HYPERPLANE SECTIONS OF CUBIC THREEFOLDS

ARNAUD BEAUVILLE

ABSTRACT. Let $X\subset \mathbb{P}^4$ be a smooth cubic hypersurface. We prove that a general cubic surface is isomorphic to a hyperplane section of X.

1. INTRODUCTION

A classical result of Sylvester states that a general cubic surface $S\subset\mathbb{P}^3$ admits an equation of the form $L_0^3+\ldots+L_4^3=0$, where the L_i are linear forms (see e.g. [D, §9.4]). An equivalent formulation is that S is isomorphic to a hyperplane section of the Fermat cubic threefold $X_0^3+\ldots+X_4^3=0$. We will show in this note that this actually holds for *any* smooth cubic threefold. Surprisingly, the proof reduces to a particular case of the *weak Lefschetz property* for the Jacobian ring of X — a case which miraculously is treated in [A-R].

2. PROOFS

Let $X\subset\mathbb{P}^4=\mathbb{P}$ be a smooth cubic hypersurface, defined by an equation $F=0$. Let $X^*\subset\mathbb{P}^*$ be the dual hypersurface. For any $H\in\mathbb{P}^*\smallsetminus X^*$, the hyperplane section $X\cap H$ is a smooth cubic surface; we get in this way a morphism s_X from $\mathbb{P}^*\smallsetminus X^*$ into the moduli space $\mathscr{M}_3$ of smooth cubic surfaces. Our aim is to determine when this map is dominant.

Let R denote the graded ring $\mathbb{C}[X_0,\ldots,X_4]$. Let J be the ideal $(F'_0,\ldots,F'_4)$ in R, where we put $F'_i:=\dfrac{\partial F}{\partial X_i}$, and let $\mathfrak{J}:=R/J$ be the Jacobian ring of F.

Proposition. *Let $H\in\mathbb{P}^*\smallsetminus X^*$, given by a linear form $L\in R_1$. The map $s_X:\mathbb{P}^*\smallsetminus X^*\to\mathscr{M}_3$ is étale at H if and only if the multiplication map $\times L:\mathfrak{J}_2\to\mathfrak{J}_3$ is injective.*

Proof : We put $S:=X\cap H$. The tangent space to $\mathbb{P}^*$ at H is naturally isomorphic to $H^0(H,\mathcal{O}_H(1))=H^0(S,\mathcal{O}_S(1))$, and the tangent map to s_X at H is the coboundary map

$$\partial : H^0(S,\mathcal{O}_S(1))\to H^1(S,T_S)$$

deduced from the exact sequence

$$0\to T_S\to T_{X|S}\to\mathcal{O}_S(1)\to 0\,.$$

Since $\dim H^0(S,\mathcal{O}_S(1))=\dim H^1(S,T_S)=4$ and $H^0(S,T_S)=0$, s_X is étale at H if and only if $H^0(S,T_{X|S})=0$.

By restricting to S the exact sequence

$$0\to T_X\to T_{\mathbb{P}|X}\to\mathcal{O}_X(3)\to 0\,,$$

I am indebted to A. Dimca for pointing out the reference [A-R].

we see that $H^0(S,T_{X|S})$ is the kernel of the map $\varphi : H^0(S,T_{\mathbb{P}|S}) \to H^0(S,\mathcal{O}_S(3))$. We choose the coordinates in $\mathbb{P}$ so that $L = X_0$. The space $H^0(S,T_{X|S})$ is generated by the vector fields (restricted to S) $X_i\dfrac{\partial}{\partial X_j}$ $(i>0, j\geq 0)$, with the relation $\sum\limits_{i\geq 1} X_i\dfrac{\partial}{\partial X_i} = 0$; we have $\varphi(X_i\dfrac{\partial}{\partial X_j}) = X_iF'_j$.

Let $V = \sum\limits_{i=0}^{4} L_i\dfrac{\partial}{\partial X_i}$ be a nonzero element of $H^0(S,T_{X|S})$, where the L_i are linear forms in $X_1,\ldots,X_4$. Assume that $\varphi(V)=0$. This means that there exists $Q\in R_2$ and $a\in\mathbb{C}$ such that

$$\sum_{i=0}^{4} L_iF'_i = X_0Q + 3aF\,; \tag{1}$$

this implies that the class of Q in $\mathfrak{J}_2$ is mapped to zero in $\mathfrak{J}_3$ under multiplication by X_0.

Let us show that this class is nonzero. Suppose $Q\in J_2$, so that $Q=\sum a_iF'_i$. Using the Euler relation $\sum X_iF'_i = 3F$, (1) becomes

$$\sum_{i=0}^{4}(L_i - a_iX_0 - aX_i)F'_i = 0\,.$$

Now the only nontrivial relation $\sum M_iF'_i = 0$ with the M_i in R_1 is the Euler relation (this follows from the cohomology exact sequence associated to $0\to T_X\to T_{\mathbb{P}|X}\to \mathcal{O}_X(3)\to 0$, plus the fact that $H^0(X,T_X)=0$).

Therefore there exists $b\in\mathbb{C}$ such that $L_i - a_iX_0 - aX_i = bX_i$ for all i; this implies $a_i = 0$ for $i\geq 1$, hence $L_i = (a+b)X_i$, and $L_0 = 0$. Therefore $V = (a+b)\sum\limits_{i\geq 1} X_i\dfrac{\partial}{\partial X_i} = 0$, a contradiction.

Conversely, if $\times X_0 : \mathfrak{J}_2\to\mathfrak{J}_3$ is not injective, there exist forms $Q\in R_2\smallsetminus J_2$ and $L_0,\ldots,L_4$ in R_1 such that

$$X_0Q = \sum L_iF'_i\,. \tag{2}$$

Replacing Q by $Q-\sum a_iF'_i$, where a_i is the coefficient of X_0 in L_i, we can assume that the L_i are linear forms in $X_1,\ldots,X_4$. Then the vector field $V := \sum L_i\dfrac{\partial}{\partial X_i}$ satisfies $\varphi(V)=0$.

We just have to check that V is nonzero. If it is, there exists $b\in\mathbb{C}$ such that $L_i = bX_i$ for $i\geq 1$ and $L_0 = 0$. Then (2) becomes

$$X_0(Q+bF'_0) = \sum_{i\geq 0} bX_iF'_i = 3bF\,.$$

Since F is irreducible, this implies $b=0$ and $Q=0$, contradiction. ■

Remark.– It is easy to give examples of pairs (X,H) such that s_X is not étale at H. For instance if X is the Fermat cubic and H is the hyperplane $X_0=0$, the elements $X_i\dfrac{\partial}{\partial X_0}$ $(i\geq 1)$ of $H^0(S,T_{\mathbb{P}|S})$ belong to $\operatorname{Ker}\varphi$, so the tangent map to s_X at H is zero. In fact, s_X contracts the lines in $\mathbb{P}^*$ consisting of the hyperplanes $X_0 = tX_i$, $t\in\mathbb{C}$, $i\geq 1$.

Theorem. *The map $s_X : \mathbb{P}^*\smallsetminus X^*\to\mathscr{M}_3$ is dominant.*

Proof : By [A-R, Theorem 1], the multiplication map $\times L : \mathfrak{J}_2\to\mathfrak{J}_3$ is injective for L general in R_1 (weak Lefschetz property in degree 2). Thus by the Proposition, s_X is étale at a general point of $\mathbb{P}^*$, hence dominant. ■

Remarks. – 1) The proof of the Proposition applies identically to a smooth hypersurface X of degree d in $\mathbb{P}^n$, with $d,n\geq 3$. With the previous notation, we get that *s_X is unramified at H if and only if* *the multiplication map* $\times L : \mathfrak{J}_{d-1} \to \mathfrak{J}_d$ *is injective*. If $\mathfrak{J}$ satisfies the weak Lefschetz property in degree $d-1$, we conclude that s_X *is generically finite onto its image*. This is the case for instance when X is the Fermat hypersurface $\sum_{i=0}^{n} X_i^d = 0$, hence also for a general hypersurface of degree d in $\mathbb{P}^n$. However there seems to be very few cases where this is known for an arbitrary smooth hypersurface.

2) The proof of the Proposition works over any algebraically closed field k. However, the weak Lefschetz property used in the Theorem does not always hold. For instance if X is the Fermat cubic threefold in characteristic 2, the Jacobian ideal is generated by the squares of all linear forms; if ℓ and m are linearly independent linear forms, we have $\ell \cdot m \neq 0$ in $\mathfrak{J}_2$, but $\ell \cdot (\ell \cdot m) = 0$ in $\mathfrak{J}_3$, so the map $\times \ell : \mathfrak{J}_2 \to \mathfrak{J}_3$ is not injective.

In fact in this case all smooth hyperplane sections of X are isomorphic, see [B, Théorème 2].

References

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UNIVERSITÉ CÔTE D'AZUR, CNRS – LABORATOIRE J.-A. DIEUDONNÉ, UMR 7351 DU CNRS, PARC VALROSE, F-06108 NICE CEDEX 2, FRANCE

E-mail address: arnaud.beauville@univ-cotedazur.fr