The Lüroth problem

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Abstract The Lüroth problem asks whether every unirational variety is rational. After a historical survey, we describe the methods developed in the 70's to get a negative answer, and give some easy examples. Then we discuss a new method introduced last year by C. Voisin.

1 Some history

1.1 Curves and surfaces

In 1876 appears a three pages note by J. Lüroth [L], where he proves that if a complex algebraic curve C can be parametrized by rational functions, one can find another parametrization which is generically one-to-one. In geometric language, if we have a dominant rational map $f:\mathbb{P}^1 \dashrightarrow C$, then C is a rational curve.

By now this is a standard exercise : we can assume that C is smooth projective, then f is a morphism, which induces an injective homomorphism $f^*: H^0(C,\Omega^1_C)\rightarrow H^0(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1})=0$. Thus C has no nontrivial holomorphic 1-form, hence has genus 0, and this implies $C\cong\mathbb{P}^1$.

Actually Lüroth does not mention at all Riemann surfaces, but uses instead an ingenious and somewhat sophisticated algebraic argument. I must say that I find

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somewhat surprising that he did not consider applying Riemann's theory, which had appeared 20 years before.

Anyhow, clearly Lüroth's paper had an important impact. When Castelnuovo and Enriques develop the theory of algebraic surfaces, in the last decade of the 19th century, one of the first questions they attack is whether the analogous statement holds for surfaces. Suppose we have a smooth projective surface S (over $\mathbb{C}$) and a dominant rational map $f : \mathbb{P}^2 \dashrightarrow S$. As in the curve case, this implies $H^0(S,\Omega^1_S) = H^0(S,K_S) = 0$ (note that f is well-defined outside a finite subset). At first Castelnuovo hoped that this vanishing would be sufficient to characterize rational surfaces, but Enriques suggested a counter-example, now known as the Enriques surface. Then Castelnuovo found the right condition, namely $H^0(S,\Omega^1_S) = H^0(S,K^2_S) = 0$; this is satisfied by our surface S, and Castelnuovo proves that it implies that S is rational. After more than one century, even if the proof has been somewhat simplified, this is still a highly nontrivial result.

1.2 Attempts in dimension 3

At this point it becomes very natural to ask what happens in higher dimension. Let us first recall the basic definitions (see §3 for a more elaborate discussion): a complex variety X of dimension n is *unirational* if there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$; it is *rational* if there is a birational such map. The *Lüroth problem* asks whether every unirational veriety is rational.

In 1912, Enriques proposed a counter-example in dimension 3 [E], namely a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^5$ – we will use the notation $V_{2,3}$ for such a complete intersection. Actually what Enriques does in this two pages paper is to prove the unirationality of $V_{2,3}$, in a clever (and correct) way; for the non-rationality he refers to a 1908 paper by Fano [F1].

In the course of his thorough study of what we now call Fano manifolds, Fano made various attempts to prove that some of them are not rational [F2, F4]. Unfortunately the birational geometry of threefolds is considerably more complicated than that of surfaces; while the intuitive methods of the Italian geometers were sufficient to handle surfaces, they could not treat adequately higher-dimensional manifolds. None of Fano's attempted proofs is acceptable by modern standards.

A detailed criticism of these attempts can be found in the book [R]. It is amusing that after concluding that none of them can be considered as correct, Roth goes on and proposes a new counter-example, which is not simply connected and therefore not rational (the fundamental group is a birational invariant). Alas, a few years later Serre (motivated in part by Roth's claim) proved that a unirational variety is simply connected [S].

1.3 The modern era

Finally, in 1971-72, three different (indisputable) counter-examples appeared. We will discuss at length these results in the rest of the paper; let us indicate briefly here the authors, their examples and the methods they use to prove non-rationality :

Authors	Example	Method
Clemens-Griffiths	$V_3 \subset \mathbb{P}^4$	JV
Iskovskikh-Manin	some $V_4 \subset \mathbb{P}^4$	$\mathrm{Bir}(V)$
Artin-Mumford	specific	Tors $H^3(V,\mathbb{Z})$

More precisely :

• Clemens-Griffiths [C-G] proved the longstanding conjecture that a smooth cubic threefold $V_3 \subset \mathbb{P}^4$ is not rational – it had long been known that it is unirational. They showed that the *intermediate Jacobian* of V_3 is not a Jacobian (*Clemens-Griffiths criterion*, see Theorem 1 below).

• Iskovskikh-Manin [I-M] proved that any smooth quartic threefold $V_4 \subset \mathbb{P}^4$ is not rational. Some unirational quartic threefolds had been constructed by B. Segre [Sg2], so these provide counter-examples to the Lüroth problem. They showed that the group of birational automorphisms of V_4 is finite, while the corresponding group for $\mathbb{P}^3$ is huge.

• Artin-Mumford [A-M] proved that a particular double covering X of $\mathbb{P}^3$, branched along a quartic surface in $\mathbb{P}^3$ with 10 nodes, is unirational but not rational. They showed that the torsion subgroup of $H^3(X,\mathbb{Z})$ is nontrivial, and is a birational invariant.

These three papers have been extremely influential. Though they appeared around the same time, they use very different ideas; in fact, as we will see, the methods tend to apply to different types of varieties. They have been developed and extended, and applied to a number of interesting examples. Each of them has its advantages and its drawbacks; very roughly:

• The intermediate Jacobian method is quite efficient, but applies only in dimension 3;

• The computation of birational automorphisms leads to the important notion of *birational rigidity*. However it is not easy to work out; so far it applies essentially to Fano varieties of index 1 (see 2.3), which are not known to be unirational in dimension > 3.

• Torsion in H^3 gives an obstruction to a property weaker than rationality, called *stable rationality* (§5). Unfortunately it applies only to very particular varieties, and not to the standard examples of unirational varieties, like hypersurfaces or complete intersections. However we will discuss in §7 a new idea of C. Voisin which extends considerably the range of that method.

They are still essentially the basic methods to prove non-rationality results. A notable exception is the method of Kollár using reduction modulo p; however it applies only to rather specific examples, which are not known to be unirational. We will describe briefly his results in (4.2).

A final remark : at the time they were discovered the three methods used the difficult *resolution of indeterminacies* due to Hironaka. This is a good reason why the Italian algebraic geometers could not succeed! It was later realized that the birational invariance of $\operatorname{Tors} H^3(V,\mathbb{Z})$ can be proved without appealing to the resolution of singularities, see (6.4) – but this still requires some highly nontrivial algebraic apparatus.

2 The candidates

In this section we will introduce various classes of varieties which are natural candidates to be counter-examples to the Lüroth problem.

2.1 Rationality and unirationality

Let us first recall the basic definitions which appear in the Lüroth problem. We work over the complex numbers. A *variety* is an integral scheme of finite type over $\mathbb{C}$.

Definition 1. 1) A variety V is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow V$.

2) V is *rational* if there exists a birational map $\mathbb{P}^n \overset{\sim}{\dashrightarrow} V$.

In the definition of unirationality we can take $n = \dim V$: indeed, if we have a dominant rational map $\mathbb{P}^N \dashrightarrow V$, its restriction to a general linear subspace of dimension $\dim(V)$ is still dominant.

We may rephrase these definitions in terms of the function field $\mathbb{C}(V)$ of V : V is unirational if $\mathbb{C}(V)$ is contained in a purely transcendental extension of $\mathbb{C}$; V is rational if $\mathbb{C}(V)$ is a purely transcendental extension of $\mathbb{C}$. Thus the Lüroth problem asks whether every extension of $\mathbb{C}$ contained in $\mathbb{C}(t_1,\ldots,t_n)$ is purely transcendental.

2.2 Rational connectedness

Though the notion of unirationality is quite natural, it is very difficult to handle. The crucial problem is that so far there is no known method to prove non-unirationality, like the ones we mentioned in (1.3) for non-rationality.

There is a weaker notion which behaves much better than unirationality, and which covers all varieties we will be interested in :

Definition 2. A smooth projective variety V is *rationally connected* (RC for short) if any two points of V can be joined by a rational curve.

It is enough to ask that two general points of V can be joined by a rational curve, or even by a chain of rational curves. In particular, rational connectedness is a birational property.

In contrast to unirationality, rational connectedness has extremely good properties (see for instance [Ar] for proofs and references) :

$a)$ It is an open and closed property; that is, given a smooth projective morphism $f : V \to B$ with B connected, if some fiber of f is RC, *all* the fibers are RC.

$b)$ Let $f : V \dashrightarrow B$ be a rational dominant map. If B and the general fibers of f are RC, V is RC.

$c)$ If V is RC, all contravariant tensor fields vanish; that is, $H^0(V,(\Omega^1_V)^{\otimes n}) = 0$ for all n. It is conjectured that the converse holds; this is proved in dimension ≤ 3.

Neither $a)$ nor $b)$ are expected to hold when we replace rational connectedness by unirationality or rationality. For $a)$, it is expected that the general quartic threefold is *not* unirational (see [R, V.9]), though some particular V_4 are; so unirationality should not be stable under deformation. Similarly it is expected that the general cubic fourfold is not rational, though some of them are known to be rational.

Projecting a cubic threefold V_3 from a line contained in V_3 gives a rational dominant map to $\mathbb{P}^2$ whose generic fiber is a rational curve, so $b)$ does not hold for rationality. The same property holds more generally for a general hypersurface of degree d in $\mathbb{P}^4$ with a $(d-2)$-uple line; it is expected that it is not even unirational for $d \geq 5$ [R, IV.6].

2.3 Fano manifolds

A more restricted class than RC varieties is that of *Fano manifolds* – which were extensively studied by Fano in dimension 3. A smooth projective variety V is *Fano* if the anticanonical bundle K_V^{-1} is ample. This implies that V is RC; but contrary to the notions considered so far, this is not a property of the birational class of V.

A Fano variety V is called *prime* if $\mathrm{Pic}(V)=\mathbb{Z}$ (the classical terminology is "of the first species"). In that case we have $K_V=L^{-r}$, where L is the positive generator of $\mathrm{Pic}(V)$. The integer r is called the *index* of V. Prime Fano varieties are somehow minimal among RC varieties : they do not admit a Mori type contraction or morphisms to smaller-dimensional varieties.

In the following table we list what is known about rationality issues for prime Fano threefolds, using their classification by Iskovskikh [I1] : for each of them, whether it is unirational or rational, and, if it is not rational, the method of proof and the corresponding reference. The only Fano threefolds of index ≥ 3 are $\mathbb{P}^3$ and the smooth quadric $V_2\subset\mathbb{P}^4$, so we start with index 2, then 1:

variety	unirational	rational	method	reference
$V_6\subset\mathbb{P}(1,1,1,2,3)$	?	no	$\mathrm{Bir}(V)$	[Gr]
quartic double $\mathbb{P}^3$	yes	no	JV	[V1]
$V_3\subset\mathbb{P}^4$	"	no	JV	[C-G]
$V_{2,2}\subset\mathbb{P}^5$, $X_5\subset\mathbb{P}^6$	"	yes		
sextic double $\mathbb{P}^3$	?	no	$\mathrm{Bir}(V)$	[I-M]
$V_4\subset\mathbb{P}^4$	some	no	$\mathrm{Bir}(V)$	[I-M]
$V_{2,3}\subset\mathbb{P}^5$	yes	no (generic)	JV, $\mathrm{Bir}(V)$	[B1, P]
$V_{2,2,2}\subset\mathbb{P}^6$	"	no	JV	[B1]
$X_{10}\subset\mathbb{P}^7$	"	no (generic)	JV	[B1]
$X_{12},X_{16},X_{18},X_{22}$	"	yes		
$X_{14}\subset\mathbb{P}^9$	"	no	JV	[C-G] + [F3][1]

A few words about notation : as before $V_{d_1,\ldots,d_p}$ denotes a smooth complete intersection of multidegree $(d_1,\ldots,d_p)$ in $\mathbb{P}^{p+3}$, or, for the first row, in the weighted projective space $\mathbb{P}(1,1,1,2,3)$. A quartic (resp. sextic) double $\mathbb{P}^3$ is a double cover of $\mathbb{P}^3$ branched along a smooth quartic (resp. sextic) surface. The notation $X_d\subset\mathbb{P}^m$ means a smooth threefold of degree d in $\mathbb{P}^m$. The mention "(generic)" means that non-rationality is known only for those varieties belonging to a certain Zariski open subset of the moduli space.

[1] Fano proved in [F3] that the variety X_{14} is birational to a smooth cubic threefold.

2.4 Linear quotients

An important source of unirational varieties is provided by the quotients V/G, where G is an algebraic group (possibly finite) acting linearly on the vector space V. These varieties, and the question whether they are rational or not, appear naturally in various situations. The case G finite is known as the Noether problem (over $\mathbb{C}$); we will see below (6.4) that a counter-example has been given by Saltman [Sa], using an elaboration of the Artin-Mumford method. The case where G is a connected linear group appears in a number of moduli problems, but there is still no example where the quotient V/G is known to be non-rational – in fact the general expectation is that all these quotients should be rational, but this seems out of reach at the moment.

A typical case is the moduli space $\mathscr{H}_{d,n}$ of hypersurfaces of degree $d \geq 3$ in $\mathbb{P}^n$, which is birational to $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))/\mathrm{GL}_{n+1}$ – more precisely, it is the quotient of the open subset of forms defining a smooth hypersurface. For $n = 2$ the rationality is now known except for a few small values of d, see for instance [BBK] for an up-to-date summary; for $n \geq 3$ there are only a few cases where $\mathscr{H}_{d,n}$ is known to be rational. We refer to [D] for a survey of results and problems, and to [C-S] for a more recent text.

3 The intermediate Jacobian

In this section we discuss our first non-rationality criterion, using the intermediate Jacobian. Then we will give an easy example of a cubic threefold which satisfies this criterion, hence gives a counter-example to the Lüroth problem.

3.1 The Clemens-Griffiths criterion

In order to define the intermediate Jacobian, let us first recall the Hodge-theoretic construction of the Jacobian of a (smooth, projective) curve C. We start from the Hodge decomposition

$$H^1(C,\mathbb{Z}) \subset H^1(C,\mathbb{C}) = H^{1,0} \oplus H^{0,1}$$

with $H^{0,1} = \overline{H^{1,0}}$. The latter condition implies that the projection $H^1(C,\mathbb{R}) \to H^{0,1}$ is a ($\mathbb{R}$-linear) isomorphism, hence that the image Γ of $H^1(C,\mathbb{Z})$ in $H^{0,1}$ is a *lattice* (that is, any basis of Γ is a basis of $H^{0,1}$ over $\mathbb{R}$). The quotient $JC := H^{0,1}/\Gamma$ is a complex torus. But we have more structure. For $\alpha, \beta \in H^{0,1}$, put $H(\alpha,\beta) = 2i\int_C \bar{\alpha} \wedge \beta$.

Then H is a positive hermitian form on $H^{0,1}$, and the restriction of $\mathrm{Im}(H)$ to $\Gamma \cong H^1(C,\mathbb{Z})$ coincides with the cup-product

$$H^1(C,\mathbb{Z}) \otimes H^1(C,\mathbb{Z}) \to H^2(C,\mathbb{Z}) = \mathbb{Z} \ ;$$

thus it induces on Γ a skew-symmetric, integer-valued form, which is moreover *unimodular*. In other words, H is a *principal polarization* on JC (see [B-L], or [B5] for an elementary treatment). This is equivalent to the data of an ample divisor $\Theta \subset JC$ (defined up to translation) satisfying $\dim H^0(JC, \mathscr{O}_{JC}(\Theta)) = 1$. Thus (JC, Θ) is a *principally polarized abelian variety* (p.p.a.v. for short), called the *Jacobian* of C.

One can mimic this definition for higher dimensional varieties, starting from the odd degree cohomology; this defines the general notion of intermediate Jacobian. In general it is only a complex torus, not an abelian variety. But the situation is much nicer in the case of interest for us, namely rationally connected threefolds. For such a threefold V we have $H^{3,0}(V) = H^0(V, K_V) = 0$, hence the Hodge decomposition for H^3 becomes :

$$H^3(V,\mathbb{Z})_{\mathrm{tf}} \subset H^3(V,\mathbb{C}) = H^{2,1} \oplus H^{1,2}$$

with $H^{1,2} = \overline{H^{2,1}}$ ($H^3(V,\mathbb{Z})_{\mathrm{tf}}$ denotes the quotient of $H^3(V,\mathbb{Z})$ by its torsion subgroup). As above $H^{1,2}/H^3(V,\mathbb{Z})_{\mathrm{tf}}$ is a complex torus, with a principal polarization defined by the hermitian form $(\alpha, \beta) \mapsto -2i \int_V \bar{\alpha} \wedge \beta$: this is the *intermediate Jacobian JV* of V.

We will use several times the following well-known and easy lemma, see for instance [V2, Thm. 7.31] :

Lemma 1. *Let X be a complex manifold, $Y \subset X$ a closed submanifold of codimension c, $\hat{X}$ the variety obtained by blowing up X along Y. There are natural isomorphisms*

$$H^p(\hat{X},\mathbb{Z}) \xrightarrow{\sim} H^p(X,\mathbb{Z}) \oplus \sum_{k=1}^{c-1} H^{p-2k}(Y,\mathbb{Z}) \ .$$

Theorem 1 (Clemens-Griffiths criterion). *Let V be a smooth rational projective threefold. The intermediate Jacobian JV is isomorphic (as p.p.a.v.) to the Jacobian of a curve or to a product of Jacobians.*

Sketch of proof : Let $\varphi : \mathbb{P}^3 \overset{\sim}{\dashrightarrow} V$ be a birational map. Hironaka's resolution of indeterminacies provides us with a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, "b"'] \arrow[dr, "f"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\varphi"] & & V \end{tikzcd}$$

where $b : P \to \mathbb{P}^3$ is a composition of blowing up, either of points or of smooth curves, and f is a birational *morphism*.

We claim that JP is a product of Jacobians of curves. Indeed by Lemma 1, blowing up a point in a threefold V does not change $H^3(V,\mathbb{Z})$, hence does not change JV either. If we blow up a smooth curve $C \subset V$ to get a variety $\hat{V}$, Lemma 1 gives a canonical isomorphism $H^3(\hat{V},\mathbb{Z}) \cong H^3(V,\mathbb{Z}) \oplus H^1(C,\mathbb{Z})$, compatible in an appropriate sense with the Hodge decomposition and the cup-products; this implies $J\hat{V} \cong JV \times JC$ as p.p.a.v. Thus going back to our diagram, we see that JP is isomorphic to $JC_1 \times \ldots \times JC_p$, where $C_1, \ldots, C_p$ are the (smooth) curves which we have blown up in the process.

How do we go back to JV ? Now we have a birational *morphism* $f : P \to V$, so we have homomorphisms $f^* : H^3(V,\mathbb{Z}) \to H^3(P,\mathbb{Z})$ and $f_* : H^3(P,\mathbb{Z}) \to H^3(V,\mathbb{Z})$ with $f_*f^* = 1$, again compatible with the Hodge decomposition and the cup-products in an appropriate sense. Thus $H^3(V,\mathbb{Z})$, with its polarized Hodge structure, is a direct factor of $H^3(P,\mathbb{Z})$; this implies that JV is a direct factor of $JP \cong JC_1 \times \ldots \times JC_p$, in other words there exists a p.p.a.v. A such that $JV \times A \cong JC_1 \times \ldots \times JC_p$.

How can we conclude? In most categories the decomposition of an object as a product is not unique (think of vector spaces!). However here a miracle occurs. Let us say that a p.p.a.v. is *indecomposable* if it is not isomorphic to a product of nontrivial p.p.a.v.

Lemma 2. 1) *A p.p.a.v.* (A,Θ) *is indecomposable if and only if the divisor* Θ *is irreducible.*

2) *Any p.p.a.v. admits a* unique *decomposition as a product of indecomposable p.p.a.v.*

Sketch of proof : We start by recalling some classical properties of abelian varieties, for which we refer to [M]. Let D be a divisor on an abelian variety A; for $a \in A$ we denote by D_a the translated divisor $D+a$. The map $\varphi_D : a \mapsto \mathscr{O}_A(D_a - D)$ is a homomorphism from A into its dual variety $\hat{A}$, which parametrizes topologically trivial line bundles on A. If D defines a principal polarization, this map is an isomorphism.

Now suppose our p.p.a.v. (A,Θ) is a product $(A_1,\Theta_1) \times \ldots \times (A_p,\Theta_p)$. Then $\Theta = \Theta^{(1)} + \ldots + \Theta^{(p)}$, with $\Theta^{(i)} := A_1 \times \ldots \Theta_i \times \ldots \times A_p$; we recover the summand $A_i \subset A$ as $\varphi_\Theta^{-1}(\varphi_{\Theta^{(i)}}(A))$. Conversely, let (A,Θ) be a p.p.a.v., and let $\Theta^{(1)}, \ldots, \Theta^{(p)}$ be the irreducible components of Θ (each of them occurs with multiplicity one, since otherwise one would have $h^0(A;\mathscr{O}_A(\Theta)) > 1$). Putting $A_i := \varphi_\Theta^{-1}(\varphi_{\Theta^{(i)}}(A))$ and $\Theta_i := \Theta^{(i)}_{|A_i}$, it is not difficult to check that (A,Θ) is the product of the (A_i,Θ_i) – see [C-G], Lemma 3.20 for the details. $\square$

Once we have this, we conclude as follows. The Theta divisor of a Jacobian JC is the image of the Abel-Jacobi map $C^{(g-1)} \to JC$, and therefore is irreducible. From the isomorphism $JV \times A \cong JC_1 \times \ldots \times JC_p$ and the Lemma we conclude that JV is isomorphic to $JC_{i_1} \times \ldots \times JC_{i_r}$ for some subset $\{i_1, \ldots, i_r\}$ of $[1,p]$. $\square$

Remark .– One might think that products of Jacobians are more general than Jacobians, but it goes the other way around: in the moduli space $\mathscr{A}_g$ of g-dimensional p.p.a.v., the boundary $\bar{\mathscr{J}}_g \smallsetminus \mathscr{J}_g$ of the Jacobian locus is precisely the locus of products of lower-dimensional Jacobians.

3.2 The Schottky problem

Thus to show that a threefold V is not rational, it suffices to prove that its intermediate Jacobian is not the Jacobian of a curve, or a product of Jacobians. Here we come across the classical Schottky problem : the characterization of Jacobians among all p.p.a.v. (the usual formulation of the Schottky problem asks for equations of the Jacobian locus inside the moduli space of p.p.a.v.; here we are more interested in special geometric properties of Jacobians). One frequently used approach is through the singularities of the Theta divisor : the dimension of $\operatorname{Sing}(\Theta)$ is $\geq g-4$ for a Jacobian (JC,Θ) of dimension g, and $g-2$ for a product. However controlling $\operatorname{Sing}(\Theta)$ for an intermediate Jacobian is quite difficult, and requires a lot of information on the geometry of V. Let us just give a sample :

Theorem 2. *Let $V_3 \subset \mathbb{P}^4$ be a smooth cubic threefold. The divisor $\Theta \subset JV_3$ has a unique singular point p, which is a triple point. The tangent cone $\mathbb{P}T_p(\Theta) \subset \mathbb{P}T_p(JV_3) \cong \mathbb{P}^4$ is isomorphic to V_3.*

This elegant result, apparently due to Mumford (see [B2] for a proof), implies both the non-rationality of V_3 (because $\dim \operatorname{Sing}(\Theta) = 0$ and $\dim JV_3 = 5$) and the *Torelli theorem* : the cubic V_3 can be recovered from its (polarized) intermediate Jacobian.

There are actually few cases where we can control so well the singular locus of the Theta divisor. One of these is the quartic double solid, for which $\operatorname{Sing}(\Theta)$ has a component of codimension 5 in JV [V1]. Another case is that of *conic bundles*, that is, threefolds V with a flat morphism $p : V \to \mathbb{P}^2$, such that for each closed point $s \in \mathbb{P}^2$ the fiber $p^{-1}(s)$ is isomorphic to a plane conic (possibly singular). In that case JV is a *Prym variety*, associated to a natural double covering of the *discriminant curve* $\Delta \subset \mathbb{P}^2$ (the locus of $s \in \mathbb{P}^2$ such that $p^{-1}(s)$ is singular). Thanks to Mumford we have some control on the singularities of the Theta divisor of a Prym variety, enough to show that JV *is not a Jacobian (or a product of Jacobians) if* $\deg(\Delta) \geq 6$ [B1, thm. 4.9].

Unfortunately, apart from the cubic, the only prime Fano threefold to which this result applies is the $V_{2,2,2}$ in $\mathbb{P}^6$. However, the Clemens-Griffiths criterion of non-rationality is an *open* condition. In fact, we have a stronger result, which follows from the properties of the *Satake compactification* of $\mathscr{A}_g$ [B1, lemme 5.6.1] :

Lemma 3. *Let* $\pi : V \to B$ *be a flat family of projective threefolds over a smooth curve* B*. Let* $\mathrm{o} \in B$*; assume that :*

- *The fiber* V_b *is smooth for* $b \neq \mathrm{o}$*;*
- V_{o} *has only ordinary double points;*
- *For a desingularization* $\tilde{V}_{\mathrm{o}}$ *of* V_{o}*,* $J\tilde{V}_{\mathrm{o}}$ *is not a Jacobian or a product of Jacobians.*

Then for b *outside a finite subset of* B*,* V_b *is not rational.*

From this we deduce the generic non-rationality statements of (2.3) [B1, Thm. 5.6] : in each case one finds a degeneration as in the Lemma, such that $\tilde{V}_{\mathrm{o}}$ is a conic bundle with a discriminant curve of degree ≥ 6, hence the Lemma applies.

3.3 An easy counter-example

The results of the previous section require rather involved methods. We will now discuss a much more elementary approach, which unfortunately applies only to specific varieties.

Theorem 3. *The cubic threefold* $V \subset \mathbb{P}^4$ *defined by* $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$ *is not rational.*

Proof : Let us first prove that JV is not a Jacobian. Let ζ be a primitive 11-th root of unity. The key point is that V admits the automorphisms
$\delta : (X_0, X_1, X_2, X_3, X_4) \mapsto (X_0, \zeta X_1, \zeta^{-1} X_2, \zeta^3 X_3, \zeta^6 X_4)$,
$\sigma : (X_0, X_1, X_2, X_3, X_4) \mapsto (X_1, X_2, X_3, X_4, X_0)$,
which satisfy $\delta^{11} = \sigma^5 = 1$ and $\sigma \delta \sigma^{-1} = \delta^{-2}$.

They induce automorphisms δ^*, σ^* of JV. Suppose that JV is isomorphic (as p.p.a.v.) to the Jacobian JC of a curve C. The Torelli theorem for curves gives an exact sequence

$$1 \to \mathrm{Aut}(C) \to \mathrm{Aut}(JC) \to \mathbb{Z}/2 \;;$$

since δ^* and σ^* have odd order, they are induced by automorphisms δ_C, σ_C of C, satisfying $\sigma_C \delta_C \sigma_C^{-1} = \delta_C^{-2}$.

Now we apply the Lefschetz fixed point formula. The automorphism δ of V fixes the 5 points corresponding to the basis vectors of $\mathbb{C}^5$; it acts trivially on $H^{2i}(V, \mathbb{Q})$ for $i = 0, \ldots, 3$. Therefore we find $\mathrm{Tr}\, \delta^*_{|H^3(V,\mathbb{Q})} = -5 + 4 = -1$. Similarly σ fixes the 4 points $(1, \alpha, \alpha^2, \alpha^3, \alpha^4)$ of V with $\alpha^5 = 1$, $\alpha \neq 1$, so $\mathrm{Tr}\, \sigma^*_{|H^3(V,\mathbb{Q})} = -4 + 4 = 0$.

Applying now the Lefschetz formula to C, we find that σ_C has two fixed points on C and δ_C three. But since σ_C normalizes the subgroup generated by δ_C, it preserves the 3-points set $\mathrm{Fix}(\delta_C)$; since it is of order 5, it must fix each of these 3 points, which gives a contradiction.

Finally suppose JV is isomorphic to a product $A_1 \times \ldots \times A_p$ of p.p.a.v. By the unicity lemma (Lemma 2), the automorphism δ^* permutes the factors A_i. Since δ has order 11 and $p \leq 5$, this permutation must be trivial, so δ^* induces an automorphism of A_i for each i, hence of $H^1(A_i,\mathbb{Q})$; but the group $\mathbb{Z}/11$ has only one nontrivial irreducible representation defined over $\mathbb{Q}$, given by the cyclotomic field $\mathbb{Q}(\zeta)$, with $[\mathbb{Q}(\zeta):\mathbb{Q}] = 10$. Since $\dim(A_i) < 5$ we see that the action of δ^* on each A_i, and therefore on JV, is trivial. But this contradicts the relation $\operatorname{Tr} \delta^*_{|H^3(V,\mathbb{Q})} = -1$. $\square$

Remarks.— 1) The cubic V is the *Klein cubic threefold*; it is birational to the moduli space of abelian surfaces with a polarization of type $(1,11)$ [G-P]. In particular it admits an action of the group $\mathrm{PSL}_2(\mathbb{F}_{11})$ of order 660, which is in fact its automorphism group [A]. From this one could immediately conclude by using the Hurwitz bound $\#\mathrm{Aut}(C) \leq 84(g(C)-1)$ (see [B4]).

2) This method applies to other threefolds for which the non-rationality was not previously known, in particular the $\mathfrak{S}_7$-symmetric $V_{2,3}$ given by $\sum X_i = \sum X_i^2 = \sum X_i^3 = 0$ in $\mathbb{P}^6$ [B4] or the $\mathfrak{S}_6$-symmetric V_4 with 30 nodes given by $\sum X_i = \sum X_i^4 = 0$ in $\mathbb{P}^5$ [B6].

4 Two other methods

In this section we will briefly present two other ways to get non-rationality results for certain Fano varieties. Let us stress that in dimension ≥ 4 these varieties are not known to be unirational, so these methods do not give us new counter-examples to the Lüroth problem.

4.1 Birational rigidity

As mentioned in the introduction, Iskovskikh and Manin proved that a smooth quartic threefold $V_4 \subset \mathbb{P}^4$ is not rational by proving that any birational automorphism of V_4 is actually biregular. But they proved much more, namely that V_4 is *birationally superrigid* in the following sense :

Definition 3. Let V be a prime Fano variety (2.3). We say that V is *birationally rigid* if :

a) There is no rational dominant map $V \dashrightarrow S$ with $0 < \dim(S) < \dim(V)$ and with general fibers of Kodaira dimension $-\infty$;

b) If V is birational to another prime Fano variety W, then V is isomorphic to W.

We say that V is *birationally superrigid* if any birational map $V \overset{\sim}{\dashrightarrow} W$ as in $b)$ is an isomorphism.

(The variety W in $b)$ is allowed to have certain mild singularities, the so-called $\mathbb{Q}$-factorial terminal singularities.)

After the pioneering work [I-M], birational (super)rigidity has been proved for a number of Fano varieties of index 1. Here is a sample; we refer to the surveys [P] and [Ch] for ideas of proofs and for many more examples.

• Any smooth hypersurface of degree n in $\mathbb{P}^n$ is birationally superrigid [dF].

• A general $V_{2,3}$ in $\mathbb{P}^5$ is birationally rigid. It is not birationally superrigid, since it contains a curve of lines, and each line defines by projection a 2-to-1 map to $\mathbb{P}^3$, hence a birational involution of $V_{2,3}$.

• A general $V_{d_1,\ldots,d_c}$ in $\mathbb{P}^n$ of index 1 (that is, $\sum d_i = n$) with $n > 3c$ is birationally superrigid.

• A double cover of $\mathbb{P}^n$ branched along a smooth hypersurface of degree $2n$ is birationally superrigid.

4.2 Reduction to characteristic p

Theorem 4. [K] *For $d \geq 2\lceil \frac{n+3}{3} \rceil$, a very general hypersurface $V_d \subset \mathbb{P}^{n+1}$ is not ruled, and in particular not rational.*

A variety is *ruled* if it is birational to $W \times \mathbb{P}^1$ for some variety W. “Very general” means that the corresponding point in the space parametrizing our hypersurfaces lies outside a countable union of strict closed subvarieties.

The bound $d \geq 2\lceil \frac{n+3}{3} \rceil$ has been lowered to $d \geq 2\lceil \frac{n+2}{3} \rceil$ by Totaro [T]; this implies in particular that a very general quartic fourfold is not rational. More important, by combining Kollár’s method with a new idea of Voisin (see §7), Totaro shows that a very general $V_d \subset \mathbb{P}^{n+1}$ with d as above is not *stably rational* (§5).

Let us give a very rough idea of Kollár’s proof, in the case d is even. It starts from the well-known fact that the hypersurface V_d specializes to a double covering Y of a hypersurface of degree $d/2$. This can be still done in characteristic 2, at the price of getting some singularities on Y, which must be resolved. The reward is that the resolution Y' of Y has a very unstable tangent bundle; more precisely, $\Omega^{n-1}_{Y'}$ $(\cong T_{Y'} \otimes K_{Y'})$ contains a positive line bundle, and this prevents Y' to be ruled. Then a general result of Matsusaka implies that a very general V_d cannot be ruled.

5 Stable rationality

There is an intermediate notion between rationality and unirationality which turns out to be important :

Definition 4. A variety V is *stably rational* if $V \times \mathbb{P}^n$ is rational for some $n \geq 0$.

In terms of field theory, this means that $\mathbb{C}(V)(t_1, \ldots, t_n)$ is a purely transcendantal extension of $\mathbb{C}$.

Clearly, rational $\Rightarrow$ stably rational $\Rightarrow$ unirational. We will see that these implications are strict. For the first one, we have :

Theorem 5. [BCSS] *Let $P(x,t) = x^3 + p(t)x + q(t)$ be an irreducible polynomial in $\mathbb{C}[x,t]$, whose discriminant $\delta(t) := 4p(t)^3 + 27q(t)^2$ has degree ≥ 5. The affine hypersurface $V \subset \mathbb{C}^4$ defined by $y^2 - \delta(t)z^2 = P(x,t)$ is stably rational but not rational.*

This answered a question asked by Zariski in 1949 [Sg1].

The non-rationality of V is proved using the intermediate Jacobian, which turns out to be the Prym variety associated to an admissible double covering of nodal curves. The stable rationality, more precisely the fact that $V \times \mathbb{P}^3$ is rational, was proved in [BCSS] using some particular torsors under certain algebraic tori. A slightly different approach due to Shepherd-Barron shows that actually $V \times \mathbb{P}^2$ is rational [SB]; we do not know whether $V \times \mathbb{P}^1$ is rational.

To find unirational varieties which are not stably rational, we cannot use the Clemens-Griffiths criterion since it applies only in dimension 3. The group of birational automorphisms is very complicated for a variety of the form $V \times \mathbb{P}^n$; so the only available method is the torsion of $H^3(V, \mathbb{Z})$ and its subsequent refinements, which we will examine in the next sections.

Remark.— There are other notions lying between unirationality and rationality. Let us say that a variety V is

- retract rational if there exists a rational dominant map $\mathbb{P}^N \dashrightarrow V$ which admits a rational section;
- factor-rational if there exists another variety V' such that $V \times V'$ is rational.

We have the implications :

rational $\Rightarrow$ stably rational $\Rightarrow$ factor-rational $\Rightarrow$ retract rational $\Rightarrow$ unirational.

Unfortunately at the moment we have no examples (even conjectural) of varieties which are retract rational but not stably rational. For this reason we will focus on the stable rationality, which seems at this time the most useful of these notions. Indeed we will see now that there are some classes of linear quotients V/G (see 2.4) for which we can prove stable rationality.

Let G be a reductive group acting on a variety V. We say that the action is *almost free* if there is a nonempty Zariski open subset U of V such that the stabilizer of each point of U is trivial.

Proposition 1. *Suppose that there exists an almost free linear representation V of G such that the quotient V/G is rational. Then for every almost free representation W of G, the quotient W/G is stably rational.*

The proof goes as follows [D] : let V^{o} be a Zariski open subset of V where G acts freely. Consider the diagonal action of G on $V^{\mathrm{o}} \times W$; standard arguments (the "no-name lemma") show that the projection $(V^{\mathrm{o}} \times W)/G \to V^{\mathrm{o}}/G$ defines a vector bundle over V^{o}/G. Thus $(V \times W)/G$ is birational to $(V/G) \times W$ (which is a rational variety), and symmetrically to $V \times (W/G)$, so W/G is stably rational. □

For many groups it is easy to find an almost free representation with rational quotient : this is the case for instance for a subgroup G of GL_n such that the quotient $\mathrm{GL_n}/G$ is rational (use the linear action of GL_n on $\mathrm{M}_n(\mathbb{C})$ by multiplication). This applies to GL_n, SL_n, O_n ($\mathrm{GL}_n/\mathrm{O}_n$ is the space of non-degenerate quadratic forms), SO_n, Sp_n etc.

This gives many examples of stably rational varieties. For instance, the moduli space $\mathscr{H}_{d,n}$ of hypersurfaces of degree d in $\mathbb{P}^n$ (2.4) is stably rational when $d \equiv 1$ mod. $(n+1)$: the standard representation ρ of GL_{n+1} on $H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ is not almost free, but the representation $\rho \otimes \det^k$, with $k = \frac{1-d}{n+1}$, is almost free and gives the same quotient.

6 The torsion of $H^3(V,\mathbb{Z})$ and the Brauer group

6.1 Birational invariance

Artin and Mumford used the following property of stably rational varieties :

Proposition 2. *Let V be a stably rational projective manifold. Then $H^3(V,\mathbb{Z})$ is torsion free.*

Proof : The Künneth formula gives an isomorphism

$$H^3(V \times \mathbb{P}^m, \mathbb{Z}) \cong H^3(V,\mathbb{Z}) \oplus H^1(V,\mathbb{Z}) ;$$

since $H^1(V,\mathbb{Z})$ is torsion free the torsion subgroups of $H^3(V,\mathbb{Z})$ and $H^3(V \times \mathbb{P}^m, \mathbb{Z})$ are isomorphic, hence replacing V by $V \times \mathbb{P}^m$ we may assume that V is rational. Let $\varphi : \mathbb{P}^n \overset{\sim}{\dashrightarrow} V$ be a birational map. As in the proof of the Clemens-Griffiths criterion, we have Hironaka's "little roof"

$$\begin{tikzcd} & P \arrow[dl, "b"'] \arrow[dr, "f"] & \\ \mathbb{P}^n \arrow[rr, dashed, "\varphi"] & & V \end{tikzcd}$$

where $b : P \to \mathbb{P}^n$ is a composition of blowing up of smooth subvarieties, and f is a birational *morphism*.

By Lemma 1, we have $H^3(P,\mathbb{Z}) \cong H^1(Y_1,\mathbb{Z}) \oplus \ldots \oplus H^1(Y_p,\mathbb{Z})$, where $Y_1,\ldots,Y_p$ are the subvarieties successively blown up by b; therefore $H^3(P,\mathbb{Z})$ is torsion free. As in the proof of Theorem 1, $H^3(V,\mathbb{Z})$ is a direct summand of $H^3(P,\mathbb{Z})$, hence is also torsion free. □

We will indicate below (6.4) another proof which does not use Hironaka's difficult theorem.

6.2 The Brauer group

The torsion of $H^3(V,\mathbb{Z})$ is strongly related to the *Brauer group* of V. There is a huge literature on the Brauer group in algebraic geometry, starting with the three "exposés" by Grothendieck in [G]. We recall here the cohomological definition(s) of this group; we refer to [G] for the relation with Azumaya algebras.

Proposition 3. *Let V be a smooth variety. The following definitions are equivalent, and define the* Brauer group *of V :*
(i) $\mathrm{Br}(V) = \mathrm{Coker}\, c_1 : \mathrm{Pic}(V) \otimes \mathbb{Q}/\mathbb{Z} \to H^2(V,\mathbb{Q}/\mathbb{Z})$*;*
(ii) $\mathrm{Br}(V) = H^2_{\text{ét}}(V,\mathbb{G}_m)$ *(étale cohomology).*

Proof : Let $n \in \mathbb{N}$. The exact sequence of étale sheaves $1 \to \mathbb{Z}/n \to \mathbb{G}_m \xrightarrow{\times n} \mathbb{G}_m \to 1$ gives a cohomology exact sequence

$$0 \to \mathrm{Pic}(V) \otimes \mathbb{Z}/n \xrightarrow{c_1} H^2(V,\mathbb{Z}/n) \longrightarrow \mathrm{Br}(V) \xrightarrow{\times n} \mathrm{Br}(V) \ .$$

(Note that the étale cohomology $H^*_{\text{ét}}(V,\mathbb{Z}/n)$ is canonically isomorphic to the classical cohomology).

Taking the direct limit with respect to n gives an exact sequence

$$0 \to \mathrm{Pic}(V) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{c_1} H^2(V,\mathbb{Q}/\mathbb{Z}) \longrightarrow \mathrm{Tors}\,\mathrm{Br}(V) \to 0 \ ; \qquad (1)$$

it is not difficult to prove that $\mathrm{Br}(V)$ is a torsion group [G, II, Prop. 1.4], hence the equivalence of the definitions (i) and (ii). □

Remark.– If V is compact, the same argument shows that $\mathrm{Br}(V)$ is also isomorphic to the torsion subgroup of $H^2(V,\mathscr{O}_h^*)$, where $\mathscr{O}_h$ is the sheaf of holomorphic functions on V (for the classical topology).

Proposition 4. *There is a surjective homomorphism* $\mathrm{Br}(V)\to\mathrm{Tors}\,H^3(V,\mathbb{Z})$*, which is bijective if* $c_1:\mathrm{Pic}(V)\to H^2(V,\mathbb{Z})$ *is surjective.*

The latter condition is satisfied in particular if V is projective and $H^2(V,\mathscr{O}_V)=0$.

Proof : The exact sequence $0\to\mathbb{Z}\to\mathbb{Q}\to\mathbb{Q}/\mathbb{Z}\to 0$ gives a cohomology exact sequence

$$0\to H^2(V,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z}\longrightarrow H^2(V,\mathbb{Q}/\mathbb{Z})\longrightarrow \mathrm{Tors}\,H^3(V,\mathbb{Z})\to 0\ .$$

Together with (1) we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(V)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "c_1"'] & H^2(V,\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, equal] & \mathrm{Br}(V) \arrow[r] & 0 \\
0 \arrow[r] & H^2(V,\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] & H^2(V,\mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{Tors}\,H^3(V,\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

which implies the Proposition. $\square$

We will now describe a geometric way to construct nontrivial elements of the Brauer group.

Definition 5. Let V be a complex variety. A $\mathbb{P}^m$*-bundle* over V is a smooth map $p:P\to V$ whose geometric fibers are isomorphic to $\mathbb{P}^m$.

An obvious example is the projective bundle $\mathbb{P}_V(E)$ associated to a vector bundle E of rank $m+1$ on V; we will actually be interested in those $\mathbb{P}^m$-bundles which are *not* projective.

It is not difficult to see that a $\mathbb{P}^m$-bundle is locally trivial for the étale topology. This implies that isomorphism classes of $\mathbb{P}^{n-1}$-bundles over V are parametrized by the étale cohomology set $H^1(V,PGL_n)$, where for an algebraic group G we denote by G the sheaf of local maps to G. The exact sequence of sheaves of groups

$$1\to\mathbb{G}_m\to GL_n\to PGL_n\to 1$$

gives rise to a sequence of pointed sets

$$H^1(V,GL_n)\xrightarrow{\pi}H^1(V,PGL_n)\xrightarrow{\partial}H^2(V,\mathbb{G}_m)$$

which is exact in the sense that $\partial^{-1}(1)=\mathrm{Im}\,\pi$. Thus ∂ associates to each $\mathbb{P}^{n-1}$-bundle $p:P\to V$ a class in $H^2(V,\mathbb{G}_m)$, which is trivial if and only if p is a projective bundle. Moreover, by comparing with the exact sequence

$0 \to \mathbb{Z}/n \to SL_n \to PGL_n \to 1$ we get a commutative diagram

$$\begin{tikzcd}
H^1(V,SL_n) \arrow[r] \arrow[d] & H^1(V,PGL_n) \arrow[r] \arrow[d, equal] & H^2(V,\mathbb{Z}/n) \arrow[d] \\
H^1(V,GL_n) \arrow[r] & H^1(V,PGL_n) \arrow[r, "\partial"] & H^2(V,\mathbb{G}_m)
\end{tikzcd}$$

which shows that the image of ∂ is contained in the n-torsion subgroup of $\mathrm{Br}(V)$.

6.3 The Artin-Mumford example

The Artin-Mumford counter-example is a double cover of $\mathbb{P}^3$ branched along a *quartic symmetroid*, that is, a quartic surface defined by the vanishing of a symmetric determinant.

We start with a web Π of quadrics in $\mathbb{P}^3$; its elements are defined by quadratic forms $\lambda_0 q_0 + \ldots + \lambda_3 q_3$. We assume that :

(i) Π is base point free;

(ii) If a line in $\mathbb{P}^3$ is singular for a quadric of Π, it is not contained in another quadric of Π.

Let $\Delta \subset \Pi$ be the discriminant locus, corresponding to quadrics of rank ≤ 3. It is a quartic surface (defined by $\det(\sum \lambda_i q_i) = 0$); under our hypotheses, its has 10 ordinary double points, corresponding to quadrics of rank 2, and no other singularity (see for instance [Co]). Let $\pi : V' \to \Pi$ be the double covering branched along Δ. Again V' has 10 ordinary double points; blowing up these points we obtain the Artin-Mumford threefold V.

Observe that a quadric $q \in \Pi$ has two systems of generatrices (= lines contained in q) if $q \in \Pi \smallsetminus \Delta$, and one if $q \in \Delta \smallsetminus \mathrm{Sing}(\Delta)$. Thus the smooth part V^{o} of V parametrizes pairs (q,λ), where $q \in \Pi$ and λ is a family of generatrices of q.

Theorem 6. *The threefold V is unirational but not stably rational.*

Proof : Let $\mathbb{G}$ be the Grassmannian of lines in $\mathbb{P}^3$. A general line is contained in a unique quadric of Π, and in a unique system of generatrices of this quadric; this defines a dominant rational map $\gamma : \mathbb{G} \dashrightarrow V'$, thus V is unirational. We will deduce from Proposition 2 that V is not stably rational, by proving that $H^3(V,\mathbb{Z})$ contains an element of order 2. This is done by a direct calculation in [A-M] and, with a different method, in [B3]; here we will use a more elaborate approach based on the Brauer group.

Consider the variety $P \subset \mathbb{G} \times \Pi$ consisting of pairs (ℓ, q) with $\ell \subset q$. The projection $P \to \Pi$ factors through a morphism $p' : P \to V'$. Put $V^{\mathrm{o}} := V' \smallsetminus \mathrm{Sing}(V')$, and $P^{\mathrm{o}} := p'^{-1}(V^{\mathrm{o}})$. The restriction $p : P^{\mathrm{o}} \to V^{\mathrm{o}}$ is a $\mathbb{P}^1$-bundle: a point of V^{o} is a pair (q,σ), where q is a quadric in Π and σ a system of generatrices of q; the fiber $p^{-1}(q,\sigma)$ is the smooth rational curve parametrizing the lines of σ.

Proposition 5. *The $\mathbb{P}^1$-bundle $p : P^{\mathrm{o}} \to V^{\mathrm{o}}$ does not admit a rational section.*

Proof : Suppose it does. For a general point q of Π, the section maps the two points of $\pi^{-1}(q)$ to two generatrices of the quadric q, one in each system. These two generatrices intersect in one point $s(q)$ of q; the map $q \mapsto s(q)$ is a rational section of the universal family of quadrics $\mathscr{Q} \to \Pi$, defined by $\mathscr{Q} := \{(q,x) \in \Pi \times \mathbb{P}^3 \mid x \in q\}$. This contradicts the following lemma:

Lemma 4. *Let $\Pi \subset \mathbb{P}(H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$ be a base point free linear system of hypersurfaces, of degree $d \geq 2$. Consider the universal family $p : \mathscr{H} \to \Pi$, with $\mathscr{H} := \{(h,x) \in \Pi \times \mathbb{P}^n \mid x \in h\}$. Then p has no rational section.*

Proof : Since Π is base point free, the second projection $q : \mathscr{H} \to \mathbb{P}^n$ is a projective bundle, hence $\mathscr{H}$ is smooth. If p has a rational section, the closure $Z \subset \mathscr{H}$ of its image gives a cohomology class $[Z] \in H^{2n-2}(\mathscr{H},\mathbb{Z})$ such that $p_*([Z]) = 1$ in $H^0(\Pi,\mathbb{Z})$. Let us show that this is impossible.

We have $\dim(\Pi) \geq n$, hence $2n-2 < n-1+\dim(\Pi) = \dim(\mathscr{H})$. By the Lefschetz hyperplane theorem, the restriction map $H^{2n-2}(\Pi \times \mathbb{P}^n,\mathbb{Z}) \to H^{2n-2}(\mathscr{H},\mathbb{Z})$ is an isomorphism. Thus $H^{2n-2}(\mathscr{H},\mathbb{Z})$ is spanned by the classes $p^*h_\Pi^i \cdot q^*h_{\mathbb{P}}^{n-1-i}$ for $0 \leq i \leq n-1$, where h_Π and $h_{\mathbb{P}}$ are the hyperplane classes. All these classes go to 0 under p_* except $q^*h_{\mathbb{P}}^{n-1}$, whose degree on each fiber is d. Thus the image of $p_* : H^{2n-2}(\mathscr{H},\mathbb{Z}) \to H^0(\Pi,\mathbb{Z}) = \mathbb{Z}$ is $d\mathbb{Z}$. This proves the lemma, hence the Proposition. □

Thus the $\mathbb{P}^1$-bundle p over V^{o} is not a projective bundle, hence gives a nonzero 2-torsion class in $\mathrm{Br}(V^{\mathrm{o}})$. In the commutative diagram

$$\begin{tikzcd}
\mathrm{Pic}(V) \arrow[r, "c_1"] \arrow[d] & H^2(V,\mathbb{Z}) \arrow[d, "r"] \\
\mathrm{Pic}(V^{\mathrm{o}}) \arrow[r, "c_1"] & H^2(V^{\mathrm{o}},\mathbb{Z})
\end{tikzcd}$$

the top horizontal arrow is surjective because $H^2(V,\mathscr{O}_V) = 0$. Since $Q := V \smallsetminus V^{\mathrm{o}}$ is a disjoint union of quadrics, the Gysin exact sequence

$$H^2(V,\mathbb{Z}) \xrightarrow{r} H^2(V^{\mathrm{o}},\mathbb{Z}) \to H^1(Q,\mathbb{Z}) = 0$$

shows that r is surjective. Therefore the map $c_1 : \mathrm{Pic}(V^{\mathrm{o}}) \to H^2(V^{\mathrm{o}},\mathbb{Z})$ is surjective, and by Proposition 4 we get a nonzero 2-torsion class in $H^3(V^{\mathrm{o}},\mathbb{Z})$. Using again the Gysin exact sequence

$$0 \to H^3(V,\mathbb{Z}) \to H^3(V^{\mathrm{o}},\mathbb{Z}) \to H^2(Q,\mathbb{Z})$$

we find that $\operatorname{Tors} H^3(V,\mathbb{Z})$ is isomorphic to $\operatorname{Tors} H^3(V^{\mathrm{o}},\mathbb{Z})$, hence nonzero. $\square$

6.4 The unramified Brauer group

An advantage of the group $\mathrm{Br}(V)$ is that it can be identified with the *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V))$, which is defined purely in terms of the field $\mathbb{C}(V)$; this gives directly its birational invariance, without using Hironaka's theorem. Let us explain briefly how this works.

Proposition 6. *Let V be a smooth projective variety, and $\mathscr{D}$ be the set of integral divisors on V. There is an exact sequence*

$$0 \to \mathrm{Br}(V) \to \varinjlim_{U} \mathrm{Br}(U) \to \bigoplus_{D\in\mathscr{D}} H^1_{\text{ét}}(\mathbb{C}(D),\mathbb{Q}/\mathbb{Z})$$

where the direct limit is taken over the set of Zariski open subsets $U \subset V$.

Proof : Let D be an effective reduced divisor on V, and let $U = V \smallsetminus D$. Since $\mathrm{Sing}(D)$ has codimension ≥ 2 in V, the restriction map

$$H^2(V,\mathbb{Q}/\mathbb{Z}) \to H^2(V \smallsetminus \mathrm{Sing}(D),\mathbb{Q}/\mathbb{Z})$$

is an isomorphism. Thus, putting $D_{sm} := D \smallsetminus \mathrm{Sing}(D)$, we can write part of the Gysin exact sequence as

$$H^0(D_{sm},\mathbb{Q}/\mathbb{Z}) \to H^2(V,\mathbb{Q}/\mathbb{Z}) \to H^2(U,\mathbb{Q}/\mathbb{Z}) \to H^1(D_{sm},\mathbb{Q}/\mathbb{Z})\ .$$

Comparing with the analogous exact sequence for Picard groups gives a commutative diagram

$$\begin{tikzcd}
H^0(D_{sm},\mathbb{Z})\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "\wr"] & \mathrm{Pic}(V)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & \mathrm{Pic}(U)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
H^0(D_{sm},\mathbb{Q}/\mathbb{Z}) \arrow[r] & H^2(V,\mathbb{Q}/\mathbb{Z}) \arrow[r] & H^2(U,\mathbb{Q}/\mathbb{Z}) \arrow[r] & H^1(D_{sm},\mathbb{Q}/\mathbb{Z})
\end{tikzcd}$$

from which we get an exact sequence $0 \to \mathrm{Br}(V) \to \mathrm{Br}(U) \to H^1(D_{sm},\mathbb{Q}/\mathbb{Z})$.

Let $D_1,\ldots,D_k$ be the irreducible components of D_{sm}; we have $H^1(D_{sm},\mathbb{Q}/\mathbb{Z}) = \oplus H^1(D_{sm}\cap D_i,\mathbb{Q}/\mathbb{Z})$, and the group $H^1(D_{sm}\cap D_i,\mathbb{Q}/\mathbb{Z})$ embeds into the étale cohomology group $H^1_{\text{ét}}(\mathbb{C}(D_i),\mathbb{Q}/\mathbb{Z})$. Thus we can write our exact sequence

$$0 \to \mathrm{Br}(V) \to \mathrm{Br}(U) \to \bigoplus_i H^1_{\text{ét}}(\mathbb{C}(D_i),\mathbb{Q}/\mathbb{Z})\ .$$

Passing to the limit over D gives the Proposition. $\square$

Let K be a field. For each discrete valuation ring (DVR) R with quotient field K and residue field κ_R, there is a natural exact sequence [G, III, Prop. 2.1] :

$$0 \to \mathrm{Br}(R) \to \mathrm{Br}(K) \xrightarrow{\rho_R} H^1_{\text{ét}}(\kappa_R, \mathbb{Q}/\mathbb{Z}) \ .$$

The group $\mathrm{Br}_{\mathrm{nr}}(K)$ is defined as the intersection of the subgroups $\mathrm{Ker}\ \rho_R$, where R runs through all DVR with quotient field K.

Now consider the exact sequence of Proposition 6. The group $\varinjlim_U \mathrm{Br}(U)$ can be identified with the Brauer group $\mathrm{Br}(\mathbb{C}(V))$, and the homomorphism $\mathrm{Br}(\mathbb{C}(V)) \to H^1_{\text{ét}}(\mathbb{C}(D), \mathbb{Q}/\mathbb{Z})$ coincides with the homomorphism $\rho_{\mathscr{O}_{V,D}}$ associated to the DVR $\mathscr{O}_{V,D}$. Thus we have $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V)) \subset \mathrm{Br}(V)$. But if R is any DVR with quotient field $\mathbb{C}(V)$, the inclusion $\mathrm{Spec}\,\mathbb{C}(V) \hookrightarrow V$ factors as $\mathrm{Spec}\,\mathbb{C}(V) \hookrightarrow \mathrm{Spec}\,R \to V$ by the valuative criterion of properness, hence $\mathrm{Br}(V)$ is contained in the image of $\mathrm{Br}(R)$ in $\mathrm{Br}(K)$, that is, in $\mathrm{Ker}\ \rho_R$. Thus we have $\mathrm{Br}(V) = \mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V))$ as claimed.

The big advantage of working with $\mathrm{Br}_{\mathrm{nr}}(K)$ is that to compute it, we do not need to find a smooth projective model of the function field K. This was used first by Saltman to give his celebrated counter-example to the Noether problem [Sa] : there exists a finite group G and a linear representation V of G such that the variety V/G is not rational. In such a situation Bogomolov has given a very explicit formula for $\mathrm{Br}_{\mathrm{nr}}(\mathbb{C}(V/G))$ in terms of the Schur multiplier of G [Bo].

The idea of using the unramified Brauer group to prove non-rationality results has been extended to higher *unramified cohomology groups*, starting with the paper [C-O]. We refer to [C] for a survey about these more general invariants.

7 The Chow group of 0-cycles

In this section we discuss another property of (stably) rational varieties, namely the fact that their Chow group CH_0 parametrizing 0-cycles is *universally trivial*. While the idea goes back to the end of the 70's (see [Bl]), its use for rationality questions is recent [V4].

This property implies that $H^3(X, \mathbb{Z})$ is torsion free, but not conversely. Moreover it behaves well under deformation, even if we accept mild singularities (Proposition 9 below).

In this section we will need to work over non-algebraically closed fields (of characteristic 0). We use the language of schemes.

Let X be a smooth algebraic variety over a field k, of dimension n. Recall that the Chow group $CH^p(X)$ is the group of codimension p cycles on X modulo linear equivalence. More precisely, let us denote by $\Sigma^p(X)$ the set of codimension p

closed integral subvarieties of X. Then $CH^p(X)$ is defined by the exact sequence

$$\bigoplus_{W\in\Sigma^{p-1}(X)} k(W)^* \longrightarrow \mathbb{Z}^{(\Sigma^p(X))} \longrightarrow CH^p(X) \to 0\ , \tag{2}$$

where the first arrow associates to $f\in k(W)^*$ its divisor [Fu, 1.3].

We will be particularly interested in the group $CH_0(X) := CH^n(X)$ of 0-cycles. Associating to a 0-cycle $\sum n_i[p_i]$ $(n_i\in\mathbb{Z}, p_i\in X)$ the number $\sum n_i[k(p_i):k]$ defines a homomorphism $\deg : CH_0(X)\to\mathbb{Z}$. We denote its kernel by $CH_0(X)_0$.

Proposition 7. *Let X be a smooth complex projective variety, of dimension n, and let $\Delta_X\subset X\times X$ be the diagonal. The following conditions are equivalent :*

(i) *For every extension $\mathbb{C}\to K$, $CH_0(X_K)_0 = 0$;*

(ii) $CH_0(X_{\mathbb{C}(X)})_0 = 0$;

(iii) *There exists a point $x\in X$ and a nonempty Zariski open subset $U\subset X$ such that $\Delta_X - X\times\{x\}$ restricts to 0 in $CH(U\times X)$;*

(iv) *there exists a point $x\in X$, a smooth projective variety T of dimension $<n$ (not necessarily connected), a generically injective map $i:T\to X$, and a cycle class $\alpha\in CH(T\times X)$ such that*

$$\Delta_X - X\times\{x\} = (i\times 1)_*\alpha \quad \textit{in}\ \ CH(X\times X)\ . \tag{3}$$

*When these properties hold, we say that X is CH_0-*trivial.

Proof : The implication (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii) : Let η be the generic point of X. The point (η,η) of $\{\eta\}\times X = X_{\mathbb{C}(X)}$ is rational (over $\mathbb{C}(X)$), hence is linearly equivalent to (η,x) for any closed point $x\in X$. The class $\Delta_X - X\times\{x\}$ restricts to $(\eta,\eta)-(\eta,x)$ in $CH_0(\eta\times X)$, hence to 0. We want to show that this implies (iii).

An element of $\Sigma^p(\eta\times X)$ extends to an element of $\Sigma^p(U\times X)$ for some Zariski open subset U of X; in other words, the natural map $\varinjlim_U \Sigma^p(U\times X)\to\Sigma^p(\eta\times X)$ is an isomorphism. Thus writing down the exact sequence (2) for $U\times X$ and passing to the direct limit over U we get a commutative diagram of exact sequences

$$\begin{array}{ccccccc}
\varinjlim_U \bigoplus_{W\in\Sigma^{p-1}(U\times X)} k(W)^* & \longrightarrow & \varinjlim_U \mathbb{Z}^{(\Sigma^p(U\times X))} & \longrightarrow & \varinjlim_U CH^p(U\times X) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{W\in\Sigma^{p-1}(\eta\times X)} k(W)^* & \longrightarrow & \mathbb{Z}^{(\Sigma^p(\eta\times X))} & \longrightarrow & CH^p(\eta\times X) & \longrightarrow & 0
\end{array}$$

where the first two vertical arrows are isomorphisms; therefore the third one is also an isomorphism. We conclude that the class $\Delta - X\times\{x\}$ is zero in $CH^n(U\times X)$ for some U.

(iii) $\Rightarrow$ (iv) : Put $T' := X \smallsetminus U$. The localization exact sequence [Fu, Prop. 1.8]

$$CH(T'\times X)\to CH(X\times X)\to CH(U\times X)\to 0$$

implies that $\Delta - X\times\{x\}$ comes from the class in $CH(T'\times X)$ of a cycle $\sum n_i Z'_i$. For each i, let T'_i be the image of Z_i in T', and let T_i be a desingularization of T'_i. Since Z'_i is not contained in the singular locus $\mathrm{Sing}(T'_i)\times X$, it is the pushforward of an irreducible subvariety $Z_i\subset T_i\times X$. Putting $T=\coprod T_i$ and $\alpha=\sum n_i[Z_i]$ does the job.

(iv) $\Rightarrow$ (i) : Assume that (3) holds; then it holds in $CH(X_K\times X_K)$ for any extension K of $\mathbb{C}$, so it suffices to prove $CH_0(X)_0=0$.

Denote by p and q the two projections from $X\times X$ to X, and put $n:=\dim(X)$. Any class $\delta\in CH^n(X\times X)$ induces a homomorphism $\delta_*: CH_0(X)\to CH_0(X)$, defined by $\delta_*(z)=q_*(\delta\cdot p^*z)$. Let us consider the classes which appear in (3). The diagonal induces the identity of $CH_0(X)$; the class of $X\times\{x\}$ maps $z\in CH_0(X)$ to $\deg(z)\,[x]$, hence is 0 on $CH_0(X)_0$.

Now consider $\delta:=(i\times 1)_*\alpha$. Let p',q' be the projections from $T\times X$ to T and X. Then, for $z\in CH_0(X)$,

$$\delta_* z = q_*((i\times 1)_*\alpha\cdot p^*z) = q'_*(\alpha\cdot p'^*i^*z)\ .$$

Since $\dim T<\dim X$, i^*z is zero, hence also $\delta_* z$. We conclude from (3) that $CH_0(X)_0=0$. $\square$

Example.– The group $CH_0(X)$ is a birational invariant [Fu, ex. 16.1.11], thus the above properties depend only on the birational equivalence class of X. In particular a rational variety is CH_0-trivial. More generally, since $CH_0(X\times\mathbb{P}^n)\cong CH_0(X)$ for any variety X, a stably rational variety is CH_0-trivial.

Despite its technical aspect, Proposition 7 has remarkable consequences (see e.g. [B-S]) :

Proposition 8. *Suppose X is CH_0-trivial.*
1) $H^0(X,\Omega^r_X)=0$ *for all* $r>0$.
2) *The group $H^3(X,\mathbb{Z})$ is torsion free.*

Proof : The proof is very similar to that of the implication (iv) $\Rightarrow$ (i) in the previous Proposition; we use the same notation. Again a class δ in $CH^n(X\times X)$ induces a homomorphism $\delta^*: H^r(X,\mathbb{Z})\to H^r(X,\mathbb{Z})$, defined by $\delta^*z:=p_*(\delta\cdot q^*z)$. The diagonal induces the identity, the class $[X\times\{p\}]$ gives 0 for $r>0$, and the class

$(i\times 1)_*\alpha$ gives the homomorphism $z\mapsto i_*p'_*(\alpha\cdot q'^*z)$. Thus formula (3) gives for $r>0$ a commutative diagram

$$\begin{tikzcd} & H^*(T,\mathbb{Z}) \arrow[dr, "i_*"] & \\ H^r(X,\mathbb{Z}) \arrow[ur] \arrow[rr, "\mathrm{Id}"] & & H^r(X,\mathbb{Z}) \end{tikzcd} \quad . \qquad (4)$$

On each component T_k of T the homomorphism $i_*: H^*(T_k,\mathbb{C})\to H^*(X,\mathbb{C})$ is a morphism of Hodge structures of bidegree (c,c), with $c:=\dim X-\dim T_k>0$. Therefore its image intersects trivially the subspace $H^{r,0}$ of $H^r(X,\mathbb{C})$. Since i_* is surjective by (4), we get $H^{r,0}=0$.

Now we take $r=3$ in (4). The only part of $H^*(T,\mathbb{Z})$ with a nontrivial contribution in (4) is $H^1(T,\mathbb{Z})$, which is torsion free. Any torsion element in $H^3(X,\mathbb{Z})$ goes to 0 in $H^1(T,\mathbb{Z})$, hence is zero. □

Observe that in the proof we use only formula (3) in $H^*(X\times X)$ and not in the Chow group. The relation between these two properties is discussed in Voisin's papers [V3, V4, V5].

As the Clemens-Griffiths criterion, the triviality of $CH_0(X)$ behaves well under deformation (compare with Lemma 3) :

Proposition 9. [V4] *Let $\pi: X\to B$ be a flat, proper family over a smooth variety B, with* $\dim(X)\geq 3$. *Let* $o\in B$*; assume that :*

- *The general fiber X_b is smooth;*
- *X_o has only ordinary double points, and its desingularization $\tilde{X}_o$ is not CH_0-trivial.*

Then X_b is not CH_0-trivial for a very general point b of B.

Recall that 'very general' means 'outside a countable union of strict subvarieties of B' (4.2).

We refer to [V4] for the proof. The idea is that there cannot exist a decomposition (3) of Proposition 7 for b general in B, because it would extend to an analogous decomposition over X, then specialize to X_o, and finally extend to $\tilde{X}_o$. One concludes by observing that the locus of points $b\in B$ such that X_b is smooth and CH_0-trivial is a countable union of subvarieties.

Corollary 1. *The double cover of $\mathbb{P}^3$ branched along a very general quartic surface is not stably rational.*

Proof : Consider the pencil of quartic surfaces in $\mathbb{P}^3$ spanned by a smooth quartic and a quartic symmetroid, and the family of double covers of $\mathbb{P}^3$ branched along the members of this pencil. By Proposition 8.2), the Artin-Mumford threefold is not CH_0-trivial. Applying the Proposition we conclude that a very general quartic double solid is not CH_0-trivial, hence not stably rational. □

More generally, Voisin shows that the desingularization of a very general quartic double solid with at most seven nodes is not stably rational.

Voisin's idea has given rise to a number of new results. Colliot-Thélène and Pirutka have extended Proposition 9 to the case where the singular fiber X_o has (sufficiently nice) non-isolated singularities, and applied this to prove that a very general quartic threefold is not stably rational [C-P1]. I proved that a very general sextic double solid is not stably rational [B7]. As already mentioned, combining the methods of Kollár and Colliot-Thélène-Pirutka, Totaro has proved that a very general hypersurface of degree d and dimension n is not stably rational for $d \geq 2\lceil \frac{n+2}{3} \rceil$ [T]; Colliot-Thélène and Pirutka have extended this to certain cyclic coverings [C-P2]. Hassett, Kresch and Tschinkel have shown that a conic bundle (see 3.2) with discriminant a very general plane curve of degree ≥ 6 is not stably rational [HKT]. Finally, using the result of [HKT], Hassett and Tschinkel have proved that a very general Fano threefold which is not rational or birational to a cubic threefold is not stably rational [HT].

We do not know whether there exist smooth quartic double solids which are CH_0-trivial. In contrast, Voisin has constructed families of smooth cubic threefolds wich are CH_0-trivial [V5] – we do not know what happens for a general cubic threefold.

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