# On the Brauer group of Enriques surfaces

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# The Brauer group

S smooth projective variety over $\mathbb{C}$

$$\mathrm{Br}(S) := H^2_{\text{ét}}(S, \mathbb{G}_m) \cong H^2(S, \mathcal{O}^*_{hol})_{\text{tors}}$$

## Why is it interesting?

- $\mathrm{Br}(K)$ important in number theory and algebra.
- $\mathrm{Br}(S)$ birational invariant – detects non-rational varieties (Artin-Mumford).
- $\mathrm{Br}(S)$ classifies Severi-Brauer fibrations ($= \mathbb{P}^n$-bundles) over $S$
- $\alpha \in \mathrm{Br}(S) \rightsquigarrow$ an abelian category $\mathrm{Coh}_\alpha(S)$ of "$\alpha$-twisted coherent sheaves".
- For $S$ over a number field, $\mathrm{Br}(S)$ provides the Brauer-Manin obstruction to the existence of rational points.

# How to compute $\mathrm{Br}(S)$?

$\mathrm{Br}(S)$ can be computed from the exponential exact sequence:

$$H^2(S,\mathbb{Z}) \to H^2(S,\mathcal{O}_S) \to H^2(S,\mathcal{O}_S^*) \to H^3(S,\mathbb{Z}) \to H^3(S,\mathcal{O}_S)$$

- If $H^{2,0}(S) = 0$, $\quad \mathrm{Br}(S) \cong H^3(S,\mathbb{Z})_{\mathrm{tors}}$ ;
- If $S$ simply-connected surface, $\quad \mathrm{Br}(S) \cong \mathrm{Hom}(T_S, \mathbb{Q}/\mathbb{Z})$,

  where $T_S := \mathrm{Pic}(S)^{\perp} \subset H^2(S,\mathbb{Z})$ = transcendental lattice.

# Enriques and K3 surfaces

$X$ = K3 surface, $\sigma$ fixed-point free involution of $X$,

$S := X/\sigma$ Enriques surface, $\pi : X \to S$.

$\mathrm{Br}(S) = H^3(S,\mathbb{Z}) = \mathbb{Z}/2$ has a unique nonzero element $b_S$;

$\mathrm{Br}(X) = \mathrm{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$.

Question (Harari, Skorobogatov): What is $\pi^* b_S$? Is it $\neq 0$?

**Proposition**

*$\pi^* b_S \in \mathrm{Hom}(T_X, \mathbb{Q}/\mathbb{Z})$ is the map*

$$T_X \subset H^2(X,\mathbb{Z}) \xrightarrow{\pi_*} H^2(S,\mathbb{Z}) \xrightarrow{\cdot\beta} \mathbb{Z}/2 \subset \mathbb{Q}/\mathbb{Z}$$

*where $\beta$ is any element of $H^2(S,\mathbb{Z}/2)$ not coming from $H^2(S,\mathbb{Z})$.*

But that does not say whether $\pi^* b_S = 0$ or not.

# Main result

**Theorem**

1. $\pi^* b_S = 0 \iff \exists L \in \mathrm{Pic}(X),\ \sigma^* L \cong L^{-1},\ L^2 \equiv 2 \mod. 4$
2. *In the moduli space of Enriques surfaces, the surfaces $S$ with $\pi^* b_S = 0$ form a countable union of hypersurfaces.*

**Remark**

The proof does not give any concrete example. But :

- Harari and Skorobogatov give an explicit example of $S$ over $\mathbb{Q}$ with $\pi^* b_S \neq 0$;
- Garbagnati and Van Geemen gave recently an explicit example of $S$ over $\mathbb{Q}$ with $\pi^* b_S = 0$.

# Brauer groups and cyclic coverings

**Proposition 1**

*$\pi : X \to S$ étale cyclic covering of smooth projective varieties, Galois group $G = \langle\sigma\rangle$. Consider*

$$\mathrm{Nm} : \begin{cases} \mathrm{Pic}(X) \to \mathrm{Pic}(S) \\ \mathcal{O}_X(D) \mapsto \mathcal{O}_S(\pi_* D) \end{cases} . \textit{ Then } :$$

$$\mathrm{Ker}\big(\pi^* : \mathrm{Br}(S) \to \mathrm{Br}(X)\big) = \mathrm{Ker}\,\mathrm{Nm}\,/(1-\sigma^*)(\mathrm{Pic}(X))$$

(valid over any algebraically closed field.)

# Sketch of proof

Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(G, H^q(X, \mathbb{G}_m)) \implies H^{p+q}(S, \mathbb{G}_m)$$

$E_2^{2,0} = H^2(G, \mathbb{C}^*) = 0$; then

$$\left.\begin{array}{l} 0 \to E_\infty^{1,1} \longrightarrow H^2(S, \mathbb{G}_m) \xrightarrow{\pi^*} H^2(X, \mathbb{G}_m) \\ \text{and} \qquad 0 \to E_\infty^{1,1} \longrightarrow E_2^{1,1} \xrightarrow{d_2} E_2^{3,0} \end{array}\right\} \Rightarrow \operatorname{Ker} \pi^* = \operatorname{Ker} d_2$$

$$d_2 : H^1(G, \operatorname{Pic}(X)) \to H^3(G, \mathbb{C}^*) = \operatorname{Hom}(G, \mathbb{C}^*) = \hat{G}$$

In general, $H^1(G, -) = \operatorname{Ker} \mathsf{S} / \operatorname{Im}(1-\sigma)$, with $\mathsf{S} = \sum_{g \in G} g$ ;

here $\mathsf{S}(L) = \bigotimes_{g \in G} g^* L = \pi^* \operatorname{Nm}(L)$. Hence $\operatorname{Nm}$ induces

$$\overline{\operatorname{Nm}} : H^1(G, \operatorname{Pic}(X)) \to \operatorname{Ker} \pi^* \cong \hat{G}$$

**Claim** : $d_2 = \overline{\mathrm{Nm}}$ $\quad (\Longrightarrow$ Prop. 1)

Follows from: some general non-sense on spectral sequences,

+ explicit computation. ∎

- Back to $X$ K3, $S$ Enriques: Prop. 1 $\Rightarrow$

$$\pi^* b_S = 0 \iff \exists\, L \in \mathrm{Pic}(X),\ \mathrm{Nm}(L) = \mathcal{O}_S,\ L \notin (1-\sigma^*)(\mathrm{Pic}(X))\,.$$

**Proposition 2**

$L \in \mathrm{Pic}(X)$, $\sigma^* L \cong L^{-1}$. *Then*

$$\mathrm{Nm}(L) = \mathcal{O}_S \ \textit{ and } \ L \notin (1-\sigma^*)(\mathrm{Pic}(X)) \iff L^2 \equiv 2 \bmod 4$$

**Thus :** $\pi^* b_S = 0 \iff \exists L \in \mathrm{Pic}(X),\ \sigma^* L \cong L^{-1},\ L^2 \equiv 2 \bmod 4$

= part ① of the Theorem.

# Partial proof of Proposition 2

$L^2 \equiv 2 \mod 4 \implies L \notin (1-\sigma^*)(\mathrm{Pic}(X))$ *and* $\mathrm{Nm}(L) = \mathcal{O}_S$ :

ⓐ If $L = \mathcal{O}_X(D - \sigma^* D)$ for some divisor $D$,

$$L^2 = (D-\sigma^* D)^2 \equiv (D+\sigma^* D)^2 = (\pi^*\pi_* D)^2 = 2(\pi_* D)^2 \equiv 0 \mod 4$$

ⓑ $\pi^* \mathrm{Nm}(L) = L \otimes \sigma^* L = \mathcal{O}_X$. If $\mathrm{Nm}(L) \neq \mathcal{O}_S$,

then $\mathrm{Nm}(L) = K_S \Rightarrow$

$E := \pi_* L$ has a non-degenerate symmetric $\varphi : E \otimes E \to K_S$,

hence $h^2(E) = h^0(E^* \otimes K_S) = h^0(E)$.

$$\varphi \rightsquigarrow \quad H^1(E) \otimes H^1(E) \longrightarrow H^2(K_S) \cong \mathbb{C}$$

non-degenerate skew-symmetric, hence $h^1(E)$ even.

Thus $\chi(E) = \chi(L)$ even $\Rightarrow L^2 \equiv 0 \mod 4$ (Riemann-Roch). ■

# The period map for Enriques surfaces

Fix $X_0$ K3, $\sigma_0$ fixed-point free involution. Put

$$L := H^2(X_0, \mathbb{Z}) \quad \sigma_L := \sigma_0^* \quad L^- := \mathrm{Ker}(1 + \sigma_L)$$

independent of the choice of $(X_0, \sigma_0)$; $\mathrm{rk}\ L^- = 12$.

$S$ Enriques surface, $\pi : X \to S$ double cover, $\sigma$ involution,

$\omega$ holomorphic 2-form on $X$ $(\omega \neq 0)$; then $\sigma^*\omega = -\omega$.

**Definition**

1. A marking of $S$ is an isometry $\varphi : H^2(X, \mathbb{Z}) \xrightarrow{\sim} L$ which conjugates the actions of $\sigma^*$ and $\sigma_L$.
2. The period $\wp(S, \varphi)$ is $\varphi_{\mathbb{C}}(\mathbb{C}\omega) \in \mathbb{P}(L^-_{\mathbb{C}})$.

# Horikawa's theorem

Let $\Omega := \{x \in \mathbb{P}(L^-_{\mathbb{C}}) \mid (x \cdot x) = 0 \quad (x \cdot \bar{x}) > 0\} \subset \mathbb{P}(L^-_{\mathbb{C}}) \cong \mathbb{P}^{11}$

and $\Omega^{\mathrm{o}} := \Omega - \bigcup \lambda^{\perp}$ for all $\lambda \in L^-$, $\lambda^2 = -2$.

**Claim :** $\wp(S, \varphi) \in \Omega^{\mathrm{o}}$.

ⓐ $(\omega \cdot \omega) = 0$ and $(\omega \cdot \bar{\omega}) > 0$, hence $\wp(S, \varphi) \in \Omega$;

ⓑ $\wp(S, \varphi) \in \lambda^{\perp} \Leftrightarrow \varphi^{-1}(\lambda) = [D]$, $D \in \mathrm{Div}(X)$;

$D^2 = -2 \Rightarrow D$ or $-D \geq 0$. But $\sigma^* D \equiv -D$, impossible.

**Theorem (Horikawa)**

1. $\wp(S, \varphi) = \wp(S', \varphi') \Rightarrow S \cong S'$;
2. *Every point of* $\Omega^{\mathrm{o}}$ *is the period point of some* $(S, \varphi)$.

# Theorem, part ②

Hence $\wp$ induces a bijection

$$\{\text{isom. classes of Enriques surfaces}\} \xrightarrow{\sim} \Omega^{\mathrm{o}}/O(L^-)\ .$$

$\mathcal{M} := \Omega^{\mathrm{o}}/O(L^-)$ is the coarse moduli space of Enriques surfaces; it is a quasi-projective variety (actually quasi-affine (Borcherds), and rational (Kondō)).

For $\lambda \in L^-$, the image $\mathcal{H}_\lambda$ of $\lambda^\perp$ in $\mathcal{M}$ is an algebraic hypersurface.

**Theorem**

*In $\mathcal{M}$, $\{S \mid \pi^* b_S = 0\} = \bigcup \mathcal{H}_\lambda$ for $\lambda \in L^-$, $\lambda^2 \equiv 2 \pmod{4}$.*

**Proof**

Recall: $\pi^* b_S = 0 \iff \exists D$ with $\sigma^* D \equiv -D$ and $D^2 \equiv 2 \pmod{4}$

$$\iff \exists \lambda \in L^-, (\lambda \cdot \omega) = 0, \lambda^2 \equiv 2 \pmod{4}. \quad \blacksquare$$

# Complements and questions

**Remark** (H. Martinez):

- When $\pi^* b_S = 0$, explicit construction of $b_S$:

$$L \in \mathrm{Pic}(X)\ ,\ \sigma^* L \cong L^{-1}\ ,\ L^2 \equiv 2 \pmod{4}.\ \text{Then}$$

$$\sigma^*\mathbb{P}(\mathcal{O}_X \oplus L) = \mathbb{P}(\mathcal{O}_X \oplus L^{-1}) \cong \mathbb{P}(\mathcal{O}_X \oplus L) \qquad \rightsquigarrow$$

$\mathbb{P}(\mathcal{O}_X \oplus L) \to X$ descends to Severi-Brauer $P \to S$ representing $b_S$.

**Questions**:

- Extend to other surfaces; typical case, Godeaux surfaces:

  $\sigma =$ automorphism $(X_i) \mapsto (\zeta^i X_i)$ of $\mathbb{P}^3$, $\zeta^5 = 1$,

  $X :=$ invariant quintic (e.g. $\sum X_i^5 = 0$), $\pi : X \to S := X/\langle\sigma\rangle$.

  $Br(S) = \mathbb{Z}/5$, $b_S$ generator, when is $\pi^* b_S = 0$ ??

# Questions (continued)

- Extend to higher-dimensional varieties.

  Good case: $X$ holomorphic symplectic, e.g. fourfold.

  A fixed-point free automorphism must be of order 3 $(\chi(\mathcal{O}_X) = 3)$. Example ??

**THE END**