Vector bundles on Fano threefolds and K3 surfaces

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Tyurin's observation

X Fano threefold (over $\mathbb{C}$), $S \in |K_X^{-1}|$ smooth K3 surface.

$\mathcal{M}_X, \mathcal{M}_S$ components of the moduli space of simple vector bundles on X and S (exist as an algebraic space).

$\mathcal{M}_S$ smooth, admits a **symplectic structure** (Mukai).

Theorem (Tyurin, 1990)

Assume $H^2(X, \mathcal{E}nd(E)) = 0$ for all E in $\mathcal{M}_X$.

1) *E_S is simple, hence* $\mathrm{res} : \begin{cases} \mathcal{M}_X \to \mathcal{M}_S \\ E \mapsto E_S \end{cases}$ *defined.*

2) res *is a local isomorphism onto a Lagrangian subvariety of $\mathcal{M}_S$ (:=* **Lagrangian immersion***).*

Problem: How to check $H^2(X, \mathcal{E}nd(E)) = 0$?

Proof of Tyurin's result

Exact sequence $0 \to K_X \to \mathcal{O}_X \to \mathcal{O}_S \to 0 \quad \rightsquigarrow$

$$0 \to K_X \otimes \mathcal{E}nd(E) \to \mathcal{E}nd(E) \to \mathcal{E}nd(E_S) \to 0\,.$$

Since $H^2(\mathcal{E}nd(E)) = H^1(K_X \otimes \mathcal{E}nd(E)) = 0$, get $\mathrm{End}(E_S) = \mathbb{C}$ and

$$\begin{array}{ccccccc} 0 \to & H^1(X, \mathcal{E}nd(E)) & \to & H^1(S, \mathcal{E}nd(E_S)) & \to H^1(X, \mathcal{E}nd(E))^* \to 0\,, \\ & \| & & \| & \\ \text{hence} & T_E(\mathcal{M}_X) & \hookrightarrow & T_{E_S}(\mathcal{M}_S), & \text{and } \dim \mathcal{M}_X = \tfrac{1}{2} \dim \mathcal{M}_S\,. \end{array}$$

$T_E(\mathcal{M}_X)$ isotropic :

$$\begin{array}{ccc} H^1(X, \mathcal{E}nd(E))^{\otimes 2} & \longrightarrow & H^1(S, \mathcal{E}nd(E_S))^{\otimes 2} \\ \downarrow & & \downarrow \\ H^2(X, \mathcal{E}nd(E)) & \xrightarrow{\ 0\ } & H^2(S, \mathcal{E}nd(E_S)) \\ & & \mathrm{Tr} \downarrow \wr \\ & & H^2(S, \mathcal{O}_S)\,. \end{array}$$

■

Serre construction

$C \subset X$ smooth, $K_C = (K_X \otimes L)_{|C}$ for L ample on X $\rightsquigarrow$ extension:

$0 \to \mathcal{O}_X \xrightarrow{s} E \to \mathcal{I}_C L \to 0\,, \quad \mathrm{rk}(E) = 2\,, \quad Z(s) = C\,, \quad E_{|C} = N_C\,.$

Proposition

Assume: $\mathbf{H^1(N_C) = 0}$, *and*

$H^0(X, K_X \otimes L) \to H^0(C, K_C)$ *and* $H^0(X, L) \to H^0(C, L_{|C})$ *onto.*

Then : $H^2(X, \mathcal{E}nd(E)) = 0$. *Moreover the E obtained in this way fill up a Zariski open subset of* $\mathcal{M}_X$.

Idea of proof : Straightforward. $\otimes$ extension by $E^* \cong E \otimes L^{-1}$:

$$0 \to E^* \to \mathcal{E}nd(E) \to \mathcal{I}_C E \to 0$$

and prove $H^2(E^*) = H^2(\mathcal{I}_C E) = 0$ using exact sequences.

Last statement $\Leftarrow$ deformation theory. ■

Remark : Surjectivity easy. Serious condition: $H^1(N_C) = 0$.

The examples

The rest of the talk will be devoted to examples. Set-up:

$X \subset \mathbb{P}$, $K_X = \mathcal{O}_X(-i)$: $i =$ index $= 1$ or 2. $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$.

Index 2: $X_d \subset \mathbb{P}^{d+1}$, $3 \leqslant d \leqslant 5$: $X_3 \subset \mathbb{P}^4$, $X_{2,2} \subset \mathbb{P}^5$, $X_5 \subset \mathbb{P}^6$.

Index 1: $X_{2g-2} \subset \mathbb{P}^{g+1}$, $g \in \{3, 10\} \cup \{12\}$: $X_4 \subset \mathbb{P}^4$, $X_{2,3} \subset \mathbb{P}^5$, ...

Fix $L = \mathcal{O}_X(j)$, $j = 1$ or 2. $K_C = (K_X \otimes L)_{|C} \Rightarrow K_C = \mathcal{O}_C(j-i)$.

Then $j - i \in \{-1, 0, 1\}$. This gives 3 possibilities for C:

1. $i = 2, j = 1:$ C conic in X_d (index 2);
2. $i = j = 1$ or $2:$ C elliptic, $L = \mathcal{O}_X(i)$;
3. $i = 1, j = 2:$ C canonical in X_{2g-2} (index 1).

Hypothesis: To simplify, $\mathrm{Pic}(S) = \mathbb{Z} \cdot \mathcal{O}_S(1) \Rightarrow E$ and E_S stable.

Warm-up: conics in index 2 Fano threefolds

$X_d \subset \mathbb{P}^{d+1}$, $K_X = \mathcal{O}_X(-2)$, $S := X \cap Q$, $L = \mathcal{O}_X(1)$, C conic.

Proposition

- $H^1(N_C) = 0$, $\dim \mathcal{M}_X = 5 - d$.
- $\boldsymbol{d = 3}$ **:** $\mathcal{M}_X \cong F(X)$ *(by conic $\mapsto$ residual line).*

  $\mathcal{M}_S \cong S^{[2]}$ *(4 coplanar points $\mapsto$ residual subscheme).*

  $\mathrm{res} : F(X) \hookrightarrow S^{[2]}, \ \ell \mapsto \ell \cap Q.$
- $\boldsymbol{d = 4}$ **:** $S_{2,2,2} \subset X_{2,2}$:

$$\begin{array}{ccc} \mathcal{M}_X & \xrightarrow{\ \mathrm{res}\ } & \mathcal{M}_S \\ \downarrow{\scriptstyle 2:1} & & \downarrow{\scriptstyle 2:1} \\ \ell = \{Q \supset X\} & \hookrightarrow & \Pi = \{Q \supset S\} \end{array}$$

- $\boldsymbol{d = 5}$ **:** $X_5 = \mathbb{G}(2,5) \cap \mathbb{P}^6$. $\mathcal{M}_X = \mathcal{M}_S = \{[E]\}$, *restriction of universal quotient bundle on* $\mathbb{G}(2,5)$.

Elliptic curves in index 2 Fano threefolds

C normal elliptic curve: $C_{d+2} \subset X_d \subset \mathbb{P}^{d+1}$.

Then $(K_X \otimes L)_{|C} = \mathcal{O}_C \ \Rightarrow \ L = \mathcal{O}_X(2)$, extension

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_C(2) \to 0\,. \qquad S = X_d \cap Q \subset \mathbb{P}^{d+1}\,.$$

Proposition

1) *Every X_d contains an elliptic curve C_{d+2} .*

2) *E* **Ulrich bundle :** *$H^\bullet(E(-i)) = 0$ for $1 \leqslant i \leqslant 3$.*

$$\Big(\iff \ \exists \textit{ linear resolution of length } c = \operatorname{codim}(X, \mathbb{P}^{d+1})$$
$$0 \to \mathcal{O}_{\mathbb{P}}(-c)^\bullet \to \cdots \to \mathcal{O}_{\mathbb{P}}(-1)^\bullet \to \mathcal{O}_{\mathbb{P}}^\bullet \to E \to 0 \Big)$$

3) $H^1(N_C) = 0$, res : $\mathcal{M}_X \longrightarrow \mathcal{M}_S$ *Lagrangian immersion;*

$\dim \mathcal{M}_X = 5$, $\mathcal{M}_S \cong_{bir} \mathrm{OG}_{10}$.

Sketch of proof

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Ideas of the proof: 1) Deform $C_{d+1} \bigcup_p \ell$.

2) $H^{\bullet}(E(-i)) = 0$ follows from $0 \to \mathcal{O}_X \to E \to \mathcal{I}_C(2) \to 0$.

3) $H^1(N_C) \cong H^0(N_C^*) = H^0(E_{|C}^*)$; $= 0$ using linear resolution.

Then $E_S(-1)$ has det $= \mathcal{O}_S$, $c_2 = 4$ $\rightsquigarrow \mathcal{M}_S \cong_{bir} \mathrm{OG}_{10}$. ■

Example: the cubic threefold

For $X_3 \subset \mathbb{P}^4$, linear resolution $0 \to \mathcal{O}_{\mathbb{P}}(-1)^6 \xrightarrow{M} \mathcal{O}_{\mathbb{P}}^6 \to E \to 0$ with M skew-symmetric, hence X given by $\mathrm{Pf}(M) = 0$.

Proposition (Iliev, Markushevich, Tikhomirov): $\mathcal{M}_X \cong_{bir} J(X)$.

(More precisely (Druel): $\overline{\mathcal{M}}_X \xrightarrow{\sim} \mathrm{Bl}_{F(X)} J(X)$.)

Now fix $S = S_{2,3}$, defined by $Q = F = 0$ in $\mathbb{P}^4$.
X varies in $\Pi_5 := |\mathcal{I}_S(3)| = \{aF + LQ = 0\} \cong \mathbb{P}^5$, L linear.

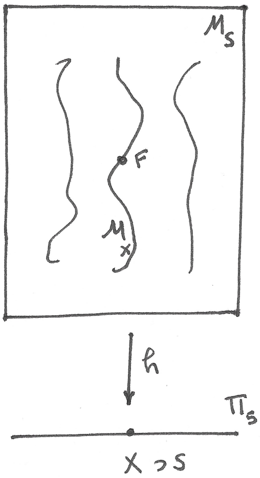
Proposition

$\exists$ *Lagrangian fibration* $h : \mathcal{M}_S \dashrightarrow \Pi_5$, $h^{-1}(X) = \mathcal{M}_X \ (\cong_{bir} J(X))$.

Proof : For F general in $\mathcal{M}_S$, $\mathcal{O}_Q(-1)^6 \xrightarrow{M} \mathcal{O}_Q^6 \to F \to 0$,
M skew-symmetric with entries in $H^0(\mathcal{O}_Q(1)) = H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \rightsquigarrow$
$0 \to \mathcal{O}_{\mathbb{P}}(-1)^6 \xrightarrow{M} \mathcal{O}_{\mathbb{P}}^6 \to E \to 0$, $E_{|S} = F$, $X = \mathrm{Supp}(E)$). ■

The (rational) Lagrangian fibration

That is: a general vector bundle $F \in \mathcal{M}_S$ determines a cubic $h(F) = X \supset S$, plus a vector bundle $E \in \mathcal{M}_X$ such that $E_S = F$.



A recent improvement

Theorem (Saccà, February 2020)

There exists a projective hyperkähler manifold $\mathcal{M}'_S$ and a birational map $\varphi : \mathcal{M}'_S \overset{\sim}{\dashrightarrow} \mathcal{M}_S$ such that $h \circ \varphi : \mathcal{M}'_S \to \Pi_5$ is a Lagrangian fibration, with fiber $J(X)$ over $X \in \Pi_5$.

Idea : $S \subset \mathbb{P}^4 : Q = F = 0$. Rational Lagrangian fibration $h : \mathcal{M}_S \cong_{bir} \mathrm{OG}_{10} \dashrightarrow \Pi_5$, with $\Pi_5 = \{X : aF + LQ = 0\}$.

Consider the nodal cubic **fourfold** $V : \; F + TQ = 0$ in $\mathbb{P}^5$. Then $X = V \cap H$, with $H : \{aT = L\}$. Thus $\Pi_5 \cong (\mathbb{P}^5)^*$.

For a smooth or 1-nodal cubic fourfold $V \subset \mathbb{P}^5$, Saccà constructs a projective hyperkähler manifold $\mathcal{M}$ and a Lagrangian fibration $h : \mathcal{M} \to (\mathbb{P}^5)^*$ with $h^{-1}(H) = J(V \cap H)$ for $V \cap H$ smooth.

(This extends the construction of Laza-Saccà-Voisin for general V).

Elliptic curves in index 1 Fano threefolds: an example

$X = X_{2,2,2} \subset \mathbb{P}^6$, $S = X \cap H = S_{2,2,2} \subset \mathbb{P}^5$. Some geometry:

$|\mathcal{I}_X(2)| := \Pi \, (\cong \mathbb{P}^2) \supset \Delta_7 =$ {singular quadrics in Π}.

ρ: $\tilde{\boldsymbol{\Delta}} \xrightarrow{2:1} \boldsymbol{\Delta_7}$: $\tilde{\Delta} = \{(q,\sigma)\}$, $q \in \Delta_7, \sigma$ family of 3-planes $\subset q$.

$|\mathcal{I}_S(2)| = \Pi \supset \Delta_6$. $\boldsymbol{\pi} : \boldsymbol{\Sigma} \xrightarrow{2:1} \boldsymbol{\Pi}$ branched along Δ_6,

$\Sigma = \{(q,\tau)\}$, $q \in \Pi, \tau$ family of 2-planes $\subset q$.

$$\begin{array}{ccc} \tilde{\Delta} & \xrightarrow{r} & \Sigma \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\pi} \\ \Delta_7 & \hookrightarrow & \Pi \end{array}$$

$r : \tilde{\Delta} \to \Sigma$,

$(q,\sigma) \mapsto (q \cap H, \sigma_H)$.

Image $= \pi^{-1}(\Delta_7)$.

Fact: Δ_7 everywhere tangent to Δ_6, hence $\pi^{-1}(\Delta_7)$ has 21 nodes.

Proposition

Take $C = C_{2,2} \subset \mathbb{P}^3$. Then: $H^1(N_C) = 0$; $\mathcal{M}_X = \tilde{\Delta}$, $\mathcal{M}_S = \Sigma$,

res : $\mathcal{M}_X \to \mathcal{M}_S$ *identified with $r : \tilde{\Delta} \to \Sigma$, thus* **not injective**.

Index 1, canonical curve

$X_{2g-2} \subset \mathbb{P}^{g+1}$, $K_X = \mathcal{O}_X(1)$: $X_4 \subset \mathbb{P}^4$, $X_{2,3} \subset \mathbb{P}^6$, ...

$C \subset \mathbb{P}^{g+1}$ canonical, genus $g+2$. Then $L = \mathcal{O}_X(2) \rightsquigarrow$ extension

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_C(2) \to 0. \quad S = X \cap H.$$

Proposition

1) *$X \supset C$ canonical of genus $g+2$, with $H^1(N_C) = 0$.*

2) res : $\mathcal{M}_X \to \mathcal{M}_S$ *is a Lagrangian immersion.*

3) $\dim \mathcal{M}_X = 5$, $\mathcal{M}_S \cong_{bir} \mathrm{OG}_{10}$.

(*Note: for $g = 3$, we must assume that X is* general.)

Sketch of proof

The proof for $g \geqslant 4$ relies on a result of Brambilla and Faenzi, who construct directly the vector bundle E as a flat deformation of $\mathcal{E}$:

$$0 \to \mathcal{I}_A(1) \to \mathcal{E} \to \mathcal{I}_B(1) \to 0$$

where A and B are general conics.

Since $g \geqslant 4$, $\mathcal{I}_A(1)$ and $\mathcal{I}_B(1)$ are globally generated, hence so are $\mathcal{E}$ and E. Hence for general $s \in H^0(E)$, $Z(s)$ is a smooth curve C.

$K_C = K_{X_{|C}} \otimes \det N_C = \mathcal{O}_C(1)$, $H^0(\mathcal{I}_C(1)) = H^1(\mathcal{I}_C(1)) = 0 \Rightarrow$ C canonically embedded in $\mathbb{P}^{g+1}$.

Brambilla and Faenzi prove $H^2(\mathcal{E}nd(E)) = 0$ ($\Leftrightarrow$ $H^1(N_C) = 0$).

Tyurin's theorem $\Rightarrow$ $\mathrm{res} : \mathcal{M}_X \to \mathcal{M}_S$ Lagrangian immersion.

$E(-1)_{|S}$ has $c_1 = 0$, $c_2 = 4 \Rightarrow \mathcal{M}_S \cong_{bir} \mathrm{OG}_{10}$.

Example: $X_{2,3}$

$X = V_3 \cap Q \subset \mathbb{P}^5$. Another way to construct $C \subset X \subset \mathbb{P}^5$ canonical of genus 6: $C = S_5 \cap Q$, with $S_5 \subset V_3$.

Facts

1) For V cubic fourfold, $V \supset S_5 \iff V$ *pfaffian*, i.e:
$\exists\ 0 \to \mathcal{O}_{\mathbb{P}}^6(-1) \xrightarrow{M} \mathcal{O}_{\mathbb{P}}^6 \to F \to 0$, M skew-symmetric, F rank 2 bundle on V, V defined by $\mathrm{Pf}(M) = 0$.

2) For s general in $H^0(F)$, $Z(s) = S_5$. Hence $F_{|X} \in \mathcal{M}_X$.

3) The pfaffian cubics form a hypersurface $\mathcal{C}_{14}$ in $|\mathcal{O}_{\mathbb{P}}(3)|$.

4) For V general in $\mathcal{C}_{14}$, F unique.

Remark : I do not know if this gives the same component $\mathcal{M}_X$ as the Brambilla-Faenzi construction.

$X_{2,3}$ (continued)

Let $\Pi_6 := |\mathcal{I}_X(3)| = \{V_3' \supset X\} \cong \mathbb{P}^6$, and $\mathcal{Pf} := \Pi_6 \cap \mathcal{C}_{14}$. Then

Proposition

$$\mathcal{M}_X \cong_{bir} \mathcal{Pf} .$$

Proof : $\mathcal{Pf} \dashrightarrow \mathcal{M}_X$: V general $\rightsquigarrow$ unique F on V $\rightsquigarrow$ $F_{|X}$.

$\mathcal{M}_X \dashrightarrow \mathcal{Pf}$: For general $E \in \mathcal{M}_X$, resolution

$0 \to \mathcal{O}_Q(-1)^6 \xrightarrow{M} \mathcal{O}_Q^6 \to E \to 0$, V defined by $\mathrm{Pf}(M) = 0$. ■

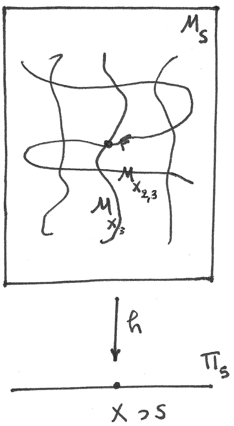
$S = X_{2,3} \cap H$: $\mathrm{Im}(\mathcal{M}_X)$ Lagrangian subvariety in $\mathcal{M}_S = \mathrm{OG}_{10}(S)$.

Recall Lagrangian fibration $h : \mathcal{M}_S \dashrightarrow \Pi_5$.

Proposition

$$p : \mathcal{M}_X \xrightarrow{\text{res}} \mathcal{M}_S \overset{h}{\dashrightarrow} \Pi_5 \textit{ is generically finite.}$$

$X_{2,3}$ (continued)



Proof : $p \cong_{bir} \bar{q} : \mathcal{Pf} \dashrightarrow \Pi_5$,

$V\,(\supset X) \mapsto V \cap H \supset S$.

$q : \Pi_6 \dashrightarrow \Pi_5$, $V \mapsto V \cap H$.

Not defined only at $\mathrm{o} := H \cup Q$

$\Longrightarrow \;\; q =$ projection from o ,

$q^{-1}(x) = \langle \mathrm{o}, x \rangle$.

$\bar{q}^{-1}(x) = \langle \mathrm{o}, x \rangle \cap \mathcal{Pf}$ finite for x general. ■

THE END