The action of $\mathbf{SL}_2$ on abelian varieties

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Synopsis

$$\begin{array}{ccc} \mathbf{SL}_2 & \longrightarrow & \mathrm{Aut}(A) \\ & \searrow & \cap \\ & & \mathrm{Corr}(A) \end{array}$$

- Interest: $\mathrm{Corr}(A)$ acts on functorial invariants of A: $H^*(A)$, $CH(A)$, ... , hence action of $\mathbf{SL}_2$ on these spaces.
- On $H^*(A)$: classical action of $\mathbf{SL}_2 \iff$ Hard Lefschetz.
- On $CH(A)$: gives a twisted version of Hard Lefschetz.

NOTE : Action of $\mathbf{SL}_2$ on $CH(A)$ already known (Künnemann, Polishchuk).

Reminder on cycles and correspondences

$$CH(A) := \{ \sum_i n_i Z_i \mid n_i \in \mathbf{Q} \} \ / \ \text{rational equivalence}$$

Infinite-dimensional **Q**-vector space, rather poorly understood.

$\mathrm{Corr}(A) := CH(A \times A)$, with **Q**-algebra structure given by composition $(\alpha, \beta) \mapsto \alpha \circ \beta$ such that

$$\Gamma_u \circ \Gamma_v = \Gamma_{u \circ v} \quad \text{for} \quad u, v \in \mathrm{Aut}(A) \ .$$

Action of $\mathrm{Corr}(A)$ on $CH(A)$: for $\alpha \in \mathrm{Corr}(A)$, $z \in CH(A)$:

$$\begin{array}{ccc} & A \times A & \\ {}^{p}\swarrow & & \searrow^{q} \\ A & & A \end{array} \qquad \alpha_* z := q_*(p^* z \cdot \alpha)$$

Main theorem

Main theorem : $\mathbf{SL}_2 \longrightarrow \mathrm{Corr}(A)^* \longrightarrow \mathrm{Aut}_{\mathbf{Q}}(CH(A))$.

I will concentrate on $\mathbf{SL}_2 \longrightarrow \mathrm{Aut}_{\mathbf{Q}}(CH(A))$. Slight refinement of the proof gives the map $\mathbf{SL}_2 \to \mathrm{Corr}(A)^*$.

Some history: Mukai

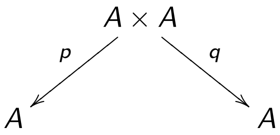
Mukai (1981): action of $SL_2(\mathbf{Z})$ on $\mathbf{D}(A)$ "up to shift".

$\mathbf{D}(A)$ = (bounded) derived category of A

= an extension $\mathrm{Coh(A)} \subset \mathbf{D}(A)$

not abelian, but notion of exact functors; all classical functors $f_*, f^*, \otimes$ become exact.

For $K \in \mathrm{Ob}\,\mathbf{D}(A \times A)$, define



$$K_*(-) = q_*(p^*(-) \otimes K)$$

$K_* = \mathbf{D}(A) \to \mathbf{D}(A)$ is the Fourier-Mukai functor associated to K.

Mukai, II

We assume that A has a polarization, i.e. an ample line bundle L (defined up to translation). For simplicity we will assume that the polarization is *principal*: it defines an isomorphism $A \xrightarrow{\sim} \hat{A}$.

In particular, we have a Poincaré line bundle $\mathcal{P}$ on $A \times A$.

Mukai 1981: $\mathcal{P}_* : \mathbf{D}(A) \to \mathbf{D}(A)$ is an equivalence. Moreover:

$$\mathcal{P}_*^2 = \big(\mathcal{P}_* \circ (\otimes L)\big)^3 = (-1_A)^*[-g] \qquad (g = \dim A)\ .$$

Recall: $SL_2(\mathbf{Z})$ is generated by

$$w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with the relations $w^2 = (uw)^3$, $w^4 = 1$.

Mukai, III

A way of formulating Mukai's observation:

Introduce $\widetilde{SL}_2(\mathbf{Z})$ generated by $\tilde{w}, \tilde{u}$ with $\tilde{w}^2 = (\tilde{u}\tilde{w})^3 \stackrel{\text{def}}{=} z$:

$$\text{Central extension :} \qquad 0 \to \mathbf{Z} \cdot z^2 \longrightarrow \widetilde{SL}_2(\mathbf{Z}) \longrightarrow SL_2(\mathbf{Z}) \to 1$$

$$(\widetilde{SL}_2(\mathbf{Z}) = \text{ trefoil knot group } = \text{ braid group } B_3)$$

We have $\widetilde{SL}_2(\mathbf{Z}) \to \mathrm{Aut}(\mathbf{D}(A))$ with $\begin{cases} w \mapsto \mathcal{P}_* \\ u \mapsto \otimes L \end{cases}$

From $\mathbf{D}(A)$ to $CH(A)$

The Chern character provides $\mathrm{Aut}(\mathbf{D}(A)) \to \mathrm{Aut}_{\mathbf{Q}}(CH(A))$:

$$\begin{array}{ccc} \mathbf{D}(A) & \xrightarrow{\mathcal{P}_*} & \mathbf{D}(A) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ CH(A) & \xrightarrow{\mathcal{F}} & CH(A) \end{array} \qquad \begin{array}{ccc} \mathbf{D}(A) & \xrightarrow{\otimes L} & \mathbf{D}(A) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ CH(A) & \xrightarrow{\cdot e^{\theta}} & CH(A) \end{array}$$

where $\theta = [L]$ in $CH^1(A)$. Since z^2 acts by an even shift:

$$\begin{array}{ccc} \widetilde{SL}_2(\mathbf{Z}) & \longrightarrow & \mathrm{Aut}(\mathbf{D}(A)) \\ \downarrow & & \downarrow \\ SL_2(\mathbf{Z}) & \xrightarrow{\tau} & \mathrm{Aut}(CH(A)) \end{array}$$

with $\tau(w) = \mathcal{F}$, $\tau(u) = \times\, e^{\theta}$.

From $SL_2(\mathbf{Z})$ to $\mathbf{SL}_2$

Theorem

The action of $SL_2(\mathbf{Z})$ on $CH(A)$ extends to an action of $\mathbf{SL}_2$, such that $CH(A)$ is a direct sum of finite-dimensional representations.

We have

$$\begin{pmatrix} n & 0 \\ 0 & n^{-1} \end{pmatrix} \cdot z = n^{-g} n_A^* z \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot z = \mathcal{F}(z)$$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot z = e^{a\theta} z \qquad \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \cdot z = a^g e^{\theta/a} * z$$

Sketch of proof

Key point : description of $\mathbf{SL}_2$ by generators and relations (Demazure, SGA 3).

$$T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \quad U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \quad B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

Lemma

H algebraic group over $\mathbf{Q}$.

Given $\left\{ \begin{array}{l} \tau : SL_2(\mathbf{Z}) \to H(\mathbf{Q}) \\ \beta : B \to H \end{array} \right.$ *which coincide on $B(\mathbf{Z})$*

and $\tau(w)\beta(t)\tau(w)^{-1} = \beta(t^{-1})$ for $t \in T$,

$\exists$ a unique morphism $f : \mathbf{SL}_2 \to H$ extending τ and β.

Sketch of proof, II

Need to define $\beta : B \to \mathrm{Aut}(CH(A))$. Use $B = U \rtimes T$.

On U, must have $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \cdot z = e^{a\theta} z$.

$\beta : T \to \mathrm{Aut}(CH(A))$ given by graduation $CH(A) = \bigoplus_s CH_s(A)$

$$CH_s^p(A) = \{z \in CH^p(A) \mid n_A^* z = n^{2p-s} z \quad \forall n \in \mathbf{Z}\}$$

Relations in the lemma are satisfied $\Longrightarrow$ action extends to $\mathbf{SL}_2$.

Differentiating gives action of $\mathfrak{sl}_2(\mathbf{Q})$, for $z \in CH_s^p(A)$:

$$X \cdot z = \theta z \qquad H \cdot z = (2g - p - s) z \qquad Y \cdot z = \frac{\theta^g}{g!} * z \ .$$

H diagonal, X, Y nilpotent $\Rightarrow$ $CH(A) = \oplus V_i$, $\dim V_i < \infty$. ∎

The "level" grading

$$
\begin{array}{ccccccc}
CH^0(A) & = & \mathbf{Q} & & & & \\
CH^1(A) & = & CH^1_0(A) & \oplus & CH^1_1(A) & & \\
\vdots & \mathbf{?} & \vdots & & \vdots & \ddots & \\
CH^g(A) & = & CH^g_0(A) & \oplus & CH^g_1(A) & \ldots \oplus & CH^g_g(A)
\end{array}
$$

VANISHING CONJECTURE (1986): *$CH_s(A) = 0$ for $s < 0$.*

Follows from the Beilinson conjectures – hopefully easier?

Applications

The well-known structure of finite-dimensional representations of $\mathbf{SL}_2$ gives:

Proposition ("Twisted" Hard Lefschetz)

The multiplication map

$$\times\,\theta^{g-2p+s} : CH^p_s(A) \longrightarrow CH^{g-p+s}_s(A) \qquad \textit{is bijective.}$$

What about "standard" Hard Lefschetz? Cannot expect surjectivity (see above), but:

Proposition

$CH_s(A) = 0$ *for* $s < 0 \iff \times\,\theta^{g-2p} : CH^p(A) \longrightarrow CH^{g-p}(A)$ *injective.*

NOTE : Right hand side makes sense for any smooth projective variety.