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A remark on the generalized Franchetta conjecture for K3 surfaces

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Abstract

A family of K3 surfaces $\mathscr{X} \to B$ has the *Franchetta property* if the Chow group of 0-cycles on the generic fiber is cyclic. The generalized Franchetta conjecture proposed by O'Grady asserts that the universal family $\mathscr{X}_g \to \mathscr{F}_g$ of polarized K3 of degree $2g-2$ has the Franchetta property. While this is known only for small g thanks to [7], we prove that for all g there is a hypersurface in $\mathscr{F}_g$ such that the corresponding family has the Franchetta property.

1 Introduction

In 1954, Franchetta stated that the only line bundles defined on the generic curve of genus $g \geq 2$ are the powers of the canonical bundle [3]. Since the proof was insufficient, the result became known as the *Franchetta conjecture*; it was proved by Harer in [5], see also [1].

In [6], O'Grady proposed an analogue of this result for 0-cycles on K3 surfaces. Recall that the Chow group $\mathrm{CH}^2(X)$ of 0-cycles on a K3 surface X contains a canonical class $\mathfrak{o}_X$, the class of any point lying on some rational curve in X; for any divisors D and D' on X, the product $D \cdot D'$ in $\mathrm{CH}^2(X)$ is a multiple of $\mathfrak{o}_X$ [2]. Let $p : \mathscr{X} \to B$ be a map of smooth varieties whose general fiber is a K3 surface. We say that the family $\mathscr{X} \to B$ has the Franchetta property if for every smooth fiber X of p the image of the restriction map $\mathrm{CH}^2(\mathscr{X}) \to \mathrm{CH}^2(X)$ is contained in $\mathbb{Z} \cdot \mathfrak{o}_X$. Equivalently, the Chow group $\mathrm{CH}^2(\mathscr{X}_\eta)$ of the generic fiber is cyclic.

For $g \geq 2$, let $\mathscr{X}_g \to \mathscr{F}_g$ be the universal family of polarized K3 surfaces of degree $2g-2$. The generalized Franchetta conjecture of O'Grady is the assertion that this family has the Franchetta property. [1] It is proved for $g \leq 10$ and some higher values of g in [7]; the general case seems far out of reach. We prove in this note a much weaker (and much easier) statement:

[1] Here one can view $\mathscr{F}_g$ as a stack, or restrict to the open subset parametrizing K3 with trivial automorphism group.

Pour Olivier – 40 ans déjà...

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Theorem *There exists for every g a hypersurface in $\mathscr{F}_g$ such that the corresponding family satisfies the Franchetta property.*

The key point of the proof is the construction, for each g, of a 18-dimensional family of polarized K3 surfaces of degree $2g-2$, which can be realized as complete intersections in $\mathbb{P}^1 \times \mathbb{P}^n$ for $n = 2, 3$ or 4 (Sect. 3). Then a simple argument, already used in [7], shows that these families have the Franchetta property (Sect. 2). Here the crucial property of our families is that they are parameterized by a linear space (in particular, they give unirational hypersurfaces in $\mathscr{F}_g$ for every g); thus there is no chance of extending the method to the whole moduli space $\mathscr{F}_g$, which is of general type for g large enough [4].

2 The method

We use the method of [7], based on the following result. Let P be a smooth complex projective variety, E a vector bundle on P, globally generated by a subspace V of $H^0(E)$. Consider the subvariety $\mathscr{X} \subset \mathbb{P}(V) \times P$ of pairs $(\mathbb{C}s, x)$ with $s(x) = 0$[2]; let p, q be the projections onto $\mathbb{P}(V)$ and P. For $s \in V \smallsetminus \{0\}$, the fiber $p^{-1}(\mathbb{C}s)$ is the zero locus of s in P; for $x \in P$, the fiber $q^{-1}(x)$ is the space of lines $\mathbb{C}s \subset V$ such that $s(x) = 0$. Since V generates E, the projection $q : \mathscr{X} \to P$ is a projective bundle (in particular, $\mathscr{X}$ is smooth).

Proposition *For any smooth fiber X of p, the image of the restriction map* $\mathrm{CH}(\mathscr{X}) \to \mathrm{CH}(X)$ *is equal to the image of* $\mathrm{CH}(P)$.

Proof Let $h \in \mathrm{CH}^1(\mathbb{P}(V))$ be the class of a hyperplane section. The class $p^*h \in CH^1(\mathscr{X})$ induces the hyperplane class on a general fiber of q; since q is a projective bundle, it follows that $\mathrm{CH}(\mathscr{X})$ is generated by $q^*\,\mathrm{CH}(P)$ and the powers of p^*h. But p^*h vanishes on the fibers of p, hence the result. □

Corollary *Assume that the smooth fibers of p are K3 surfaces, and that the multiplication map*
$m_P : \mathrm{Sym}^2 \mathrm{CH}^1(P) \to \mathrm{CH}^2(P)$ *is surjective. Then the family* $\mathscr{X} \to \mathbb{P}(V)$ *has the Franchetta property.*

Proof Let X be a smooth fiber of p. The commutative diagram

$$\begin{array}{ccc} \mathrm{Sym}^2 \mathrm{CH}^1(P) & \longrightarrow & \mathrm{Sym}^2 \mathrm{CH}^1(X) \\ \downarrow{\scriptstyle m_P} & & \downarrow{\scriptstyle m_X} \\ \mathrm{CH}^2(P) & \longrightarrow & \mathrm{CH}^2(X) \end{array}$$

shows that the image of $\mathrm{CH}^2(P) \to \mathrm{CH}^2(X)$ is contained in the image of m_X, hence in $\mathbb{Z} \cdot \mathfrak{o}_X$. □

3 Proof of the theorem

Since $\dim \mathscr{F}_g = 19$, we must construct for every g a family of polarized K3 surfaces (S, L) with $(L)^2 = 2g-2$ satisfying the Franchetta property, and depending on 18 moduli (this

[2] Here $\mathbb{P}(V)$ is the space of lines in V.

implies our Theorem, see [7, §2, Remark (i)]). We will need three different constructions in order to cover every $g \geq 8$ (the small genus cases follow from [7]). We will apply the Corollary with $P = \mathbb{P}^1 \times \mathbb{P}^n$ for $n = 2, 3$ or 4 — note that the surjectivity of m_P is trivially satisfied. For $i, j \in \mathbb{N}$, we put $\mathscr{O}_P(i, j) := \mathscr{O}_{\mathbb{P}^1}(i) \boxtimes \mathscr{O}_{\mathbb{P}^n}(j)$; the vector bundle E will be a direct sum of $n-1$ line bundles of this type, so S is a complete intersection of $n-1$ hypersurfaces in P. In order for S to be a K3 surface we must have $\det(E) = K_P^{-1} = \mathscr{O}_P(2, n+1)$. We will always take $V = H^0(E)$.

The polarization L on our K3 surface S will be the restriction of the very ample line bundle $\mathscr{O}_P(a, 1)$ on P, for $a \geq 1$. Let $p, h \in \mathrm{CH}^1(P)$ be the pull back of the class of a point in $\mathbb{P}^1$ and of the hyperplane class in $\mathbb{P}^n$. Then

$$2g - 2 = (L)^2 = (ap + h)^2 \cdot [S] = \big(2a(p \cdot h) + h^2\big) \cdot [S]\,.$$

Case I: $n = 2$, $E = \mathscr{O}_P(2, 3)$, hence

$$2g - 2 = \big(2a(p \cdot h) + h^2\big) \cdot (2p + 3h) = 2(3a + 1)\,.$$

Case II: $n = 3$, $E = \mathscr{O}_P(1, 1) \oplus \mathscr{O}_P(1, 3)$, hence

$$2g - 2 = \big(2a(p \cdot h) + h^2\big) \cdot (p + h)(p + 3h) = 2(3a + 2)\,.$$

Case III: $n = 4$, $E = \mathscr{O}_P(0, 3) \oplus \mathscr{O}_P(1, 1) \oplus \mathscr{O}_P(1, 1)$, hence

$$2g - 2 = \big(2a(p \cdot h) + h^2\big) \cdot 3h(p + h)^2 = 2(3a + 3)\,.$$

Thus we get all values of $g \geq 8$.

It remains to prove that the three families just constructed depend on 18 moduli. The exact sequence

$$0 \to T_S \to T_{P|S} \to N_{S/P} \to 0$$

gives rise to an exact sequence

$$0 \to H^0(T_{P|S}) \to H^0(N_{S/P}) \xrightarrow{\partial} H^1(S, T_S)\,;$$

the image of ∂ describes, inside the space of first order deformations of S, those which come from our family. Thus we want to prove $\dim \operatorname{Im} \partial = 18$, or equivalently $h^0(N_{S/P}) - h^0(T_{P|S}) = 18$.

We have $T_P = \mathrm{pr}_1^* T_{\mathbb{P}^1} \oplus \mathrm{pr}_2^* T_{\mathbb{P}^n}$; from the Euler exact sequence we get $h^0((\mathrm{pr}_1^* T_{\mathbb{P}^1})_{|S}) = h^0(\mathrm{pr}_1^* T_{\mathbb{P}^1})$, and similarly for $\mathrm{pr}_2^* T_{\mathbb{P}^n}$. Thus $h^0(T_{P|S}) = h^0(T_{\mathbb{P}^1}) + h^0(T_{\mathbb{P}^n}) = 3 + n(n+2)$.

Let us denote by d_S the restriction to S of a class $d \in \mathrm{Pic}(P)$. Using $d_S \cdot d'_S = d \cdot d' \cdot [S]$, we find

$$p_S^2 = 0\,, \quad p_S.h_S = 3\,, \quad h_S^2 = 2n - 2\,.$$

By Riemann–Roch, we have $h^0(\mathscr{O}_S(i, j)) = 2 + \dfrac{1}{2}(ip_S + jh_S)^2 = 2 + 3ij + j^2(n-1)$.

Case I: $h^0(N_{S/P}) = h^0(\mathscr{O}_S(2, 3)) = 29$, $h^0(T_{P|S}) = 11$.

Case II: $h^0(N_{S/P}) = h^0(\mathscr{O}_S(1, 1)) + h^0(\mathscr{O}_S(1, 3)) = 9 + 29 = 36$, $h^0(T_{P|S}) = 18$.

Case III: $h^0(N_{S/P}) = 2h^0(\mathscr{O}_S(1, 1)) + h^0(\mathscr{O}_S(0, 3)) = 2 \cdot 8 + 29 = 45$, $h^0(T_{P|S}) = 27$.

In each case we find $h^0(N_{S/P}) - h^0(T_{P|S}) = 18$ as required. □

Remarks.– 1) In fact, for S very general in each family, $\mathrm{Pic}(S)$ is generated by p_S and h_S: this follows from the Noether–Lefschetz theory, see [8, Thm. 3.33]. Therefore $\mathrm{Pic}(S)$ is the rank 2 lattice with intersection matrix $\begin{pmatrix} 0 & 3 \\ 3 & 2n-2 \end{pmatrix}$.

2) Our 3 families admit actually a simple geometric description. In what follows we consider a general surface S in each family. We fix homogeneous coordinates U, V on $\mathbb{P}^1$.

Case I: S is given by an equation $U^2A + 2UVB + V^2C = 0$ in $P = \mathbb{P}^1 \times \mathbb{P}^2$, with A, B, C cubic forms on $\mathbb{P}^2$. Projecting onto $\mathbb{P}^2$ gives a double covering $S \to \mathbb{P}^2$ branched along the sextic plane curve $\Gamma : B^2 - AC = 0$. Let α and γ be the divisors on Γ defined by $A = B = 0$ and $C = B = 0$; then 2α, 2γ and $\alpha + \gamma$ are induced by the cubic curves $A = 0$, $C = 0$ and $B = 0$ respectively, hence belong to the canonical system $|K_\Gamma|$. It follows that α and γ are linearly equivalent theta-characteristics, hence belong to a half-canonical g^1_9, that is, a vanishing thetanull on Γ. Conversely, it is easy to see that a smooth plane sextic with a vanishing thetanull has an equation of the above form. We conclude that *the surfaces in Case I are the double covers of* $\mathbb{P}^2$ *branched along a sextic curve with a vanishing thetanull.*

Case II: The equations of S in $P = \mathbb{P}^1 \times \mathbb{P}^3$ have the form $UL + VM = UA + VB = 0$, where L, M; A, B are forms of degree 1 and 3 on $\mathbb{P}^3$. The projection $S \to \mathbb{P}^3$ is an isomorphism onto the quartic surface $LB - MA = 0$; this is the equation of a general quartic containing a line. Thus *the surfaces in Case II are the quartic surfaces containing a line*.

Case III: The equations of S in $P = \mathbb{P}^1 \times \mathbb{P}^4$ are of the form $UA + VB = UC + VD = F = 0$, where A, B, C, D; F are forms of degree 1 and 3 on $\mathbb{P}^3$. The projection $S \to \mathbb{P}^4$ is an isomorphism onto the surface $AD - BC = F = 0$, that is, the intersection of a quadric cone (with one singular point) and a cubic in $\mathbb{P}^4$. Thus *the surfaces in Case III are the complete intersections of a quadric cone and a cubic in* $\mathbb{P}^4$.

Note that one sees easily from this description that each family depends indeed on 18 moduli.

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