Dynamics of Cremona transformations

Arnaud Beauville

Université de Nice

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The dynamical degree

$f \in \mathrm{Bir}(\mathbb{P}^2)$, $f = (f_0, f_1, f_2)$ homogeneous polynomials of degree d

$d := \deg(f)$: $f^* : H^2(\mathbb{P}^2) \to H^2(\mathbb{P}^2)$ is multiplication by d.

For $f, g \in \mathrm{Bir}(\mathbb{P}^2)$, $\deg(fg) \leq \deg(f)\deg(g) \quad \Longrightarrow$

$$\lim_{n\to\infty} (\deg f^n)^{\frac{1}{n}} = \lambda(f) :=$$ dynamical degree of f

$1 \leq \lambda(f) \leq \deg(f)$; $\lambda(f) = \deg(f)$ if f is generic,

$\lambda(f) = 1$ if f periodic or $f \in PGL(3)$.

Theorem (Diller-Favre)

1. $f \in \mathrm{Bir}(\mathbb{P}^2)$:

$\lambda(f)$	Growth of $\deg f^n$	up to conjugacy by $S \overset{\sim}{\dashrightarrow} \mathbb{P}^2$:
$=1$	$\deg f^n \leq C$	$f \in \mathrm{Aut}(S)$, $f^n \in \mathrm{Aut}^o(S)$
	$\deg f^n \sim Cn$	f preserves a rational fibration $f \notin \mathrm{Aut}(S)$
	$\deg f^n \sim Cn^2$	f preserves an elliptic fibration $f \in \mathrm{Aut}(S)$
>1	$\deg f^n \sim C\lambda(f)^n$	

Diller-Favre 2

1. x
2. $\lambda(f)$ is an algebraic integer; its conjugates μ_i have $|\mu_i| \leq 1$.

$$\left(\begin{array}{l} \implies \lambda(f) \text{ is either a } \textit{Pisot number } (|\mu_i| < 1 \ \ \forall i) \\ \text{or a } \textit{Salem} \text{ number } \ (\text{conjugates} : \{\lambda, \lambda^{-1}, \mu_i\} \text{ with } |\mu_i| = 1). \end{array} \right)$$

Example

$SL(2, \mathbb{Z})$ embeds into $\mathrm{Bir}(\mathbb{P}^2)$:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f_A : (x, y) \mapsto (x^a y^b, x^c y^d)$$

- either $|\,\mathrm{Tr}\,A| \leq 2$, and $\lambda(f_A) = 1$;
- or A has eigenvalues $\lambda^{-1} < 1 < \lambda$, and $\lambda(f_A) = \lambda$.

Ingredients

- $\lambda(f)$ can be defined on any rational S: replace $\deg(f)$ by
  - (f^*h, h), for any ample h;
  - or $\|f^*\|$ for any norm on $H^2(S,\mathbb{R})$.

Then: $\lambda(f)$ invariant under conjugacy. Recall: $(fg)^* \neq g^*f^*$.

- Key ingredient: $f \in \mathrm{Bir}(S)$ *algebraically stable* (AS) if

$$(f^n)^* = (f^*)^n \text{ on } H^2(S,\mathbb{R}) \ \text{ for all } n \geq 1 \ .$$

Proposition

$f \in \mathrm{Bir}(S)$ $\exists\, \varphi : S \overset{\sim}{\dashrightarrow} S'$ *such that* $\varphi f \varphi^{-1}$ *is AS.*

Corollary

$\lambda(f) = \lim_{n\to\infty} \|(f^*)^n\|^{\frac{1}{n}} =$ *spectral radius of* $f^* =$ *algebraic integer.*

Algebraic stability

REMARK: f^* is not an isometry (in general).

Example (continued)

$$f_A : (x, y) \mapsto (x^a y^b, x^c y^d) \text{ for } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) ,$$

with $a, b, c, d > 0$.

Then $\deg(f_A^2) < (\deg f_A)^2$ (exercise) $\Rightarrow$ $f_A \in \mathrm{Bir}(\mathbb{P}^2)$ not AS.

View $f_A \in \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$, with $(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1$.

$$H^2(\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{Z}) = \mathbb{Z}\, L_x \oplus \mathbb{Z}\, L_y ,$$

with $L_x = \{0\} \times \mathbb{P}^1$, $L_y = \mathbb{P}^1 \times \{0\}$. Then :

$$f_A^* L_x = aL_x + bL_y \ , \ f_A^* L_y = cL_x + dL_y \ \Rightarrow \ \mathrm{Mat}(f_A^*) = {}^t A$$

$\Rightarrow$ $f_A \in \mathrm{Bir}(\mathbb{P}^1 \times \mathbb{P}^1)$ AS, $\lambda(f_A) =$ largest eigenvalue of A.

Samples of proof of the theorem

1) : deg(f^n) bounded $\Rightarrow$ $f^N \in \mathrm{Aut}^{\mathrm{o}}(S)$ (up to conjugacy)?

Conjugate f so that it is AS. Then $\|(f^*)^n\| = \|(f^n)^*\|$ bounded.

Since f^* preserves $H^2(S,\mathbb{Z})$, $\exists\ p, q > 0$ such that $(f^*)^p = (f^*)^{p+q}$.

f^* automorphism (to be proved) $\Rightarrow$ $(f^*)^q = I$ $\Rightarrow$ $(f^q)^* = I$

$\Rightarrow$ f^q preserves all exceptional curves

$\Rightarrow$ f^q descends to an automorphism of the minimal model. ■

2) : All conjugates μ_i of $\lambda(f)$ have $|\mu_i| \leq 1$? Follows from

Lemma

1. (geometry) $(f^*v)^2 \geq v^2$ *for all* v *in* $H^2(S,\mathbb{R})$.
2. (linear algebra) *V real vector space, q quadratic form of signature* $(1,n)$, *$u \in \mathrm{End}(V)$ such that* $q(u(v)) \geq q(v)\ \ \forall v$. *Then u has at most one eigenvalue λ with* $|\lambda| > 1$. ■

The Picard-Manin space

S rational surface.

For $u : S' \to S$ birational morphism, $u^* : \mathrm{Pic}(S) \to \mathrm{Pic}(S')$

injective, preserves intersection product and nef cone.

$$Z(S) := \varinjlim_{S' \to S} \mathrm{Pic}(S') \cong \mathrm{Pic}(S) \oplus \mathbb{Z}^{(\Sigma)} \ ,$$

where $\Sigma = S \cup \{\text{infinitesimally near points}\}$. Structures :

- intersection product $\rightsquigarrow$ quadratic form $(+, -, -, \ldots)$
- Nef cone $Z^+(S) =$ limit of the nef cones in $\mathrm{Pic}(S')$.

Picard-Manin 2

$u : S' \to S$ birational $\rightsquigarrow$ $u_* : Z(S') \xrightarrow{\sim} Z(S)$ isometry.

More generally, for $f \in \mathrm{Bir}(S)$,

$$\begin{array}{ccc} & \hat{S} & \\ {\scriptstyle g}\swarrow & & \searrow{\scriptstyle h} \\ S & \overset{f}{\dashrightarrow} & S \end{array}$$

Define $f_* := h_*(g_*)^{-1} : Z(S) \xrightarrow{\sim} Z(S)$; isometry, preserves $Z^+(S)$

$\rightsquigarrow$ homomorphism $\mathrm{Bir}(S) \hookrightarrow \mathrm{Aut}(Z(S))$.

Definition: $\mathcal{Z}(S) :=$ Hilbert completion of $Z(S) \otimes \mathbb{R}$

$= (\mathrm{Pic}(S) \otimes \mathbb{R}) \oplus \ell^2(\Sigma)$ ("Picard-Manin space")

with Lorentzian product + nef cone $\mathcal{Z}^+(S)$.

$\rightsquigarrow$ embedding $\mathrm{Bir}(S) \hookrightarrow \mathrm{Isom}(\mathcal{Z}(S))$.

Lorentzian isometries

V vector space with quadratic form q of signature $(+,-,-,\ldots)$.

Fix one component $\mathcal{C}$ of the cone $q(v)>0$.

u isometry of (V,q) preserving $\mathcal{C}$. 3 possibilities :

- *elliptic* : $\exists\ v\in\mathcal{C},\ u(v)=v$; $u_{\ |v^\perp}$ orthogonal.
- *parabolic* : $\exists$ unique (up to $\mathbb{R}_+^*$) eigenvector $v\in\bar{\mathcal{C}}$; $u(v)=v$.
- *hyperbolic* : u has 2 eigenvectors in $\bar{\mathcal{C}}-\mathcal{C}$, with eigenvalues $\lambda^{-1}<1<\lambda$.

Theorem (Cantat)

$\lambda(f)$	Growth of $\deg f^n$	up to conjugacy :	f_* on $\mathcal{Z}(\mathrm{S})$
$=1$	$\deg f^n \leq \mathrm{C}$	$f \in \mathrm{Aut}(\mathrm{S}),\ f^n \in \mathrm{Aut}^{o}(\mathrm{S})$	elliptic
	$\deg f^n \sim \mathrm{C}n$	f preserves a rational fibration $f \notin \mathrm{Aut}(\mathrm{S})$	parabolic
	$\deg f^n \sim \mathrm{C}n^2$	f preserves an elliptic fibration $f \in \mathrm{Aut}(\mathrm{S})$	
>1	$\deg f^n \sim \mathrm{C}\lambda(f)^n$		hyperbolic

Centralizers

Key point of the proof (Boucksom-Favre-Jonsson):

$$\lambda(f) > 1 \;\Rightarrow\; \exists\, v \in \mathcal{Z}^+(S)\,,\; f_* v = \lambda(f) v \quad (\Rightarrow\; f_* \text{ hyperbolic}).$$

Remark

$$f_* v = \alpha v \;\text{ for }\; v \in \mathcal{Z}^+(S) \;\Rightarrow\; \alpha = \lambda(f) \;\text{ or }\; \lambda(f)^{-1}\,.$$

Corollary (Cantat)

$\lambda(f) > 1$, *g commutes with f* $\;\Rightarrow\; \exists\, m, n \in \mathbb{Z}\,,\; g^m = f^n\,.$

Stronger conjecture (Cantat): $[\mathrm{Cent}(f) : \langle f \rangle] < \infty$.

Sketch of proof

Fix $v \in \mathcal{Z}^+(S)$ such that $f_* v = \lambda(f) v$.

Then for $g \in \mathrm{Cent}(f)$, $g_* v = \alpha(g) v$ with $\alpha(g) = \lambda(g)$ or $\lambda(g)^{-1}$.

$$\rightsquigarrow \ \alpha : \mathrm{Cent}(f) \to \mathbb{R}_+^* \ \text{ homomorphism.}$$

Suffices to prove: 1) $\mathrm{Ker}\,\alpha$ torsion; 2) $\alpha(\langle f, g \rangle)$ discrete.

Proof of 2): Let $h \in \langle f, g \rangle$ with $\alpha(h) \in [\frac{1}{2}, 2]$. Then

$\alpha(h) = \lambda(h)^{\pm 1}$ is an algebraic integer, of bounded degree,

with bounded conjugates (Diller-Favre)

$\Rightarrow$ finitely many possible minimal polynomials

$\Rightarrow \alpha(\langle f, g \rangle) \cap [\frac{1}{2}, 2]$ finite. ■

Other results

Theorem (Cantat)

Γ *finitely generated subgroup of* $\mathrm{Bir}(\mathbb{P}^2)$ *:*

1. Γ *contains either a solvable subgroup of finite index, or a non-cyclic free group (Tits alternative).*
2. *If* Γ *has Kazhdan property (T), it is conjugate to a subgroup of* $PGL(3, \mathbb{C})$*.*

(Kazhdan property (T): any action of Γ on a Hilbert space by affine isometries has a fixed point.)