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Algebraic geometry / *Geometrie algébrique*

A non-hyperelliptic curve with torsion Ceresa class

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Abstract. We exhibit a non-hyperelliptic curve C of genus 3 such that the class of the Ceresa cycle $[C]-[-C]$ in the intermediate Jacobian of JC is torsion.

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1. Introduction

Let C be a complex curve of genus $g \geq 3$, and p a point of C. We embed C into its Jacobian J by the Abel–Jacobi map $x \mapsto [x]-[p]$. The *Ceresa cycle* $\mathfrak{z}_p(C)$ is the cycle $[C]-[(-1_J)^*C]$ in the Chow group $CH_1(J)_{\mathrm{hom}}$ of homologically trivial 1-cycles. The *Ceresa class* $\mathfrak{c}_p(C)$ is the image of $\mathfrak{z}_p(C)$ in the intermediate Jacobian $\mathfrak{J}_1(J)$ parameterizing 1-cycles under the Abel–Jacobi map $CH_1(J)_{\mathrm{hom}} \to \mathfrak{J}_1(J)$.

When C is general, $\mathfrak{z}_p(C)$ is not algebraically trivial [2]. On the other hand, if C is hyperelliptic $\mathfrak{z}_p(C)$ is algebraically trivial – in fact it is zero if one chooses for p a Weierstrass point. Not much is known besides these two extreme cases. There are few curves for which $\mathfrak{z}_p(C)$ is known to be not algebraically trivial: Fermat curves of degree ≤ 1000 [4], and the Klein quartic [5]. An essential ingredient of these results is the fact that $\mathfrak{c}_p(C)$ is not a torsion class.

It is an open question whether there are non-hyperelliptic curves with $\mathfrak{z}_p(C)$ algebraically trivial. As observed in [3, Remark 2.4], this condition is equivalent to a number of interesting properties: in particular the existence of a *multiplicative Chow–Künneth decomposition* modulo algebraic equivalence, or the fact that the class $[C] \in CH_1(J)\otimes\mathbb{Q}$ is algebraically equivalent to the minimal class $\frac{\theta^{g-1}}{(g-1)!}$, where $\theta \in CH^1(J)$ is the class of the principal polarization.

In this note we exhibit a curve C of genus 3 with the weaker property that the Ceresa class $\mathfrak{c}_p(C)$ is torsion (under the Bloch–Beilinson conjectures, this actually implies the algebraic triviality of $\mathfrak{z}_p(C)$ up to torsion). The construction is very simple: the curve C has an automorphism σ which

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fixes a point p, and therefore preserves $\mathfrak{c}_p(C)$; we just have to check that the fixed point set of σ acting on $\mathfrak{J}_1(J)$ is finite.

A similar example, based on a much more sophisticated approach, appears in [1, Remark 3.6].

2. The result

Proposition 1. *Let $C \subset \mathbb{P}^2$ be the genus 3 curve defined by $X^4 + XZ^3 + Y^3Z = 0$, and let $p = (0,0,1)$. The Ceresa class $\mathfrak{c}_p(C)$ is torsion.*

Proof. Let ω be a primitive 9th root of unity. We consider the automorphism σ of C defined by $\sigma(X,Y,Z) = (X, \omega^2 Y, \omega^3 Z)$. We have $\sigma(p) = p$; therefore σ preserves the Ceresa cycle $\mathfrak{z}_p(C)$, and also its class $\mathfrak{c}_p(C)$ in $\mathfrak{J} := \mathfrak{J}_1(J)$.

Thus it suffices to prove that σ has finitely many fixed points on $\mathfrak{J}$; equivalently, that the eigenvalues of σ acting on the tangent space $T_0(\mathfrak{J})$ are $\neq 1$.

Now $T_0(\mathfrak{J})$ is identified with $H^{0,3}(J) \oplus H^{1,2}(J) = \bigwedge^3 V^* \oplus (\bigwedge^2 V^* \otimes V)$, where $V = H^{1,0}(J) = H^0(C, K_C)$. We first compute the eigenvalues of σ on V. The elements of V are of the form $L \cdot \frac{XdZ - ZdX}{Y^2Z}$, with $L \in H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}}(1))$; it follows that the eigenvalues of σ on V are $\omega^5, \omega^7, \omega^8$. Therefore the eigenvalue on $\bigwedge^3 V^*$ is ω^7, and the eigenvalues on $\bigwedge^2 V^*$ are $\omega^3, \omega^5, \omega^6$. Thus each product of an eigenvalue on $\bigwedge^2 V^*$ and one on V is $\neq 1$, hence the Proposition. □

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