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A Non-Hyperelliptic Curve with Torsion Ceresa Cycle Modulo Algebraic Equivalence

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We exhibit a non-hyperelliptic curve C of genus 3 such that the class of the Ceresa cycle $[C]-[C^-]$ in JC modulo algebraic equivalence is torsion.

1 Introduction

Let C be a complex curve of genus $g \geq 2$ and p be a point of C. We embed C into its Jacobian J by the Abel–Jacobi map $x \mapsto [x]-[p]$; we denote by C^- the image of C under the involution $(-1)_J : a \mapsto -a$ of J. The *Ceresa class* is the class $\mathfrak{z}(C) := [C]-[C^-]$ in the group $A_1(J)$ of 1-cycles on J modulo algebraic equivalence (it is independent of the choice of p). Since $(-1)_J$ acts trivially on $H^p(J,\mathbb{Z})$ for p even, $\mathfrak{z}(C)$ belongs to the *Griffiths group* $G(J)$, the kernel of the cycle class map $A_1(J) \to H^{2g-2}(J,\mathbb{Z})$.

Ceresa classes have played a prominent role in the study of Griffiths groups, especially in the development of techniques for showing that a given element is non-zero [9, 11, 17]. In addition they played an important role in showing that $G(J)$ can have infinite rank [20]. As the conjectures of Bloch and Beilinson were developed and are studied $\mathfrak{z}(C)$ appears repeatedly [5, 8], [23, §1.5], always as an element of infinite order.

When C is hyperelliptic, $\mathfrak{z}(C)=0$; in fact, $C-C^-$ is zero as a cycle when p is a Weierstrass point. In this note we will exhibit what we believe to be the 1st example

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of a non-hyperelliptic curve C with $\mathfrak{z}(C) = 0$ in $A_1(J) \otimes \mathbb{Q}$. The curve C has genus 3, and admits an automorphism σ of order 9, such that the quotient variety $J/\langle\sigma\rangle$ is uniruled. This implies that the Griffiths group of a resolution of $J/\langle\sigma\rangle$ is torsion; going back to J gives the result.

2 Main Result

Theorem. Let $C \subset \mathbb{P}^2$ be the genus 3 curve defined by $X^4 + XZ^3 + Y^3Z = 0$. Then $\mathfrak{z}(C) = 0$ in $A_1(J) \otimes \mathbb{Q}$.

Proof. Let ζ be a primitive 9th root of unity. We consider the automorphism σ of C defined by $\sigma(X, Y, Z) = (X, \zeta^2 Y, \zeta^3 Z)$. We use the fixed-point $p = (0, 0, 1)$ to embed C in its Jacobian J, so that the action of σ on J preserves C and C^-. We denote by V the quotient variety $J/\langle\sigma\rangle$ and by $\pi : J \to V$ the quotient map. Let $F \subset J$ be the subset of elements with nontrivial stabilizer; the singular locus $\operatorname{Sing} V$ of V is $\pi(F)$. We put $J^{\mathrm{o}} := J \smallsetminus F$ and $V^{\mathrm{o}} := V \smallsetminus \operatorname{Sing} V$.

Lemma 1. $\operatorname{Sing} V$ is finite; the points $\pi(x)$ for $x \in \operatorname{Ker}(1_J - \sigma)$ are non-canonical singularities.

Proof. The space $T_0(J)$ is canonically identified with $H^0(C, K_C)^*$. The elements of $H^0(C, K_C)$ are of the form $L\dfrac{XdZ - ZdX}{Y^2Z}$, with $L \in H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1))$ [7, §9.3, Corollary of Theorem 1]. It follows that the eigenvalues of σ on $H^0(C, K_C)$ are $\zeta^5, \zeta^7, \zeta^8$, and those on $T_0(J) = H^0(C, K_C)^*$ are ζ, ζ^2, ζ^4. Therefore, $\operatorname{Ker}(1_J - \sigma^d)$ is finite for $0 < d < 9$, so F is finite. Since $1 + 2 + 4 < 9$, Reid's criterion [21, Theorem 3.1] implies that the singular points $\pi(x)$ for $x \in \operatorname{Ker}(1_J - \sigma)$ are not canonical. ■

Lemma 2. The variety V is uniruled.

Proof. Let $\rho : \tilde{V} \to V$ be a resolution of singularities; it suffices to prove that $\tilde{V}$ has Kodaira dimension $-\infty$ [18]. Suppose this is not the case: there exist an integer $r \geq 1$ and a nonzero section $\tilde{\omega}$ of $K_{\tilde{V}}^r$. By restriction to $\rho^{-1}(V^{\mathrm{o}}) \cong V^{\mathrm{o}}$, we get a section ω of $K_{V^{\mathrm{o}}}^r$, whose pull back under π is a nonzero section of $K_{J^{\mathrm{o}}}^r$; therefore, ω is a generator of $K_{V^{\mathrm{o}}}^r$, hence extends to a generator of the reflexive sheaf $K_V^{[r]}$ (with the notation of [21]). By construction this generator remains regular on $\tilde{V}$, which means that the singularities of V are canonical [21, Proposition 1.2], contradicting Lemma 1. ■

Lemma 3. Let X be a uniruled smooth projective threefold. The Griffiths group $G(X)$ is torsion.

Proof. There exists a smooth projective surface S and a dominant rational map $S \times \mathbb{P}^1 \dashrightarrow X$. After blowing up some points and some smooth curves in $S \times \mathbb{P}^1$, we get a smooth projective threefold W birational to $S \times \mathbb{P}^1$ and a generically finite morphism $f : W \to X$. Since the Griffiths group is a stably birational invariant (see [22, Proposition 2.30]), we have $G(W) = G(S) = 0$. For $z \in G(X)$, we have $(\deg f)z = f_*f^*(z) = 0$, hence $G(X)$ is annihilated by $\deg f$. ■

Remark. One can actually deduce from [6, Theorem 1 (ii)] that $G(X) = 0$, but we will not need this fact.

Proof of the Theorem. We can choose the resolution $\rho : \tilde{V} \to V$ so that $E := \rho^{-1}(\mathrm{SingV})$ is a normal crossing divisor, whose irreducible components are smooth and *rational* [13, Corollary of Theorem 1].

Let $\bar{C}$ and $\bar{C}^-$ be the images in V of C and C^-, and let $\tilde{C}$ and $\tilde{C}^-$ be their proper transforms in $\tilde{V}$. We have $[\bar{C}] - [\bar{C}^-] = \frac{1}{9}\pi_*([C] - [C^-]) = 0$ in $H^4(V^o, \mathbb{Q})$. Now we have an exact sequence [12, Corollaire 8.2.8]

$$H^2(\tilde{E}, \mathbb{Q}) \xrightarrow{i_*} H^4(\tilde{V}, \mathbb{Q}) \to H^4(V^o, \mathbb{Q}) \,,$$

where $\tilde{E}$ is the normalization of E and i the composition $\tilde{E} \to E \hookrightarrow \tilde{V}$. Therefore, we have $[\tilde{C}] - [\tilde{C}^-] = i_*z$ in $H^4(\tilde{V}, \mathbb{Q})$ for some class $z \in H^2(\tilde{E}, \mathbb{Q})$. Since the components of $\tilde{E}$ are rational, z is the class of an element $\mathfrak{z}$ of $A_1(\tilde{E}) \otimes \mathbb{Q}$. Then $[\tilde{C}] - [\tilde{C}^-] - i_*\mathfrak{z} \in A_1(\tilde{V}) \otimes \mathbb{Q}$ is homologous to zero, hence equal to zero by Lemma 3. Restricting to $\tilde{V} \smallsetminus E \cong V^o$, we get $[\bar{C}] - [\bar{C}^-] = 0$ in $A_1(V^o \otimes \mathbb{Q})$, hence $[C] - [C^-] = \pi^*([\bar{C}] - [\bar{C}^-]) = 0$ in $A_1(J^o) \otimes \mathbb{Q}$. But the restriction map $A_1(J) \to A_1(J^o)$ is an isomorphism [14, Example 10.3.4], hence the theorem. ■

3 Complements

Corollary 1. Let Θ be a Theta divisor on J. We have $[C] = \dfrac{[\Theta]^2}{2}$ in $A_1(J) \otimes \mathbb{Q}$ (Poincaré formula).

Proof. Indeed for *any* genus 3 curve C we have $[\Theta]^2 = [C] + [C^-]$ in $A_1(J) \otimes \mathbb{Q}$ (if p, q are two distinct points of C, the intersection of Θ with its translate by $[p] - [q]$ is the union of a translate of C and a translate of C^{-}—see, for instance, [19, Lecture IV]). Thus, the corollary is equivalent to the theorem. ■

Recall that the *modified diagonal cycle* $\Gamma(C,p)$, first considered in [16], is the element $\Gamma(C,p)$ of $A_1(C^3)$ defined as follows. We denote by $[x,x,x], [x,x,p], [x,p,p]$, etc., the classes in $A_1(C \times C \times C)$ of the image of C by the maps $x \mapsto (x,x,x)$, $x \mapsto (x,x,p)$, $x \mapsto (x,p,p)$ etc. Then,

$$\Gamma(C,p) := [x,x,x] - [x,x,p] - [x,p,x] - [p,x,x] + [x,p,p] + [p,x,p] + [p,p,x] \, .$$

By [15, Remark 3.4], we have the following:

Corollary 2. $\Gamma(C,p) = 0$ in $A_1(C^3) \otimes \mathbb{Q}$.

Finally, let us mention the result of [3]: the class of $[C] - [C^{-}]$ in the intermediate Jacobian $\mathscr{J}_1(J)$ is torsion. It can also be deduced from our theorem, though the proof in [3] is more direct.

In [4] the authors construct a genus 7 curve with the same property and suggest that the corresponding Ceresa cycle should be torsion modulo algebraic equivalence (Remark 1.2).

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