0,$$

from which we find

$$c\big(\Omega^1_X(2)\big) = (1+h)^{n+2}(1+2h)^{-1}\big(1+(2-d)h\big)^{-1},$$

where h is the class of a hyperplane section in $H^2(X,\mathbf{Z})$. Since $h^n = d$, we have

$$c_n\big(\Omega^1_X(2)\big) = d\,\mathrm{Res}_0\,\omega \quad \text{with } \omega = \frac{(1+x)^{n+2}}{x^{n+1}(1+2x)(1+(2-d)x)}\,dx.$$

Straightforward computations give

$$\operatorname{Res}_{\infty} \omega = \frac{1}{2(d-2)}, \qquad \operatorname{Res}_{-1/2} \omega = \frac{(-1)^{n+1}}{2d}, \qquad \operatorname{Res}_{1/(d-2)} \omega = \frac{-(d-1)^{n+2}}{d(d-2)};$$

hence, by the residue theorem,

$$c_n(\Omega^1_X(2)) = \frac{2(d-1)^{n+2} - d + (-1)^n(d-2)}{2(d-2)}.$$

Using $(d-1)^2 = d(d-2) + 1$, we get $c_n(\Omega^1_X(2)) > d(d-1)^n \geq d\,2^n$; hence the result in this case.

For the case $n = 2$, we observe that the result is straightforward when K_X is ample or trivial (see Proposition 2); therefore it only remains to prove it for cubic surfaces. This can be easily done with the above method, but we deduce it from the more general case of Del Pezzo surfaces (see Proposition 3). ■

Remark. The same method applies (with some work) to complete intersections of multidegree $(d_1, \ldots, d_p)$ in $\mathbf{P}^{n+p}$, *provided one of the* d_i *is greater than* 2. On the other hand, it does not work in general for complete intersections of quadrics.

2 Other manifolds

In this section we gather a few remarks on projective manifolds admitting an endomorphism of degree greater than 1. We first recall that this has strong implications on the Kodaira dimension of the manifold.

Proposition 2. Let X be a compact manifold with an endomorphism f of degree greater than 1.

(a) The Kodaira dimension $\kappa(X)$ is less than $\dim(X)$.

(b) If $\kappa(X) \geq 0$, then f is étale. □

Proof. Assertion (a) follows for instance from [KO], and assertion (b) is stated in [SF]. Let us give the proofs for completeness.

(a) Consider the pluricanonical maps $\varphi_m : X \dashrightarrow |mK_X|^*$ associated to the linear systems $|mK_X|$ $(m \geq 1)$. The pullback map $f^* : H^0(X, mK_X) \to H^0(X, mK_X)$ is injective and therefore bijective. We have the following commutative diagram:

$$\begin{array}{ccc} X & \overset{\varphi_m}{\dashrightarrow} & |mK_X|^* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \wr\ {}^t f^*} \\ X & \overset{\varphi_m}{\dashrightarrow} & |mK_X|^* \end{array}$$

In particular, we see that f *induces an automorphism of* $\varphi_m(X)$. If $\dim \varphi_m(X) = \dim X$, then this implies $\deg f = 1$.

(b) Let m be a positive integer such that the linear system $|mK_X|$ is nonempty. Let F be the fixed divisor of this system, and let $|M|$ be its moving part, so that $mK_X \equiv F + M$. The Hurwitz formula reads $K_X \equiv f^*K_X + R$, where R is the ramification divisor of f. This gives

$$F + M \equiv \big(f^*F + mR\big) + f^*M.$$

In particular, we have $h^0(f^*M) \leq h^0(M) = h^0(mK_X)$. Since the pullback map $f^* : H^0(X, M) \to H^0(X, f^*M)$ is injective, we get $h^0(f^*M) = h^0(mK_X)$, which means that $|f^*M|$ is the moving part of $|mK_X|$ and $f^*F + mR$ is its fixed part. Thus

$$F = f^*F + mR$$

in the divisor group $\mathrm{Div}(X)$ of X. Let $\nu : \mathrm{Div}(X) \to \mathbf{Z}$ be the homomorphism that takes the value 1 on each irreducible divisor. Since $\nu(f^*F) \geq \nu(F)$, the above equality is possible only if $R = 0$. We conclude that f is étale. ■

Every Kodaira dimension less than $\dim X$ can indeed occur, as shown by the varieties $V \times A$, where A is an abelian variety. More generally, any (nontrivial) abelian scheme over a smooth projective variety admits étale endomorphisms of degree greater than 1. It seems possible that every projective manifold with an endomorphism of degree greater than 1 and $\kappa(X) \geq 0$ is of this type, up to a finite étale covering. For surfaces this follows easily from the classification. Partial results in the threefold case are announced in [SF].

Let us now turn to *ramified* endomorphisms. By Proposition 2 we must consider manifolds with $\kappa(X) = -\infty$; a natural place to look at is Fano manifolds. For surfaces we have a complete answer.

Proposition 3. A Del Pezzo surface S admits an endomorphism of degree greater than 1 if and only if $K_S^2 \geq 6$. □

Proof. (a) A Del Pezzo surface of degree greater than or equal to 6 is isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$ or $\mathbf{P}^2$ blown-up at some of the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. The first case is trivial; in the second case, the endomorphisms $(X, Y, Z) \mapsto (X^p, Y^p, Z^p)$ of $\mathbf{P}^2$ extend to the blown-up surface.

(b) Let us now consider a Del Pezzo surface S with an endomorphism $f : S \to S$ of degree $d > 1$. Let E be an exceptional curve on S, $F = f(E)$, and let δ be the degree of

$f_{|E} : E \to F$. We have $f_*E = \delta F$ and therefore $f^*F \equiv (d/\delta)E$ (see Lemma 1). Taking squares gives $F^2 = -d/\delta^2$. Because of the genus formula $C^2 + C \cdot K = 2g(C) - 2$, the only curves with negative square on a Del Pezzo surface are the exceptional ones. Thus F is exceptional, $d = \delta^2$, and $f^*F \equiv \delta E$. Since the right-hand side does not move, this is an equality of divisors. It means that f is ramified along E with ramification index δ. In other words, if we denote by $\mathcal{E}$ the (finite) set of exceptional curves on S and by R the ramification divisor of f, we have $R = \sum_{E \in \mathcal{E}} (\delta - 1)E + Z$, where Z is an effective divisor. By intersecting with $-K_S$, we get

$$-K_S \cdot R \geq (\delta - 1)\,\mathrm{Card}(\mathcal{E}).$$

For each $E \in \mathcal{E}$ we have $f^*K_S \cdot E = K_S \cdot f_*E = \delta K_S \cdot F = -\delta$, and therefore $(f^*K_S - \delta K_S) \cdot E = 0$. We can assume that $\mathcal{E}$ spans the Picard group of S. (This holds as soon as $K_S^2 \leq 7$.) Thus $f^*K_S \equiv \delta K_S$. Then the Hurwitz formula $K_S \equiv f^*K_S + R$ gives $R \equiv (\delta - 1)(-K_S)$, so that the above inequality becomes $K_S^2 \geq \mathrm{Card}(\mathcal{E})$. This is impossible for $K_S^2 \leq 5$, as the surface S then contains at least 10 exceptional curves. ∎

For Fano threefolds we know the answer in the case $b_2 = 1$, as a consequence of the more general results of [A] and [ARV]: the only Fano threefold with $b_2 = 1$ admitting an endomorphism of degree greater than 1 is $\mathbf{P}^3$. Their methods apply to some other Fano threefolds, but the general case seems to require new techniques.

Acknowledgment

I am indebted to I. Dolgachev for bringing the problem to my attention.

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