Non-abelian theta functions

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Abelian theta functions

C curve ($=$ Riemann surface) of genus g. Topologically, line bundles on C are classified by the *degree* $\deg L \in \mathbb{Z}$.

$J := \{$line bundles of degree $g-1$ on $C\}$

$J \cong J^0$ (the Jacobian of C) $\cong$ complex torus $\mathbb{C}^g/\Gamma$.

$$\Theta := \{L \in J \mid H^0(L) \neq 0\} \text{ hypersurface in } J \text{ (theta divisor)} .$$

Definition

$\{$theta functions of order $k\} := H^0(J, \mathcal{O}_J(k\Theta))$

$= \{$meromorphic functions on J with poles $\leq k\Theta\}$;

line bundles trivial on $\mathbb{C}^g$ $\Rightarrow$ theta functions lift to functions on $\mathbb{C}^g$, quasi-periodic w.r.t. Γ.

Algebro-geometric properties

Reminder

L line bundle on compact manifold X; $\dim H^0(X,L) < \infty$.

Rational map $\varphi_L : \begin{cases} X \dashrightarrow |L|^* \\ x \longmapsto s(x) \end{cases}$ where $|L| := \mathbb{P}(H^0(X,L))$.

defined outside $\mathrm{Bs}\,|L| := \bigcap_{s\in H^0(L)} Z(s)$.

Back to theta functions :

$\dim H^0(J, \mathcal{O}_J(k\Theta)) = k^g$; $\varphi_{k\Theta} : J \to |k\Theta|$ embedding for $k \geq 3$;

for $k = 2$, $\varphi_{2\Theta} : J \twoheadrightarrow J/i := K(J) \hookrightarrow |2\Theta|$, $i : L \mapsto K \otimes L^{-1}$.

Gives explicit description of J as submanifold of $\mathbb{P}^N$; much is known about its equations, geometry etc.

Non-abelian theta functions

Line bundles $\leftrightarrow$ $\mathbb{C}^*$-bundles; replace $\mathbb{C}^*$ by arbitrary semi-simple algebraic group G.

$$\mathcal{M}_G := \text{moduli space of (semi-stable) } G\text{-bundles on } C.$$

(For classical groups, G-bundle = vector bundle + quadratic or symplectic form)

$\mathrm{Pic}(\mathcal{M}_G) = \mathbb{Z}[\mathcal{L}_G]$, $\mathcal{L}_G$ determinant bundle

G-theta functions of level k = elements of $H^0(\mathcal{M}_G, \mathcal{L}^k)$

Relation with physics

These spaces appear in math. physics, in (at least) 2 ways :

1. In *topological quantum field theory* : $H^0(\mathcal{M}_G, \mathcal{L}^k)$ depends essentially only on the topology of C; $C \mapsto H^0(\mathcal{M}_G, \mathcal{L}^k)$ should be a TQFT in the sense of Atiyah.
2. In *conformal field theory* : $C \mapsto H^0(\mathcal{M}_G, \mathcal{L}^k)$ is the space of *conformal blocks* for the Wess-Zumino-Witten model.

Mathematical consequences :

① : when C varies, the $H^0(\mathcal{M}_G, \mathcal{L}^k)$ form a projectively flat vector bundle on the moduli space $\mathcal{M}_g$ (Hitchin connection).

In other words, $H^0(\mathcal{M}_G, \mathcal{L}^k)$ carries a (projective) representation of the modular group $\Gamma_g = \pi_1(\mathcal{M}_g)$.

The Verlinde formula

② gives the Verlinde formula for $\dim H^0(\mathcal{M}_G, \mathcal{L}^k)$: for $G = SL(r)$:

$$\dim H^0(\mathcal{M}_{SL(r)}, \mathcal{L}^k) = \Big(\frac{r}{r+k}\Big)^g \sum_{\substack{S\subset[1,r+k]\\ |S|=r}} \prod_{\substack{s\in S\\ t\notin S}} \Big|2\sin\pi\frac{s-t}{r+k}\Big|^{g-1} .$$

(many mathematical proofs by now.)

Aim of the talk : understand $\mathcal{L}$ and $H^0(\mathcal{M}_G, \mathcal{L})$, in particular,

the rational map $\varphi_{\mathcal{L}} : \mathcal{M}_G \dashrightarrow |\mathcal{L}|^*$.

$G = SL(r)$

$$\mathcal{M}_{SL(r)} = \{E \text{ (semi-stable) rank } r \mid \det E = \mathcal{O}_C\} \ .$$

(semi-stable: every $E' \subset E$ has degree ≤ 0.)

Key construction : associate to $E \in \mathcal{M}_{SL(r)}$ a divisor on J

$$\Theta_E := \{L \in J \mid H^0(C, E \otimes L) \neq 0\}$$

- either Θ_E is a hypersurface in J; then $\Theta_E \in |r\Theta|$,
- or $\Theta_E = J$: *E has no Theta divisor.*

Thus get the theta map $\theta : \mathcal{M}_{SL(r)} \dashrightarrow |r\Theta|$, $\theta(E) = \Theta_E$.

BNR

Theorem (Narasimhan, Ramanan, AB)

$$\begin{array}{ccc} & & |\mathcal{L}|^* \\ & \overset{\varphi_{\mathcal{L}}}{\nearrow} & \downarrow \wr \\ \mathcal{M}_{SL(r)} & \underset{\theta}{\dashrightarrow} & |r\Theta| \end{array}$$

Hence $\qquad H^0(\mathcal{M}_{SL(r)}, \mathcal{L}) \xrightarrow{\sim} H^0(J, \mathcal{O}_J(r\Theta))^*.$

Consequence :

Indeterminacy locus of $\theta = \mathrm{Bs}\ |\mathcal{L}| = \{E \in \mathcal{M}_{SL(r)} \mid \Theta_E = J\}$.

Examples first constructed by Raynaud, exist for $r \geq 4$ in any genus (Pauly). One of the major difficulties in the study of θ.

$G = SL(2)$

Theorem

- *For* $g = 2$, $\theta : \mathcal{M}_{SL(2)} \xrightarrow{\sim} |2\Theta|$ *(Narasimhan-Ramanan)*
- *For* $g \geq 3$, *C non hyperelliptic,* $\theta : \mathcal{M}_{SL(2)} \hookrightarrow |2\Theta|$ *(Brivio-Verra + van Geemen-Izadi)*
- *For* $g \geq 3$, *C hyperelliptic,* θ *2-to-1 onto explicit subvariety of* $|2\Theta|$ *(Bhosle-Ramanan).*

Example (Narasimhan-Ramanan)

$g = 3$, C non hyperelliptic : $\mathcal{M}_{SL(2)}$ quartic hypersurface $\mathcal{Q} \subset |2\Theta| \cong \mathbb{P}^7$, singular along the Kummer variety $K(J) \Longrightarrow$ $\mathcal{Q}$ is the Coble quartic.

$G = SL(r)$, $g(C) = 2$

In genus 2, $\dim \mathcal{M}_{SL(r)} = \dim |r\Theta| = r^2 - 1$.

Proposition

For $g = 2$, θ is generically finite.

NOTE : θ is not a morphism for $r \geq 4$; some fibres have dimension $\geq [\frac{r}{2}] - 1$.

Example (Ortega)

$(g = 2)$ $\theta : \mathcal{M}_{SL(3)} \to |3\Theta| \cong \mathbb{P}^8$ is a double covering, branched along a sextic hypersurface $\mathcal{S} \subset |3\Theta|$.

$\mathcal{S}^* \subset |3\Theta|^*$ is the Coble cubic, the unique cubic hypersurface in $|3\Theta|$ singular along the image of J.

$SO(r)$ versus $O(r)$

$$\mathcal{M}_{O(r)} \cong \{(E,q) \mid E \text{ semi-stable rk } r\ , q : \mathrm{Sym}^2 E \to \mathcal{O}_C \text{ non-deg.}\}$$

$$\mathcal{M}_{SO(r)} \cong \{(E,q,\omega) \mid (E,q) \in \mathcal{M}_{O(r)}\ ,\ \omega \in H^0(C, \wedge^r E)\ , q(\omega) = 1)\}$$

Map $\mathcal{M}_{SO(r)} \twoheadrightarrow \mathcal{M}^{\mathcal{O}}_{O(r)} := \{(E,q) \in \mathcal{M}_{O(r)} \mid \wedge^r E = \mathcal{O}_C\}$.

- For r odd, $-1 \in \mathrm{Aut}(E,q)$ exchanges ω and $-\omega$ $\quad \Rightarrow$

$$\mathcal{M}_{SO(r)} \xrightarrow{\sim} \mathcal{M}^{\mathcal{O}}_{O(r)}\ .$$

- For r even, $\qquad \mathcal{M}_{SO(r)} \xrightarrow{2:1} \mathcal{M}^{\mathcal{O}}_{O(r)}\ .$

$O(r)$ versus $GL(r)$

Theorem (Serman)

The map $\mathcal{M}_{O(r)} \to \mathcal{M}_{GL(r)}$ is an embedding.

Remarks

1. $\mathcal{M}_{SO(r)}$ has 2 components $\mathcal{M}^{\pm}_{SO(r)}$, distinguished by the Stiefel-Whitney class $w_2 \in \{\pm 1\}$.
2. For $(E, q) \in \mathcal{M}_{O(r)}$, $E \cong E^*$, hence $\Theta_E = \Theta_{E^*} = i^*\Theta_E$, where i is the involution $L \mapsto K \otimes L^{-1}$ of J.

   Thus $\Theta_E \in |r\Theta|^+$ or $|r\Theta|^-$, the eigenspaces of i^* in $|r\Theta|$.

$H^0(\mathcal{M}^{\pm}_{SO(r)}, \mathcal{L})$

Theorem

$$\begin{array}{ccc} \mathcal{M}^{\pm}_{SO(r)} & \overset{\theta^{\pm}}{\dashrightarrow} & |r\Theta|^{\pm} \\ \downarrow & & \downarrow \\ \mathcal{M}_{SL(r)} & \overset{\theta}{\dashrightarrow} & |r\Theta| \end{array} \qquad \theta^{\pm} = \varphi_{\mathcal{L}} \ \textit{ for } \ \mathcal{M}^{\pm}_{SO(r)} \ .$$

Hence $\quad H^0(\mathcal{M}^{\pm}_{SO(r)}, \mathcal{L}) \xrightarrow{\ \sim\ } \left(H^0(J, \mathcal{O}_J(r\Theta))^*\right)^{\pm}.$

(Essential ingredient: Verlinde formula for $SO(r)$.)

Example

Example ($g = 2, r = 3$)

$$\begin{array}{ccccccc} \mathcal{M}^-_{SO(3)} & \hookrightarrow & \mathcal{M}_{SL(3)} & \hookleftarrow & \mathcal{M}^+_{SO(3)} & & \\ \downarrow{\scriptstyle \theta^-} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta^+} & \searrow & \\ |3\Theta|^-_{(\cong\mathbb{P}^3)} & \hookrightarrow & |3\Theta|\ (\cong\mathbb{P}^8) & \hookleftarrow & |3\Theta^+|\ (\cong\mathbb{P}^4) & \hookleftarrow & \mathcal{Q} \\ \cup & & \cup & & & & \\ \mathcal{S}^- = \cup H_p & & \mathcal{S} & & & & \end{array}$$

$\mathcal{S} \cap |3\Theta^-| := \mathcal{S}^- =$ union of 6 planes

$\mathcal{S} \cap |3\Theta^+| = \mathcal{Q} + 2H$, $\mathcal{Q} =$ Igusa quartic, $H = \Theta + |2\Theta| \subset |3\Theta|^+$.

$G = Sp(2r)$

$$\mathcal{M}_{Sp(2r)} = \{(E,\varphi) \mid E \in \mathcal{M}_{SL(2r)}\ ,\ \varphi : \wedge^2 E \to \mathcal{O}_C \text{ non-deg.}\}$$

Again $\mathcal{M}_{Sp(2r)} \hookrightarrow \mathcal{M}_{SL(2r)}$ (Serman).

$$E \cong E^* \ \Rightarrow\ \Theta_E \in |2r\Theta|^+ \ \rightsquigarrow \quad \theta : \begin{cases} \mathcal{M}_{Sp(2r)} \dashrightarrow & |2r\Theta|^+ \\ E \quad \mapsto & \Theta_E \end{cases}$$

Then $\mathcal{L}_{Sp(2r)} = \theta^*\mathcal{O}(1)$, but $\theta^+ \neq \varphi_{\mathcal{L}}$ for $r \geq 3$.

$\Big($That is, $\big(H^0(J,\mathcal{O}_J(2r\Theta))^*\big)^+ \longrightarrow H^0(\mathcal{M}_{Sp(2r)},\mathcal{L})$ not bijective$\Big)$

$H^0(\mathcal{M}_{Sp(2r)}, \mathcal{L})$

Replace J by $\mathcal{N} := \{F \in \mathcal{M}_{GL(2)} \mid \det F = K_C\} \quad (\cong \mathcal{M}_{SL(2)})$
and Θ by $\Delta := \{F \in \mathcal{N} \mid H^0(C, F) \neq 0\}$.

To $E \in \mathcal{M}_{Sp(2r)}$ associate $\Delta_E := \{F \in \mathcal{N} \mid H^0(C, E \otimes F) \neq 0\}$.
Then : either $\Delta_E = \mathcal{N}$, or $\Delta_E \in |r\Delta|$.

Theorem

$$\begin{array}{ccc} & & |\mathcal{L}|^* \\ & \overset{\varphi_{\mathcal{L}}}{\nearrow} & \downarrow \wr \\ \mathcal{M}_{Sp(2r)} & \xrightarrow[E \mapsto \Delta_E]{} & |r\Delta| \end{array}$$

Hence $\qquad H^0(\mathcal{M}_{Sp(2r)}, \mathcal{L}) \xrightarrow{\sim} H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}}(r\Delta))^*$.

(Proof relies on the rank-level duality $SL(2) - GL(r)$ proved by Marian-Oprea and Belkale.)

THE END