A tale of two surfaces

Arnaud Beauville

Abstract.

We point out a link between two surfaces which have appeared recently in the literature: the surface of cuboids and the Schoen surface. Both give rise to a surface with $q = 4$, whose canonical map is 2-to-1 onto a complete intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.

Dedicated to Yujiro Kawamata on his 60th birthday

§1. Introduction

The aim of this note is to point out a link between two surfaces which have appeared recently in the literature: the *surface of cuboids* [ST, vL] and the surface (actually a family of surfaces) discovered by Schoen [S]. We will show that both surfaces give rise to a surface X with $q = 4$, whose canonical map is 2-to-1 onto a complete intersection of 4 quadrics $\Sigma \subset \mathbb{P}^6$ with 48 nodes. In the first case (§2) X is a quotient $(C \times C')/(\mathbb{Z}/2)^2$, where C and C' are genus 5 curves with a free action of $(\mathbb{Z}/2)^2$. In the second case (§3), X is a double étale cover of the Schoen surface.

When the canonical map of a surface X of general type has degree > 1 onto a surface, that surface either has $p_g = 0$ or is itself canonically embedded [B1, Th. 3.1]. Our surfaces X provide one more example of the latter case, which is rather exceptional (see [CPT] for a list of the examples known so far).

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§2. The surface of cuboids and its deformations

In $\mathbb{P}^4$, with coordinates $(x,y;u,v,w)$, we consider the curve C given by

$$(1) \qquad u^2 = a(x,y) \quad , \quad v^2 = b(x,y) \quad , \quad w^2 = c(x,y)$$

where a,b,c are quadratic forms in x,y. We assume that the zeros of a,b,c form a set $Z\subset\mathbb{P}^1$ of 6 distinct points. Then C is a smooth curve of genus 5, canonically embedded. It is preserved by the group $\Gamma_0\cong(\mathbb{Z}/2)^3$ which acts on $\mathbb{P}^4$ by changing the signs of u,v,w. Let $\Gamma\subset\Gamma_0$ be the subgroup (isomorphic to $(\mathbb{Z}/2)^2$) which changes an even number of signs. It acts freely on C, so the quotient curve $D:=C/\Gamma$ has genus 2. The subring of Γ-invariant elements in $\oplus H^0(C,K_C^n)$ is generated by x,y and $z:=uvw$, with the relation $z^2=abc$; thus D is the double cover of $\mathbb{P}^1$ branched along Z.

Let JD_2 be the group of 2-torsion line bundles on D (isomorphic to $(\mathbb{Z}/2)^4$). The Γ-covering $\pi: C\rightarrow D$ corresponds to a subgroup of JD_2 isomorphic to $(\mathbb{Z}/2)^2$, namely the kernel of $\pi^*: JD\rightarrow JC$. Let $p'_a,p''_a;p'_b,p''_b;p'_c,p''_c$ be the Weierstrass points of D lying over the zeros of a,b and c respectively. Since the divisor $\pi^*(p'_a+p''_a)$ is cut out on C by the canonical divisor $u=0$, we have $\pi^*(p'_a-p''_a)\sim 0$, and similarly for b and c; thus $\operatorname{Ker}\pi^* = \{0,p'_a-p''_a,p'_b-p''_b,p'_c-p''_c\}$. This is a *Lagrangian* subgroup of JD_2 for the Weil pairing [M2]; conversely, any Lagrangian subgroup of JD_2 is of that form. Thus the curves C we are considering are exactly the $(\mathbb{Z}/2)^2$-étale covers of a curve D of genus 2 associated to a Lagrangian subgroup of JD_2. In particular they form a 3-dimensional family.

The group $\Gamma_0/\Gamma\cong\mathbb{Z}/2$ acts on $D=C/\Gamma$ through the hyperelliptic involution, so C/Γ_0 is isomorphic to $\mathbb{P}^1$.

Proposition 1. *Let C,C' be two genus 5 curves of type (1), and let X be the quotient of $C\times C'$ by the diagonal action of $\Gamma\cong(\mathbb{Z}/2)^2$.*

1) *X is a minimal surface of general type with $q=4$, $p_g=7$, $K^2=32$.*

2) *The involution i_X of X defined by the action of $\Gamma_0/\Gamma\cong\mathbb{Z}/2$ has 48 fixed points. The canonical map $\varphi_X: X\rightarrow\mathbb{P}^6$ factors through i_X, and induces an isomorphism of X/i_X onto a complete intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.*

Proof : The computation of the numerical invariants of X is straightforward.

Let us denote by $(x',y';u',v',w')$ the coordinates on $C'\subset\mathbb{P}^4$, and by a',b',c' the corresponding quadratic forms. A basis of the space $H^0(X,K_X)=\big(H^0(C,K_C)\otimes H^0(C',K_{C'})\big)^\Gamma$ is given by the elements

$$X=x\otimes x'\quad Y=x\otimes y'\quad Z=y\otimes x'\quad T=y\otimes y'$$

$$U=u\otimes u'\quad V=v\otimes v'\quad W=w\otimes w'\ .$$

They satisfy the relations $XT-YZ=0$ and

$$U^2=A(X,Y,Z,T)\ ,\ V^2=B(X,Y,Z,T)\ ,\ W^2=C(X,Y,Z,T)\ ,$$

where A,B,C are quadratic forms satisfying $A(X,Y,Z,T) = a(x,y)\otimes a(x',y')$, and the analogous relations for B and C.

Let Σ be the surface defined by these four quadratic forms, and let $\varphi : X\to\Sigma$ be the induced map. We have $\varphi\circ i_X=\varphi$, so φ induces a map $\bar\varphi$ from $X/i_X=(C\times C')/\Gamma_0$ into Σ. We consider the commutative diagram

$$\begin{tikzcd}
(C\times C')/\Gamma_0 \arrow[rr, "\bar\varphi"] \arrow[dr, "p"'] & & \Sigma \arrow[dl, "q"] \\
& Q\cong\mathbb{P}^1\times\mathbb{P}^1 &
\end{tikzcd}$$

where $p:(C\times C')/\Gamma_0\to(C/\Gamma_0)\times(C'/\Gamma_0)$ is the quotient map by Γ_0, and q the projection $(X,Y,Z,T;U,V,W)\mapsto(X,Y,Z,T)$. The group $(\mathbb{Z}/2)^3$ acts on Σ by changing the signs of (U,V,W); then $\bar\varphi$ is an equivariant map of $(\mathbb{Z}/2)^3$-coverings, hence an isomorphism.

It remains to show that i_X has 48 fixed points. These fixed points are the images (mod. Γ) of the points of $C\times C'$ fixed by one of the elements of $\Gamma_0\smallsetminus\Gamma$. Such an element changes the sign of one of the coordinates $\ell=u,v$ or w, hence fixes the 64 points (m,m') of $C\times C'$ with $\ell(m)=\ell(m')=0$. This gives $(3\times 64)/4=48$ fixed points in X. Q.E.D.

Example. Let us take for C and C' the curve C_0 defined by

$$u^2=xy\quad,\quad v^2=x^2-y^2\quad,\quad w^2=x^2+y^2\ .$$

The set of zeros of a,b,c is $\{0,\infty,\pm 1,\pm i\}$, so the genus 2 curve D is given by $z^2=x(x^4-1)$.

We get for Σ the following equations :

$$XT=YZ=U^2\ ,\ V^2=X^2-Y^2-Z^2+T^2\ ,\ W^2=X^2+Y^2+Z^2+T^2\ ;$$

or, after the linear change of variables $X=\mathsf{x}+\mathsf{t}$, $T=\mathsf{t}-\mathsf{x}$, $Y=\mathsf{y}+\mathsf{i}\mathsf{z}$, $Z=\mathsf{y}-\mathsf{i}\mathsf{z}$, $U=\mathsf{u}$, $V=2\mathsf{v}$, $W=2\mathsf{w}$:

$$\mathsf{t}^2=\mathsf{x}^2+\mathsf{y}^2+\mathsf{z}^2\quad,\quad \mathsf{u}^2=\mathsf{y}^2+\mathsf{z}^2\quad,\quad \mathsf{v}^2=\mathsf{x}^2+\mathsf{z}^2\qquad \mathsf{w}^2=\mathsf{x}^2+\mathsf{y}^2\ .$$

These are the equations of the *surface of cuboids*, studied in [ST, vL]. It encodes the relations in a cuboid (= rectangular box) between the sides $\mathsf{x},\mathsf{y},\mathsf{z}$, the face diagonals $\mathsf{u},\mathsf{v},\mathsf{w}$, and the space diagonal t. Thus the surface of cuboids belongs to a 6-dimensional family of intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.

The curve C_0 is isomorphic to the modular curve $X(8)$, and the map $C_0\times C_0\to\Sigma$ can be described in terms of theta functions [FS].

Remark 1. In [B3] we show that the surface $X=(C_0\times C_0)/\Gamma$ has *maximum Picard number* $\rho=h^{1,1}$, by analyzing the action of Γ on JC_0; it follows that the desingularization $\tilde{\Sigma}$ of the surface of cuboids Σ has the same property – a result obtained in [ST] via a computer calculation.

Remark 2. Our surfaces X fit into a tower of $(\mathbb{Z}/2)^2$-étale coverings:

$$C\times C'\longrightarrow X\xrightarrow{\ r\ }D\times D'\ .$$

The abelian covering r is the pull back of a $(\mathbb{Z}/2)^2$-étale covering of $JD\times JD'$:

$$\begin{array}{ccc} X & \xhookrightarrow{\ \alpha\ } & A\\ \Big\downarrow{\scriptstyle r} & & \Big\downarrow\\ D\times D' & \hookrightarrow & JD\times JD'\ . \end{array}$$

The abelian variety A is the Albanese variety of X, and α is the Albanese map. Since the quotient X/i_X is regular, i_X acts as (-1) on the space $H^0(X,\Omega^1_X)$; therefore if we choose α so that it maps a fixed point of i_X to 0, i_X is induced by (-1_A).

§3. The Schoen surface

The Schoen surfaces S have been defined in [S], and studied in [CMR]. A Schoen surface S is contained in its Albanese variety A; it has the following properties:

a) $K_S^2 = 16$, $p_g = 5$, $q = 4$ (hence $\chi(\mathcal{O}_S) = 2$);

b) The canonical map $\varphi_S : S \to \mathbb{P}^4$ factors through an involution i_S with 40 fixed points, and induces an isomorphism of S/i_S onto the complete intersection of a quadric and a quartic in $\mathbb{P}^4$ with 40 nodes [CMR].

Since S/i_S is a regular surface, i_S acts as (-1) on the space $H^0(S, \Omega_S^1)$. Therefore if we choose the Albanese embedding $S \hookrightarrow A$ so that it maps a fixed point of i_S to 0, i_S *is induced by the involution* (-1_A).

Let ℓ be a line bundle of order 2 on A; we denote by $\pi : B \to A$ the corresponding étale double cover, and put $X := \pi^{-1}(S)$. The restriction of ℓ to S, which we will still denote by ℓ, is nontrivial (because the restriction map $\mathrm{Pic}^{\mathrm{o}}(A) \to \mathrm{Pic}^{\mathrm{o}}(S)$ is an isomorphism), hence X is connected.

Proposition 2. X *is a minimal surface of general type with* $q = 4$, $p_g = 7$, $K_X^2 = 32$.

Proof : The formulas $K_X^2 = 32$ and $\chi(\mathcal{O}_X) = 4$ are immediate; we must prove $q(X) = 4$, that is, $H^1(S, \ell) = 0$.

By construction [S] a Schoen surface fits into a flat family over the unit disk Δ:

$$\begin{tikzcd} \mathcal{S} \arrow[r, hook] \arrow[dr] & \mathcal{A} \arrow[d] \\ & \Delta \end{tikzcd}$$

where:

• $\mathcal{A}/\Delta$ is a smooth family of abelian varieties;

• at a point $z \neq 0$ of Δ, $\mathcal{S}_z$ is a Schoen surface, and $\mathcal{S}_z \hookrightarrow \mathcal{A}_z$ is the Albanese embedding;

• $\mathcal{A}_0 = JC \times JC$ for a genus 2 curve C; $\mathcal{S}_0$ is the union of JC embedded diagonally in $JC \times JC$, and of $C \times C \subset JC \times JC$ (we choose an Abel-Jacobi embedding $C \subset JC$). These two components intersect transversally along the diagonal $C \subset C \times C$.

The line bundle ℓ extends to a line bundle $\mathcal{L}$ of order 2 on $\mathcal{A}$. Let ℓ_0 be the restriction of $\mathcal{L}$ to $\mathcal{S}_0$; we want to compute $H^1(\mathcal{S}_0, \ell_0)$. We have an exact sequence

$$(2) \qquad 0 \to \ell_0 \longrightarrow \ell_{0\,|JC} \oplus \ell_{0\,|C\times C} \longrightarrow \ell_{0\,|C} \to 0 \ .$$

The line bundle $\mathcal{L}_0$ on $JC \times JC$ can be written $\alpha \boxtimes \beta$, where α and β are 2-torsion line bundles on JC, not both trivial; we use the same letters to denote their restriction to C. The cohomology exact sequence associated to (2) gives

$$\begin{gathered} H^0(JC, \alpha\otimes\beta) \oplus H^0(C\times C, \alpha\boxtimes\beta) \longrightarrow H^0(C, \alpha\otimes\beta) \longrightarrow H^1(\mathcal{S}_0, \ell_0) \xrightarrow{\ u\ } \\ H^1(JC, \alpha \otimes \beta) \oplus H^1(C \times C, \alpha \boxtimes \beta) \longrightarrow H^1(C, \alpha \otimes \beta) \ . \end{gathered}$$

The restriction map $H^0(JC, \alpha \otimes \beta) \to H^0(C, \alpha \otimes \beta)$ is surjective, so u is injective. If α and β are nontrivial, $H^1(C \times C, \alpha \boxtimes \beta)$ is zero, and the restriction map $H^1(JC, \alpha \otimes \beta) \to H^1(C, \alpha \otimes \beta)$ is injective, so $H^1(\mathcal{S}_0, \ell_0) = 0$. If, say, β is trivial, $H^1(JC, \alpha)$ is zero and the map $H^1(C \times C, \mathrm{pr}_1^*\alpha) \to H^1(C, \alpha)$ is bijective, hence $H^1(\mathcal{S}_0, \ell_0) = 0$ again.

By semi-continuity this implies $H^1(\mathcal{S}_z, \mathcal{L}_z) = 0$ for z general in Δ, or equivalently $q(\widetilde{\mathcal{S}}_z) = q(\mathcal{S}_z) = 4$, where $\widetilde{\mathcal{S}} \to \mathcal{S}$ is the étale double covering defined by $\mathcal{L}$. But q is a topological invariant, so this holds for all $z \neq 0$ in Δ, hence $H^1(S, \ell) = 0$. Q.E.D.

The surface X has a natural action of $(\mathbb{Z}/2)^2$, given by the involution i_X induced by (-1_B) and the involution τ associated to the double covering $X \to S$, which is induced by a translation of B. We want to determine how these involutions act on $H^0(X, K_X)$. The decomposition of $H^0(X, K_X)$ into eigenspaces for τ is

$$H^0(X, K_X) \cong H^0(S, K_S) \oplus H^0(S, K_S \otimes \ell) \ .$$

By property b) above, i_S acts trivially on $H^0(S, K_S)$. It remains to study how it acts on $H^0(S, K_S \otimes \ell)$, or equivalently on $H^2(S, \ell)$. To define this action we choose the isomorphism $u : (-1_A)^*\ell \xrightarrow{\ \sim\ } \ell$ over A such that $u(0) = 1$, and we consider the involutions

$$H^p(i_S, u) : H^p(S, \ell) \xrightarrow{\ i_S^*\ } H^p(S, i_S^*\ell) \xrightarrow{\ u_{|S}\ } H^p(S, \ell) \ .$$

Proposition 3. *There exist line bundles ℓ of order 2 on A for which i_S acts trivially on $H^2(S,\ell)$. In that case i_X has 48 fixed points.*

Proof : We will denote by A_2 and $\hat{A}_2$ the 2-torsion subgroups of A and its dual abelian variety $\hat{A}$, and similarly for B. The fixed point set of i_S is $A_2 \cap S$, and that of i_X is $B_2 \cap X$.

We apply the holomorphic Lefschetz formula to the automorphism i_S of S and the i_S-linearization $u_{|S} : i_S^*\ell \to \ell$:

$$\sum_p (-1)^p \operatorname{Tr} H^p(i_S,u) = \frac{1}{4}\sum_{a\in A_2\cap S} u(a)\ .$$

(At a point a of A_2, $u(a) : \ell_a \to \ell_a$ is the multiplication by a scalar, which we still denote $u(a)$.)

Let $a \in A_2$. By [M1], property iv) p. 304, we have $u(a) = (-1)^{\langle a,\ell\rangle}$, where $\langle\ ,\ \rangle : A_2 \times \hat{A}_2 \to \mathbb{Z}/2$ is the canonical pairing. Thus the right hand side of the Lefschetz formula is $\frac{1}{4}(f_0 - f_1)$, where f_i, for $i \in \mathbb{Z}/2$, is the number of points $a \in A_2 \cap S$ with $\langle a,\ell\rangle = i$.

We have $H^0(S,\ell) = H^1(S,\ell) = 0$ (Proposition 2), hence $\dim H^2(S,\ell) = \chi(\mathcal{O}_S) = 2$. Thus the left hand side is $\operatorname{Tr} H^2(i_S,u) \in \{2,0,-2\}$. Since $f_0 + f_1 = 40$ this gives $f_i \in \{16,20,24\}$, and we want to find ℓ such that $H^2(i_S,u) = \mathrm{Id}$, that is $f_0 = 24$.

Put $F := A_2 \cap S$. Consider the homomorphism $j : \hat{A}_2 \to (\mathbb{Z}/2)^F$ given by $j(\ell) = (\langle a,\ell\rangle)_{a\in F}$. For $\ell \neq 0$, the weight of the element $j(\ell)$ of $(\mathbb{Z}/2)^F$ (that is, the number of its nonzero coordinates) is f_1, which belongs to $\{16,20,24\}$. Therefore j is injective; its image is a 8-dimensional vector subspace of $(\mathbb{Z}/2)^F$, that is, a linear code, such that the weight of any nonzero vector belongs to $\{16,20,24\}$. A simple linear algebra lemma [B2, lemme 1] shows that a code in $(\mathbb{Z}/2)^{40}$ of dimension ≥ 7 contains elements of weight < 20; thus there exist elements ℓ in $\hat{A}_2$ with $f_1 = 16$, hence $f_0 = 24$.

It remains to compute the number of fixed points of i_X in that case. The fixed locus of i_X is $B_2 \cap X = \pi^{-1}(\pi(B_2)\cap S)$. Dualizing the exact sequence of $(\mathbb{Z}/2)$-vector spaces

$$0 \to (\mathbb{Z}/2)\ell \to \hat{A}_2 \xrightarrow{\hat{\pi}} \hat{B}_2$$

and using the canonical pairings we get an exact sequence

$$B_2 \xrightarrow{\pi} A_2 \xrightarrow{\langle\ ,\ell\rangle} \mathbb{Z}/2 \to 0\ .$$

Thus the points a of $A_2 \cap S$ which belong to $\pi(B_2)$ are those with $\langle a, \ell \rangle = 0$. There are $f_0 = 24$ such points, hence 48 fixed points of i_X. Q.E.D.

Remark 3. There exist line bundles ℓ in $\hat{A}_2$ with $f_0 = f_1 = 20$. Indeed otherwise $j(\hat{A}_2)$ would be an 8-dimensional linear code in $(\mathbb{Z}/2)^{40}$ with weights 16 and 24, projective in the sense of [CK]; this is impossible since equation (3.10) of [CK] does not hold. Thus in the next Proposition the hypothesis on ℓ is necessary.

Proposition 4. *Choose ℓ as in Proposition 3. Then the canonical map $\varphi_X : X \to \mathbb{P}^6$ factors through i_X, and induces an isomorphism of X/i_X onto a complete intersection of 4 quadrics in $\mathbb{P}^6$ with 48 nodes.*

Proof : Since i_X acts trivially on $H^0(X, K_X)$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\varphi_X"] \arrow[d, "\pi"] & \Sigma \arrow[r, hook] \arrow[d, "p_\Sigma"] & \mathbb{P}^6 \arrow[d, dashed, "p"] \\ S \arrow[r, "\varphi_S"] & \Xi \arrow[r, hook] & \mathbb{P}^4 \end{tikzcd}$$

where φ_X and φ_S are the canonical maps, Σ and Ξ their images, p the projection corresponding to the injection $H^0(S, K_S) \to H^0(X, K_X)$, p_Σ its restriction to Σ.

The map $\varphi_S \circ \pi : X \to \Xi$ gives the quotient of X by the action of $(\mathbb{Z}/2)^2$. Since τ acts non-trivially on $H^0(X, K_X)$, φ_X identifies Σ with the quotient X/i_X. Thus all the maps in the left hand square of the above diagram are double coverings, étale outside finitely many points. In particular, since $K_X^2 = 32$, we have $\deg \Sigma = 16$.

We choose bases $(x_0, \ldots, x_4)$ and (u, v) of the $(+1)$ and (-1)-eigenspaces in $H^0(X, K_X)$ with respect to τ. The elements u^2, uv, v^2 of $H^0(X, K_X^{\otimes 2})$ are invariant under τ and i_X, therefore they are pull-back of i_S-invariant forms in $H^0(S, K_S^{\otimes 2})$. Such a form comes from an element of $H^0(\Xi, \mathcal{O}_\Xi(2))$, hence from an element of $H^0(\mathbb{P}^4, \mathcal{O}_\mathbb{P}(2))$. Thus we have

$$u^2 = a(x) \qquad uv = b(x) \qquad v^2 = c(x)$$

where a,b,c are quadratic forms in $x_0,\ldots,x_4$. Moreover the irreducible quadric Q containing Ξ is defined by a quadratic form $q(x)$ which vanishes on Σ.

Thus Σ is contained in the subvariety V of $\mathbb{P}^6$ defined by these 4 quadratic forms. If V is a surface, it has degree 16 and therefore is equal to Σ. Thus it suffices to prove that the morphism $p_V : V \to Q$ induced by the projection p is not surjective.

Assume that p_V is surjective; it has degree 2, and we have a cartesian diagram

$$\begin{tikzcd} \Sigma \arrow[r, hook] \arrow[d, "p_\Sigma"] & V \arrow[d, "p_V"] \\ \Xi \arrow[r, hook] & Q\,. \end{tikzcd}$$

The variety V is irreducible: otherwise Σ is contained in one of its component, which maps birationally to Q, and p_Σ has degree 1, a contradiction. Since $Q \smallsetminus \mathrm{Sing}(Q)$ is simply connected, p_V is branched along a surface $R \subset Q$. Since Ξ is an ample divisor in Q (cut out by a quartic equation), it meets R along a curve, and p_Σ is branched along that curve, a contradiction. Q.E.D.

Remark 4. It follows that $\Xi = p(\Sigma)$ is defined by the equations $q(x) = b(x)^2 - a(x)c(x) = 0$. The 40 nodes of Ξ break into two sets: the 16 points in $\mathbb{P}^4$ defined by $a(x) = b(x) = c(x) = q(x) = 0$ are the images by p_Σ of smooth points of Σ fixed by the involution induced by τ; p_Σ is étale over the other 24 nodes of Ξ, giving rise to the 48 nodes of Σ.

Remark 5. The two families of surfaces X that we have constructed in §2 and §3 are different; in fact, a surface X_1 of the first family is not even homeomorphic to a surface X_2 of the second one. Indeed X_1 admits an irrational genus 2 pencil $X \to D$, and this is a topological property [C]. But for a general member X_2 of the second family, the Albanese variety of the corresponding Schoen surface is simple [S], so its double cover $\mathrm{Alb}(X_2)$ is also simple; therefore X_2 cannot have an irrational pencil of genus 2.

It follows that the corresponding surfaces Σ belong to two different connected components of the moduli space of complete intersections of 4 quadrics in $\mathbb{P}^6$ with an even set of 48 nodes.

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Laboratoire J.-A. Dieudonné
UMR 7351 du CNRS
Université de Nice
Parc Valrose
F-06108 Nice cedex 2, France
E-mail address: `arnaud.beauville@unice.fr`