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Spectral curves and the generalised theta divisor

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§ 1. Statement of results

Let X be a smooth, irreducible, projective curve over an algebraically closed field of characteristic 0 and let $g = g_X \geqq 2$ be its genus. We show in this paper that a generic vector bundle on X of rank n and any fixed degree can be obtained as the direct image of a line bundle on an n-sheeted (ramified) covering of X. From this we deduce results concerning linear systems on the moduli space of vector bundles of rank n and degree 0.

Let $\mathscr{U}_X(n, d)$, or simply $\mathscr{U}(n, d)$ (resp. $\mathscr{S}\mathscr{U}(n, \zeta)$) denote the moduli space of semistable vector bundles of rank n and degree d (resp. with determinant isomorphic to a fixed line bundle ζ) on X. When ζ is the trivial line bundle, we simply write $\mathscr{S}\mathscr{U}(n)$ for $\mathscr{S}\mathscr{U}(n, \zeta)$. When we deal with $\mathscr{S}\mathscr{U}(n, \zeta)$, but the particular ζ that we fix is not of importance, we denote it by $\mathscr{S}\mathscr{U}(n, d)$. Let Y be any smooth curve and $\pi\colon Y \to X$ a finite morphism. We can then associate to any line bundle ξ on Y a vector bundle, namely the direct image $\pi_*(\xi)$, on X. This yields a morphism of the open set $\mathscr{T}_Y^\delta$ of J_Y^δ into $\mathscr{U}(n, d)$, where a) J_Y^δ is the space of line bundles of degree δ on Y, b) $\mathscr{T}_Y^\delta$ is the set of those ξ for which $\pi_*(\xi)$ is semistable, and c) d and δ are connected by the relation $d = \delta + \deg(\pi_*(\mathscr{O}_Y))$.

Theorem 1. *For any pair of integers n and d, there exists an n-sheeted covering $\pi\colon Y \to X$ with Y smooth and irreducible, such that the morphism $\pi_*\colon \mathscr{T}_Y^\delta \to \mathscr{U}_X(n, d)$ is dominant with $\delta = d - \deg \pi_*(\mathscr{O}_Y)$.*

Before stating the next result, we need to introduce the notion of the *generalised theta divisor*. It is easy to define a divisor Θ in $\mathscr{U}(n, d)$ with $d = n(g-1)$ supported on the set of bundles E with $\Gamma(E) \neq 0$. This defines an invertible sheaf [2] on $\mathscr{U}(n, n(g-1))$, which we may denote by $\mathscr{O}(\Theta)$. In the case $n = 1$, this divisor is the wellknown Riemann theta divisor on the variety J^{g-1}.

Theorem 2. *We have* $\dim \Gamma(\mathscr{U}(n, n(g-1)), \mathscr{O}(\Theta)) = 1$.

This theorem is proved by considering the dominant morphism $\pi_*\colon \mathscr{T}_Y^\delta \to \mathscr{U}(n, d)$ whose existence is asserted in Theorem 1, with $d = n(g-1)$. One can determine the pullback of Θ by the morphism π_*. In fact this turns out to be the Riemann theta

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[1]) Supported by DFG.

divisor. Now one shows that the complement of $\mathscr{T}_Y^\delta$ in J_Y^δ is of codimension at least 2 so that we get an injection of $\Gamma(\mathscr{U}(n, d), \Theta)$ into $\Gamma(\mathscr{T}_Y^\delta, \Theta) = \Gamma(J_Y, \Theta)$, proving our assertion.

There is a close relationship [9] between the moduli space $\mathscr{S}\mathscr{U}(n)$ and the linear system of the invertible sheaf $\mathcal{O}(n\Theta)$ on J^{g-1}. Indeed, associate to a general vector bundle E, the divisor D_E given by $\{\xi \in J^{g-1} : \Gamma(E \otimes \xi) \neq 0\}$. This gives a rational map of $\mathscr{S}\mathscr{U}(n)$ into the linear system of $n\Theta$ on J^{g-1}. On the other hand, let ξ be an element of J^{g-1}, and let $\Delta_\xi = \{E \in \mathscr{S}\mathscr{U}(n) : \Gamma(E \otimes \xi) \neq 0\}$. One can show [2] that Δ_ξ defines a Cartier divisor and the corresponding line bundle h generates $\mathrm{Pic}(\mathscr{S}\mathscr{U}(n))$.

Theorem 3. *The (rational) map, which associates to a general point e of $\mathscr{S}\mathscr{U}(n)$ the divisor D_E in J^{g-1}, where E is a semistable bundle in the class of e, induces an isomorphism of $\Gamma(J^{g-1}, \mathcal{O}(n\Theta))$ with $\Gamma(\mathscr{S}\mathscr{U}(n), h)^*$, and the rational map is defined by this linear system. In particular, $\Gamma(\mathscr{S}\mathscr{U}(n), h)$ has dimension n^g.*

The proof of this theorem is similar to that of Theorem 2, except that the Jacobian of Y is in this case replaced by the Prym variety associated to the covering.

In the case of vector bundles of rank 2, these results have been proved by A. Beauville [1]. In the rank n case, vector bundles arising from line bundles on an étale covering have been studied by Narasimhan and Ramanan [8]. Ideas of Hitchin [4], and the existence of 'very stable' vector bundles proved by Drinfeld and Laumon [5] have been useful to us.

§ 2. Preliminaries

2. 1. Twisted endomorphisms. Let E be a vector bundle of rank n, L a line bundle on X and $\varphi: E \to L \otimes E$ a homomorphism. Then one can define its trace as an element of $\Gamma(L)$ by interpreting it to be a map $E \otimes E^* \to L$ and taking the image of the identity section of $E \otimes E^*$. More generally, its characteristic coefficients $a_i \in \Gamma(X, L^i)$, for $0 \leqq i \leqq n$, may be defined by setting $a_i = (-1)^i \mathrm{Tr}\,\Lambda^i \varphi$. The Cayley-Hamilton theorem then asserts that φ satisfies its characteristic equation. This means that $\sum a_i \varphi^{n-i}$, interpreted as a homomorphism $E \to L^n \otimes E$, is zero. In particular, if all a_i are zero for $1 \leqq i \leqq n$, then $\varphi^n = 0$, and we may say that φ is *nilpotent*.

2. 2. Generalised theta divisor. Let $\mathscr{U}(n, n(g-1))$ be the moduli space of vector bundles of rank n and degree $n(g-1)$. Then by the Riemann-Roch theorem, we see that $\chi(E) = 0$ for all E in $\mathscr{U}(n, n(g-1))$. Let $\{E_t\}_{t \in T}$ be a family of vector bundles, given by a vector bundle E on $T \times X$, with $E_t = E|_{t \times X}$ semistable for all $t \in T$. Let $p: T \times X \to T$ denote the first projection. The first direct image $R^1 p_* E$ is wellbehaved under base change and is supported on the set $\{t \in T : \Gamma(E_t) \neq 0\}$. Then $(\det R^1 p_* E) \otimes (\det p_* E)^{-1}$ is defined to be the invertible sheaf $\mathcal{O}(\Theta_T)$. It can be shown [2] that there exists an invertible sheaf $\mathcal{O}(\Theta)$ on $\mathscr{U}(n, d)$ such that $\mathcal{O}(\Theta_T)$ is the pullback of $\mathcal{O}(\Theta)$ by the canonical classifying morphism $T \to \mathscr{U}(n, d)$.

2. 3. Remarks on principally polarised Abelian varieties. Suppose (A, Θ) is a principally polarised Abelian variety and N an Abelian subvariety of A. Consider the subgroup P of A consisting of all $a \in A$ such that $T_a^* \mathcal{O}(\Theta)$ and $\mathcal{O}(\Theta)$ restrict to isomorphic line bundles on N. We call P the *orthogonal* of N. It is an Abelian subvariety of A such that the maps $P \to A/N$ and $N \to A/P$ are isogenies. Let $\mathcal{O}(\Theta_N)$

and $\mathcal{O}(\Theta_P)$ be the restrictions of $\mathcal{O}(\Theta)$ to N and P respectively. One can then define a rational map ψ of P into $|\Theta_N|$ by setting $\psi(p) = T_p^*(\Theta)|N$. On the other hand, we also have a rational map $\varphi: P \to |\Theta_P|^*$, defined by the linear system $|\Theta_P|$.

2. 4. Proposition. *There is a canonical isomorphism* $\iota: |\Theta_P|^* \to |\Theta_N|$ *such that* $\iota \circ \varphi = \psi$.

Proof. Consider the isogeny $\pi: N \times P \to A$ sending (n, p) to $n + p$. The pullback $\pi^* \mathcal{O}(\Theta)$ is isomorphic to $p_N^* \mathcal{O}(\Theta_N) \otimes p_P^* \mathcal{O}(\Theta_P)$, by the theorem of the square. The canonical section of $\mathcal{O}(\Theta)$ pulls back to an element of $\Gamma(N, \mathcal{O}(\Theta_N)) \otimes \Gamma(P, \mathcal{O}(\Theta_P))$. One can now check that the induced map $\iota: \Gamma(P, \mathcal{O}(\Theta_P))^* \to \Gamma(N, \mathcal{O}(\Theta_N))$ is an isomorphism. This follows from the fact that the theta group of $\pi^* \mathcal{O}(\Theta)$ is the natural pushout of the direct product of the theta groups of $\mathcal{O}(\Theta_N)$ and $\mathcal{O}(\Theta_P)$ with respect to the imbedding of the multiplicative group $\mathbf{G}_m$ by $\lambda \mapsto (\lambda, \lambda^{-1})$ and that the irreducible theta action is given by that on $\Gamma(N, \mathcal{O}(\Theta_N)) \otimes \Gamma(P, \mathcal{O}(\Theta_P))$. It is easy to identify the unique 1-dimensional subspace fixed by the maximal level subgroup $\operatorname{Ker}(\pi)$, and see that it gives a nondegenerate element as claimed. The rational map φ of the variety P into $|\Theta_P|^*$ is characterised by its equivariance under the action of the theta group. Now one can check that $\iota^{-1} \circ \psi$ does have this property, thus proving the equality $\iota \circ \varphi = \psi$.

2. 5. Polarisation on the Prym variety of a covering. Let $\pi: Y \to X$ be a finite morphism of smooth, irreducible curves. Let us denote by $\mathfrak{d}$ the line bundle $\det(\pi_*(\mathcal{O}_Y))^{-1}$. The relative duality theorem gives an isomorphism of

$$\pi_*(\mathcal{O}_Y)^* \text{ with } \pi_*(K_Y \otimes \pi^* K_X^{-1}).$$

The ramification divisor Δ is given by a section of $\pi^* K_X^{-1} \otimes K_Y$, so that we have $\mathfrak{d} = \det \pi_*(\mathcal{O}(\Delta))$. For any line bundle L on Y, we may associate the sheaf $\pi_*(L)$ on X. It is actually a vector bundle of rank n if π is n-sheeted. Then one can show easily that $\det \pi_*(L) = \operatorname{Nm}(L) \otimes \mathfrak{d}^{-1}$, where Nm is the norm map associated to the covering π. In particular, $\det \pi_*(\mathcal{O}(\Delta)) = \operatorname{Nm}(\mathcal{O}(\Delta)) \otimes \mathfrak{d}^{-1}$ and hence $\operatorname{Nm} \mathcal{O}(\Delta) = \mathfrak{d}^2$.

Suppose that the induced homomorphism π^* of J_X into J_Y is *injective*. Then we may set $N = \pi^* J_X$ and $A = J_Y$ and apply the considerations of 2. 3 and 2. 4. The norm homomorphism $\operatorname{Nm}: J_Y \to J_X$ can be identified with the transpose of π^* and consequently has connected kernel [7]. This is the Prym variety of the map π and we would like to take for P this kernel. However, in order to avoid making too many choices, we will consider the varieties $N' = \pi^* J_X^{g-1}$, $A' = J_Y^{h-1}$ and $P' = \operatorname{Nm}^{-1}(\mathfrak{d})$, where $g = g_X$, $h = g_Y$ and Nm is the norm map $J_Y^{\deg(\mathfrak{d})} \to J_X^{\deg(\mathfrak{d})}$. Then we have a natural addition map $N' \times P' \to A'$ (namely tensor product). Up to translation, this is the same as the map defined in 2. 3. Hence the pullback of the line bundle $\mathcal{O}(\Theta_Y)$ by this map is the tensor product of a line bundle on N' and one on P'. In fact, under our assumption, N' is isomorphic to J_X^{g-1} on which is defined Riemann's theta divisor Θ_X. Now it is easy to deduce that the restriction of $\mathcal{O}(\Theta_Y)$ to $N' \times \{\alpha\}$ is isomorphic to $\mathcal{O}(n\Theta)$ for any $\alpha \in P'$. Hence there is a natural line bundle τ on P' such that the pullback mentioned above is $p_1^* \mathcal{O}(n\Theta_X) \otimes p_2^* \tau$. Now the pullback of the natural section of $\mathcal{O}(\Theta_Y)$ gives rise, in view of our remarks above, to a nondegenerate element of $\Gamma(P', \tau) \otimes \Gamma(J_X^{g-1}, \mathcal{O}(n\Theta))$, that is to say there is a natural isomorphism of $\Gamma(P', \tau)^*$ with $\Gamma(J_X^{g-1}, \mathcal{O}(n\Theta))$. Thus we have

2. 6. Proposition. *Let* $\pi: Y \to X$ *be a finite morphism of projective nonsingular curves such that the induced morphism* $\pi^*: J_X \to J_Y$ *is injective. Then with the notation* *above, there is a natural isomorphism of* $\Gamma(J_X^{g-1}, \mathcal{O}(n\Theta))$ *with* $\Gamma(P', \tau)^*$ *such that the following diagram is commutative.*

$$\begin{array}{ccc} P' & \longrightarrow & P\Gamma(P', \tau)^* \\ & \searrow & \downarrow \approx \\ & & P\Gamma(J_X^{g-1}, \mathcal{O}(n\Theta)) \end{array}$$

Here the rational map $P' \to P\Gamma(P', \tau)^*$ is given by the linear system of τ, while the rational map $P' \to P\Gamma(J_X^{g-1}, \mathcal{O}(n\Theta))$ is given by $p \mapsto T_p^* \Theta_Y \cap \pi^* J_X^{g-1}$.

2. 7. Remark. One can show in fact that under the assumption that $\pi^*: J_X \to J_Y$ is injective, the polarisation on P (and on P' as well) is of type $(1, 1, \ldots, n, n, \ldots)$ where the n's are repeated g times. In particular its Pfaffian is n^g which is of course the dimension of $\Gamma(J^{g-1}, \mathcal{O}(n\Theta))$. It is easy to show that τ is a primitive element in the group of algebraic equivalence classes.

§ 3. Spectral curves

Spectral curves have been treated in recent years by several authors, in connection with some special kinds of nonlinear differential equations. In our context it was used by Hitchin [4]. Let L be a line bundle on X and $s=(s_k)$ be sections of L^k for $k=1, 2, \ldots, n$. Then we will construct a scheme X_s and a finite morphism $\pi: X_s \to X$. Let $p: P = \mathbf{P}(\mathcal{O} \oplus L^*) \to X$ be the natural projection and $\mathcal{O}(1)$ the relatively ample bundle, or what may be called "the hyperplane bundle along the fibres". Then $p_*(\mathcal{O}(1))$ is naturally isomorphic to $\mathcal{O} \oplus L^*$ which has a canonical section, namely, the constant section 1 of $\mathcal{O}$. This gives a section of $\mathcal{O}(1)$ over P which we denote by y. On the other hand, $p_*(p^* L \otimes \mathcal{O}(1))$ is isomorphic, by the projection formula, to $L \otimes (\mathcal{O} \oplus L^*) = L \oplus \mathcal{O}$. Hence it also has a canonical section and we denote the corresponding section of $p^* L \otimes \mathcal{O}(1)$ by x. Note that the two sections x and y have as their zero schemes the natural subvarieties $\mathbf{P}(\mathcal{O})$ and $\mathbf{P}(L^*)$ of P. Now consider the section

$$x^n + (p^* s_1) \cdot y \cdot x^{n-1} + \cdots + (p^* s_n) \cdot y^n$$

of $p^* L^n \otimes \mathcal{O}(n)$. Its zero scheme is the scheme X_s we have in mind. It is clear that the restriction π of p to X_s is finite and that at any point v of X the fibre of π is the subscheme of $\mathbf{P}^1$ given by

$$x^n + a_1 y x^{n-1} + \cdots + a_n y^n = 0,$$

where (x, y) is a homogeneous coordinate system, and a_i is the value of s_i at v on identifying the fibre of L at P with the residue field at v.

3. 1. Remark. It is easily seen that the set of all $(s) \in \Gamma(L) \oplus \Gamma(L^2) \oplus \cdots \oplus \Gamma(L^n)$ for which the scheme X_s is integral (i.e., reduced and irreducible), is open. Also, it is easy to see that it is nonempty, whenever L^n admits a section whose divisor is not of the form mD for some integer m dividing n. For, in that case we can take all s_i for $i \leq n-1$ to be 0 and s_n to be a section as postulated.

Hereafter we will assume that (s) has been so chosen that X_s is integral. Since the zeros of x and y are disjoint in P, it follows that the restriction of y to X_s is everywhere nonzero and hence the restriction of $\mathcal{O}(1)$ to X_s is trivial. The restriction of x to X_s can therefore be considered as a section of $\pi^*(L)$. An alternative description of X_s is given as follows. The direct image $\pi_*(\mathcal{O})$ is naturally isomorphic to $\mathcal{O} \oplus L^{-1} \oplus \cdots \oplus L^{-(n-1)}$. Its algebra structure is better understood by interpreting this to be $\mathrm{Sym}(L^{-1})/\mathcal{I}$ where $\mathcal{I}$ is the ideal sheaf generated by the image of the homomorphism $u: L^{-n} \to \mathrm{Sym}(L^{-1})$ given as the sum of the imbeddings $L^{-n} \to L^{-(n-i)}$ defined by s_i. In terms of this sheaf of algebras, X_s is simply $\mathrm{Spec}(\mathrm{Sym}(L^{-1})/\mathcal{I})$.

3. 2. Remark. The genus g' of X_s may be computed from the above interpretation. In fact, $1-g'$ is given by

$$\begin{aligned}\chi(X_s, \mathcal{O}) = \chi(X, \pi_*(\mathcal{O})) = \sum \chi(X, L^i)& \\ &= (\deg L) \cdot (0-1-\cdots-(n-1)) + n(1-g) \\ &= -(\deg L) \cdot n(n-1)/2 + n(1-g)\end{aligned}$$

and hence $g' = (\deg L) \cdot n(n-1)/2 + n(g-1) + 1$.

3. 3. Remark. Assume that X_s is integral. Then consider the resultant of the polynomial

$$x^n + s_1 y x^{n-1} + s_2 y^2 x^{n-2} + \cdots + s_n y^n,$$

and its derivative

$$n x^{n-1} + (n-1) s_1 y x^{n-2} + \cdots + s_{n-1} y^{n-1}.$$

This can be interpreted to be a section of $(p^* L \otimes \mathcal{O}(1))^{n(n-1)/2}$. One easily verifies that the map $\pi: X_s \to X$ is étale at a point if and only if this resultant does not vanish at that point. In other words, the resultant gives the ramification divisor of the map π.

3. 4. Example. Let L be a line bundle with L^n isomorphic to $\mathcal{O}$. We could then take s_i to be 0 for $i \leqq n-1$, and s_n to be any nonzero section of L^n. In this case one can conclude from 3. 3 that the map π is actually étale. However, the scheme X_s is connected (and therefore irreducible) if and only if the section s_n is not a power of a section of L^r for some r which divides n. This would happen if and only if L^r is trivial. In other words, X_s is irreducible if and only if L is of order n in the Jacobian of X.

3. 5 Remark. Let us assume that X_s is integral. Then using the Jacobian criterion it is not difficult to prove that X_s is smooth if and only if at every point which is a multiple zero of the section s_n, the section s_{n-1} is nonzero. In particular, the set of sections (s) such that the scheme X_s is smooth, is open on the one hand, and is nonempty if L^n admits a section without multiple zeros, on the other.

3. 6. Proposition. *Let X be a curve and L any line bundle on it. Let $s = (s_i)$ be a set of sections of L^i for $1 \leqq i \leqq n$. Assume that the corresponding scheme X_s is integral. Then there is a bijective correspondence between isomorphism classes of torsion free* *sheaves of rank 1 on X_s and isomorphism classes of pairs (E, φ) where E is a vector bundle of rank n and $\varphi: E \to L \otimes E$ a homomorphism with characteristic coefficients s_i. The correspondence is given by associating to any line bundle M on X_s the sheaf $\pi_*(M)$ on X and the natural isomorphism*

$$\pi_*(M) \to L \otimes \pi_*(M) \approx \pi_*(\pi^* L \otimes M)$$

given by the section x of $\pi^*(L)$.

Proof. If M is a torsion free sheaf of rank 1 on X_s, then the sheaf $\pi_* M$ is a vector bundle of rank n on X, endowed with a $\pi_*(\mathcal{O}_{X_s})$-module structure. Conversely, since π is affine, a vector bundle E of rank n with a $\pi_*\mathcal{O}$-module structure defines a sheaf M with $\pi_*(M)$ isomorphic to E. Now M is torsion free of rank 1 because X_s is integral.

The $\mathcal{O}_X$-Algebra $\pi_* \mathcal{O}_{X_s}$ is isomorphic to $\mathrm{Sym}(L^{-1})/\mathscr{I}$, so that the data of a $\pi_*(\mathcal{O}_{X_s})$-module structure on E is equivalent to an algebra homomorphism

$$\mathrm{Sym}(L^{-1})/\mathscr{I} \to \mathscr{E}nd(E),$$

that is to say an $\mathcal{O}_X$-homomorphism $\varphi: E \to L \otimes E$ such that $P_s(\varphi) = 0$, where P_s is the polynomial determined by (s). Since X_s is integral, P_s is irreducible over the function field of X, and is hence the characteristic polynomial of φ. Conversely, if E is a vector bundle of rank n on X, and $\varphi: E \to L \otimes E$ a linear homomorphism with characteristic coefficients s_i, then $P_s(\varphi) = 0$ by the Cayley-Hamilton theorem. This proves the proposition.

3. 7. Remark. The sequence

$$0 \longrightarrow M(-\Delta) \longrightarrow \pi^* E \xrightarrow{\pi^*\varphi - x} \pi^*(L \otimes E) \longrightarrow \pi^* L \otimes M \longrightarrow 0$$

is exact. In fact one could actually define the line bundle M corresponding to (E, φ) by this exact sequence. This is essentially the point of view in Hitchin [4].

3. 8. Remark. When X_s is nonsingular, then 'torsion free sheaves of rank 1' may be replaced by 'line bundles' in Proposition 3. 6.

3. 9. Example. Let us get back to Example 3. 5. If L is a line bundle of order n in the Jacobian of X, then according to Proposition 3. 6, there is a natural bijection between line bundles on X_s and pairs (E, φ). Now there exists a map $\varphi: E \to L \otimes E$ with the given characteristic polynomial if and only if $L \otimes E$ is isomorphic to E. In this particular case this map was studied and indeed many of the computations in this paper were carried out in [8].

3. 10. Remark. If M is a nontrivial line bundle on X, and $\pi^* M$ is trivial on X_s, then it follows that $\pi_*(\pi^* M) \cong M \otimes \pi_*(\mathcal{O}) \cong \sum M \otimes L^{-i}$ admits a nontrivial section. In particular this implies that the degree of L is 0. But since $s_i \in \Gamma(L^i)$, we must have for each i, either $s_i = 0$, or L^i is trivial. Thus we see that except in this case (which includes Example 3. 9), the induced map $\pi^*: J_X \to J_{X_s}$ is injective.

§ 4. Applications to the moduli space of vector bundles

In what follows, we will take for the line bundle L the canonical line bundle K of a smooth, projective curve X of genus $g \geqq 2$. Denoting by W the direct sum $\bigoplus_{k=1}^{k=n} \Gamma(X, K^k)$, we see immediately from the Riemann-Roch theorem that

$$\dim W = n^2(g-1)+1,$$

which is also the dimension of $\mathscr{S}\mathscr{U}(n, d)$. *Let* $s=(s_i)$ be an element of W such that the corresponding curve X_s is smooth and irreducible. By Remark 3. 2, the genus of X_s is $n^2(g-1)+1$. Moreover, for any line bundle ξ on X_s we have

$$\det \pi_*(\xi) = \mathrm{Nm}(\xi) \otimes \det \pi_*(\mathcal{O}) = \mathrm{Nm}(\xi) \otimes K^{-n(n-1)/2}$$

where $\mathrm{Nm}\colon \mathrm{Pic}(X_s) \to \mathrm{Pic}(X)$ is the norm map associated to π. In particular, the degree of $\pi_*(\xi)$ is $\deg(\xi) - n(n-1)(g-1)$.

Let $\mathscr{U}$ be the moduli space of stable vector bundles of rank n and degree $d = \deg(\xi) - n(n-1)(g-1)$. Let $\Omega = \Omega^1$ denote its cotangent bundle. If E is a stable bundle of rank n and degree d, then the tangent space to $\mathscr{U}$ at the corresponding point can be canonically identified with $H^1(X, \mathscr{E}nd\, E)$. By duality, the cotangent space may be identified with $\Gamma(X, K \otimes \mathscr{E}nd\, E)$. Therefore a point of Ω can be seen as a pair (E, φ), where E is a vector bundle in $\mathscr{U}$ and $\varphi\colon E \to K \otimes E$ is a homomorphism. By associating to (E, φ) the characteristic polynomial of φ, we define a morphism $H\colon \Omega \to W$. We will call this map the Hitchin map [4].

4. 1. *Proof of the Theorem* 1. Consider the map $\lambda\colon \Omega \to \mathscr{U} \times W$ where the first factor of λ is the projection onto $\mathscr{U}$ and the second factor is the Hitchin map H. By a theorem of Drinfeld and Laumon [5], there exists a stable bundle E_0 of rank n and degree d such that every nilpotent homomorphism $E_0 \to K \otimes E_0$ is zero. This implies that the inverse image by λ of $(E_0, 0)$ in $\mathscr{U} \times W$ consists of a single point, namely the zero cotangent vector at $E_0 \in \mathscr{U}$. Now $\dim(\Omega) = \dim(\mathscr{U} \times W) = 2(n^2(g-1)+1)$. By the dimension theorem, the morphism λ is dominant and moreover its differential is an isomorphism at the generic point of Ω. It follows that the projection $H^{-1}(s) \to \mathscr{U}$ is dominant for all point s in an open set in W. Moreover we may assume that for s in this open set the spectral curve X_s is smooth and irreducible in view of Remark 3. 5. Now by Proposition 3. 6, $H^{-1}(s)$ consists of pairs (E, φ) where E is a stable bundle obtained as the direct image of a line bundle ξ on X_s and $\varphi\colon E \to K \otimes E$ is the homomorphism induced by the canonical section of $\pi^*(K)$ on X_s. This proves Theorem 1.

§ 5. The basic linear system on $\mathscr{S}\mathscr{U}(n)$

We apply the considerations of the previous section to the case when $d = n(g-1)$. Let us choose an n-sheeted covering Y of X such that most vector bundles of rank n and degree d are direct images of line bundles of degree δ with $\delta = g_Y - 1 = n^2(g-1)$. We will denote the Riemann theta divisor in J_X^{g-1} (whose support consists of effective divisor classes of degree $(g-1)$) by Θ_X. Let $\mathcal{T}_Y^\delta$ denote the open subset of the component of the Jacobian consisting of ξ such that $\pi_*(\xi)$ is semistable. Thus there is a natural morphism $\pi_*: \mathcal{T}_Y^\delta \to \mathcal{U}(n, n(g-1))$.

5. 1. Proposition. a) *For every point ξ belonging to $J_Y^\delta - \Theta_Y$, the bundle $\pi_*(\xi)$ is semi-stable.*

(b) *There is a point ξ on the theta divisor Θ_Y such that $\pi_*(\xi)$ is semistable.*

c) *The complement of $\mathcal{T}_Y^\delta$ in J_Y^δ is of codimension greater than* 1.

d) *There is a point ξ in Θ_Y such that $\pi_*(\xi)$ is stable and the map π_* has maximal rank at ξ.*

Proof. a) By our assumption, $\Gamma(Y, \xi) = 0$. Hence $\Gamma(X, \pi_* \xi) = 0$. If $\pi_*(\xi)$ were not semistable, it would admit a subbundle F of degree greater than $\mathrm{rk}(F) \cdot (g-1)$. Now the Riemann-Roch theorem implies that $\Gamma(X, F) \neq 0$, which is a contradiction.

b) If $\alpha \in J_X$, then the projection formula implies that $\pi^*(\alpha) \otimes \xi$ has semistable direct image for every ξ as in a). Since Θ is ample and $\pi^*: J_X \to J_Y$ is a finite map, it follows that it intersects the positive dimensional variety $\pi^*(J_X)\xi$.

c) Since the complement of $\mathcal{T}_Y^\delta$ in J_Y^δ is contained properly in the irreducible divisor Θ, it follows that it is of codimension at least 2.

d) We have assumed that the map π_* is dominant so that there exists some $\eta \in J_Y^\delta$ such that $\pi_*(\eta)$ is stable and π_* is of maximal rank at η. Now as in b) above, we can translate η by $\pi^*(\alpha)$ for an $\alpha \in J_X$ so that $\xi = \eta \otimes \pi^* \alpha$ belongs to Θ_Y. It is clear from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{T}_Y^\delta & \longrightarrow & \mathcal{U}(n, d) \\ \downarrow {\scriptstyle \otimes \pi^* \alpha} & & \downarrow {\scriptstyle \otimes \alpha} \\ \mathcal{T}_Y^\delta & \longrightarrow & \mathcal{U}(n, d) \end{array}$$

that $\pi_*(\xi)$ is stable and that the map π_* is of maximal rank at ξ also.

5. 2. Remark. The complement of the set S of points ξ of J_Y^δ such that $\pi_* \xi$ is stable can be shown to have codimension at least $2g-2$, if $n \geqq 3$. In fact suppose that Z is the set of points ξ in J_Y^δ such that $\pi_* \xi$ has a quotient line bundle of degree $\leqq d/n$. We will estimate the codimension of Z in J_Y^δ. By the duality theorem applied to the map π, the existence of a nonzero homomorphism of $\pi_* \xi$ into M, where M is a line bundle on X, is equivalent to the existence of a nonzero homomorphism of ξ into $\pi^* M \otimes \omega_\pi$ where ω_π is the relative dualising sheaf, namely $K_Y \otimes \pi^* K_X^{-1}$. Thus every element of Z is of the form $\pi^* M \otimes \omega_\pi(-D)$ where M is a line bundle on X with $\deg(M) \leqq d/n$, and D is an effective divisor on Y. We have

$$\deg(D) = n \cdot \deg M + \deg \omega_\pi - \delta \leqq d - \delta + 2\{g_Y - 1 - n(g_X - 1)\}.$$

It follows that $\operatorname{codim}(Z) \geqq (n-1)(g-1) \geqq 2(g-1)$ if $n \geqq 3$. If E is a general stable bundle on X, then by [10], Lemma 2. 1, for every subbundle F of E we have

$$\deg(F)/\operatorname{rk}(F) \leqq \{\deg(E) - (\operatorname{rk}(E) - \operatorname{rk}(F))(g-1)\}/\operatorname{rk}(E).$$

Let now L be a general element of $J_Y^{\delta+(2g-3)}$ such that $\pi_* L = E$ satisfies the condition given above. If D is an effective divisor of degree $2g-3$ on Y, the bundle $\pi_*(L(-D))$ is of rank n and degree d and is a subsheaf of $\pi_* L$. If F is a subbundle of $\pi_*(L(-D))$ with $\operatorname{rk}(F) \leqq n-2$, it follows that $\deg(F)/\operatorname{rk}(F) < d/n$.

On the other hand, as D varies, the line bundles $L(-D)$ form a subvariety W_L of J_Y^δ of dimension $(2g-3)$. Changing L amounts to applying a translation on W_L. Since the variety Z defined earlier is of codimension at least $2g-2$, we can choose L such that W_L does not intersect Z. For such a choice of L, W_L is contained in S. Now W_L is cohomologous to $\Theta^k/k!$ with $k = \operatorname{codim}(W_L)$ in J_Y^δ. Hence W_L intersects every subvariety of J_Y^δ of codimension $2g-3$. This proves our assertion.

5. 3. Remark. For $n=2$, the codimension of $(J_Y^\delta - S)$ is at least $g-1$. Coupled with Remark 5. 2, this implies that with the exception of the case $g=r=2$, this codimension is at least 2. Consider the Hitchin map $H: \Omega \rightarrow W$. The restriction of the natural symplectic form ω to the fibre $H^{-1}(a)$ over a general point $a \in W$ is the boundary of a holomorphic 1-form. On the other hand, $H^{-1}(a)$ can be identified with the complement in J_Y^δ of a subvariety of codimension at least 2 and hence this 1-form can be extended to the whole of J_Y^δ. It is therefore closed and so ω restricts to 0 on $H^{-1}(a)$. In other words, the map H defines a completely integrable Hamiltonian system. Moreover the Hamiltonian vector fields given by components (with respect to a basis of W) of H also extend to the Jacobian and thus the system is linearised on the variety J_Y^δ. This gives another proof of the results of Hitchin [4].

5. 4. Remark. *The map* $\pi_*: \mathscr{T}_Y^\delta \rightarrow \mathscr{U}(n, d)$ *is of degree* $2^{3g-3} \cdot 3^{5g-5} \cdots n^{(2n-1)(g-1)}$.

Proof. Indeed, the required degree is equal to the degree of the map $H_E: \Omega_E^1 \rightarrow W$, namely the restriction of H to the cotangent space to the moduli space at a sufficiently general point E. Since this is clearly defined by $3g-3$ polynomials of degree two, $5g-5$ polynomials of degree three, ... the assertion follows from Bezout's theorem.

5. 5. *Proof of Theorem* 2. Let $d = n(g-1)$ and $\delta = n^2(g-1)$. We will compute on $\mathscr{T}_Y^\delta$ the pullback of the theta divisor on $\mathscr{U}(n, d)$. Indeed, the vector bundle on $\mathscr{T}_Y^\delta \times X$ which induces the map $\pi_*: \mathscr{T}_Y^\delta \rightarrow \mathscr{U}(n, d)$ is easy to describe. Consider a Poincaré bundle P on $\mathscr{T}_Y^\delta \times Y$. Then the required bundle is $U = (1 \times \pi)_*(P)$. In order to compute the pullback of $\mathcal{O}(\Theta)$, we need to compute only $(p_1)_*(U)$ and $R^1(p_1)_*(U)$. Since $(1 \times \pi)$ is an affine morphism these are simply the sheaves $(p_1)_*(P)$ and $R^1(p_1)_*(P)$. But the sheaf $(p_1)_*(P)$ is 0 and $\det(R^1(p_1)_*(P))$ is the Riemann theta divisor on J_Y^δ. Hence the pullback of $\mathcal{O}(\Theta)$ by π_* is isomorphic to the restriction of $\mathcal{O}(\Theta)$ to $\mathscr{T}_Y^\delta$. Since the complement of $\mathscr{T}_Y^\delta$ in J_Y^δ is of codimension $\geqq 2$ it follows that π induces an injective map $\Gamma(\mathscr{U}(n, d), \mathcal{O}(\Theta)) \rightarrow \Gamma(J_Y^\delta, \mathcal{O}(\Theta))$. This proves that the dimension of $\Gamma(\mathscr{U}(n, d), \mathcal{O}(\Theta))$ is at most 1. But it is clearly given by an effective divisor so that Theorem 2 is proved.

5. 6. *Proof of Theorem* 3. Take $d=0$ and $\delta = (n^2-n)(g-1)$ and choose a covering $\pi: Y \rightarrow X$ as in Theorem 1. We will use here the notation of 2. 3—2. 6. Firstly the rational map $J_Y^\delta \rightarrow \mathscr{U}(n, 0)$ is dominant.

5. 7. Proposition. *The direct image map induces a dominant rational map $P' \longrightarrow \mathcal{SU}(n)$. Moreover, the complement of the open set $\mathcal{T}_Y^\delta \cap P'$ is of codimension at least 2.*

Proof. If $\pi_*(\alpha)$ is semistable, so is $\pi_*(\pi^* L \otimes \alpha)$ for any line bundle L on X and we can therefore conclude that there is an induced dominant rational map $P' \longrightarrow \mathcal{SU}(n)$. The rest of the assertions is also proved in the same way, using Proposition 5. 1.

Now, from the commutative diagram

$$\begin{array}{ccc} P' \times J_X^{g_X-1} & \longrightarrow & J_Y^{g_Y-1} \\ \downarrow & & \downarrow \\ \mathcal{SU}(n) \times J_X^{g_X-1} & \longrightarrow & \mathcal{U}(n, n(g-1)) \end{array}$$

we conclude that the pullback of the line bundle $\mathcal{O}(\Theta)$ to $\mathcal{SU}(n) \times J_X^{g-1}$ is of the form $p_1^* M \otimes p_2^* \mathcal{O}(n\Theta)$ for some line bundle M. This implies that the pullback of M by the map $P' \longrightarrow \mathcal{SU}(n)$ is τ. But τ is a primitive element in the group of algebraic equivalence classes of divisors in P'. Hence the same is true of M. In other words, the line bundle is the same as the ample generator h of the Picard group, which is isomorphic to $\mathbf{Z}$ [2]. From Remark 4. 9, Proposition 2. 6 and the codimension computation in Proposition 5. 1, we deduce, as in the proof of Theorem 2, that the map $\Gamma(\mathcal{SU}(n), h) \longrightarrow \Gamma(P', \tau)$ is injective. Now the group of n division points of the Jacobian of X acts on $\mathcal{SU}(n)$ as well as on $P(\Gamma(P', \tau))$ and it is easy to see that the above map is compatible with these actions. Now it is easily seen that $\Gamma(\mathcal{SU}(n), h) \neq 0$ so that we conclude that this map is bijective and in particular that $\dim \Gamma(\mathcal{SU}(n), h) = n^g$, since the action of the theta group on $\Gamma(P', \tau)$ is irreducible [6]. Finally, it remains to use the isomorphism of $\Gamma(P', \tau)^*$ with $\Gamma(J_X^{g-1}, \mathcal{O}(n\Theta))$, and identify the induced rational map with $E \mapsto D_E$. In fact, this only amounts to computing the rational map of P' into $P\Gamma(J_X^{g-1}, \mathcal{O}(n\Theta))$. But, in view of Proposition 2. 6, this means that we have to verify that for all p belonging to an open set in P', the divisor $T_p^* \Theta \cap J^{g-1}$ is the same as $D_{\pi_* p}$. The latter is given by $\{\xi \in J^{g-1} : \Gamma(\xi \otimes \pi_* p) \neq 0\}$. The projection formula implies that this consists of those ξ with $\Gamma(\pi^* \xi \otimes p) \neq 0$. This is then the divisor $\{\xi \in J^{g-1} : T_p \pi^* \xi \in \Theta_Y\} = T_p^* \Theta \cap J^{g-1}$, as was to be proved.

5. 8. Remark. On the one hand, we have a natural rational map $\Delta : J^{g-1} \longrightarrow P\Gamma(\mathcal{SU}(n), h)$ given by mapping any $\xi \in J^{g-1}$ to the pullback of the theta divisor on $\mathcal{U}(n, n(g-1))$ by the morphism $E \mapsto E \otimes \xi$ of $\mathcal{SU}(n)$ into $\mathcal{U}(n, n(g-1))$. On the other, the linear system of $n\Theta$ gives a rational map of J^{g-1} into $P\Gamma(J^{g-1}, \mathcal{O}(n\Theta))^*$. These maps fit into a commutative diagram

$$\begin{array}{ccc} J^{g-1} & \xrightarrow{\ \Delta\ } & P\Gamma(\mathcal{SU}(n), h) \\ & \searrow & \downarrow \\ & & P\Gamma(J^{g-1}, \mathcal{O}(n\Theta))^* \end{array}$$

in which the vertical map is given by the duality in Theorem 3.

5. 9. Remark. The linear system of h in $\mathscr{S}\mathscr{U}(n)$ is base point free for $n=2$. On the other hand, there are examples when it does have base points. Indeed, by 5. 8., it is spanned by the image of J^{g-1} under Δ. Therefore a point $E \in \mathscr{S}\mathscr{U}(n)$ is a base point if and only if all the divisors in the image of Δ pass through E. This condition means that for all ξ in J^{g-1}, one has $\Gamma(X, E \otimes \xi) \neq 0$. There are examples of bundles with this property ([11], § 3).

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