Holomorphic symplectic manifolds

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Projective manifolds: a view from above

If you look from very high at the collection of projective manifolds, you see 3 dominating mountains, according to the positivity of the canonical bundle:

- $K < 0$: the Fano manifolds;
- The manifolds with $K \sim 0$;
- $K > 0$: manifolds of general type.

The first one is relatively small: a finite number of families in each dimension. The third one is enormous. The second one, which we are going to discuss, is big but still somewhat accessible.

Very roughly, the minimal model program tries to relate all projective manifolds to these 3 types, e.g. by finding fibrations where the base and the general fiber are of these types.

Manifolds with $c_1 = 0$

For the intermediate case we have the following theorem:

Decomposition theorem

Let M be a compact Kähler manifold with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$.

There exists $M' \to M$ finite étale with $M' = T \times \prod_i X_i \times \prod_j Y_j$

- T = complex torus;
- $X_i = X$ simply-connected projective, dim $\geqslant 3$, $H^0(X, \Omega_X^*) = \mathbb{C} \oplus \mathbb{C}\omega$, where ω is a generator of K_X (**Calabi-Yau** manifolds).
- $Y_j = Y$ compact simply-connected, $H^0(Y, \Omega_Y^*) = \mathbb{C}[\varphi]$, where $\varphi \in H^0(Y, \Omega_Y^2)$ is everywhere non-degenerate (**irreducible symplectic** manifolds, aka **hyperkähler**).

The theorem follows from Yau's theorem on the existence of Ricci-flat metrics, plus the Berger classification of holonomy groups.

Holomorphic symplectic manifolds

Thus a basic type of manifolds with $c_1 = 0$ is:

Definition

A *irreducible holomorphic symplectic* (IHS) manifold X is a simply-connected, compact Kähler manifold X with $H^0(X, \Omega^2_X) = \mathbb{C}\varphi$, where $\varphi : \bigwedge^2 T_X \to \mathscr{O}_X$ is everywhere non-degenerate.

Some consequences : X has even dimension $2r$; the form $\varphi^r \in H^0(K_X)$ is everywhere $\neq 0$, hence $K_X \cong \mathscr{O}_X$.

Example: For $r = 1$, IHS $=$ K3 surface (e.g. $S_4 \subset \mathbb{P}^3$).

For a long time no other example was known; Bogomolov (1978) claimed that they do not exist. But Fujiki constructed a 4-dimensional example ($\sim$1982), then I found 2 series in each dimension.

The Hilbert scheme

The idea : Start from a K3 surface S, with $\sigma \neq 0$ in $H^0(S, \Omega^2_S)$.

$H^0(S^r, \Omega^2_{S^r}) = \{\lambda_1 \,\mathrm{pr}_1\, \sigma + \ldots + \lambda_r \,\mathrm{pr}_r\, \sigma\}$, symplectic if all $\lambda_i \neq 0$.

To get **one** symplectic form, impose $\lambda_1 = \ldots = \lambda_r$, i.e. invariance under $\mathfrak{S}_r$: we look at the **symmetric product** $S^{(r)} := S^r/\mathfrak{S}_r$. An element of $S^{(r)}$ can be viewed as a 0-cycle $x_1 + \ldots + x_r \ \ (x_i \in S)$.

Unfortunately $S^{(r)}$ is singular along Δ where $x_i = x_j$ for some $i \neq j$.

But there is a miraculous desingularization, the **Hilbert scheme**

$S^{[r]} := \{Z \subset S \mid \mathrm{length}(Z) = r\}$. $\pi : S^{[r]} \to S^{(r)}, Z \mapsto \sum_{p \in Z} m(p)p$.

$E := \pi^{-1}(\Delta)$ is a divisor in $S^{[r]}$. $\qquad \pi : S^{[r]} \smallsetminus E \xrightarrow{\sim} S^{(r)} \smallsetminus \Delta$,

but π contracts E to Δ. For instance if $r = 2$:

$$Z \text{ such that } \pi(Z) = 2p \ \leftrightsquigarrow \ \text{tangent direction at } p$$

so that $\pi^{-1}(2p) = \mathbb{P}T_p(S)$. Key fact: $S^{[r]}$ is **smooth** (Fogarty).

The standard examples

Proposition

For S a K3 surface, $S^{[r]}$ is an irreducible symplectic manifold.

Sketch of proof: The 2-form $\sum \mathrm{pr}_i^* \sigma$ descends to $S^{(r)} \smallsetminus \Delta \cong S^{[r]} \smallsetminus E$, and spans $H^0(S^{[r]} \smallsetminus E, \Omega^2)$. Local computation shows that it extends to a symplectic form on $S^{[r]}$. ∎

The same construction works starting from a complex torus T. $T^{[r]}$ is not simply-connected, but consider $K_r(T) :=$ fiber of

$$T^{[r+1]} \xrightarrow{\pi} T^{(r+1)} \xrightarrow{s} T \quad \text{where} \quad s(x_0 + \ldots + x_r) = \sum x_i \ .$$

Proposition

$K_r(T)$ is an IHS manifold ("generalized Kummer manifold").

Same proof.

Reminder: deformations

X compact complex manifold. *Deformation* of X over pointed space (B, o): $f : \mathscr{X} \to B$ proper smooth, with $\mathscr{X}_{\mathrm{o}} \xrightarrow{\sim} X$.

If $H^0(X, T_X) = 0$, there exists a **universal** local deformation, parametrized by $\mathrm{Def}_X \subset H^1(X, T_X)$, with $T_{\mathrm{o}}(\mathrm{Def}_X) = H^1(X, T_X)$.

Theorem (Tian-Todorov)

If $K_X = \mathcal{O}_X$, Def_X is smooth at o (equivalently, Def_X is an open subset of $H^1(X, T_X)$).

Deformations of symplectic manifolds

For X symplectic, $T_X \cong \Omega^1_X$, so $H^1(X, T_X) = H^{1,1} \subset H^2(X, \mathbb{C})$.

Proposition

$H^2(S^{[r]}, \mathbb{C}) = H^2(S, \mathbb{C}) \oplus \mathbb{C}[E], \quad H^2(K_r, \mathbb{C}) = H^2(T, \mathbb{C}) \oplus \mathbb{C}[E].$

Corollary

The IHS $S^{[r]}$ or $K_r(T)$ form a **hypersurface** *in the deformation space.*

Indeed we get $H^{1,1}(S^{[r]}) = H^{1,1}(S) \oplus \mathbb{C}[E]$, and the analog for K_r.

So a general deformation X is **not** of the form $S^{[r]}$ – we say that X is *of type* $\mathrm{K3}^{[r]}$. Same for $K_r(T)$. We will see later examples of such deformations.

Other examples?

When I found the 2 series $S^{[r]}$ and $K_r(T)$ in 1983, I expected that many other examples would appear. This turned out to be surprisingly difficult. At the moment, the only examples known, up to deformations, are:

- $S^{[r]}$ and $K_r(T)$, of dimension $2r$;
- 2 examples OG_{10} and OG_6, of dimension 10 and 6, due to O'Grady – again starting from a K3 or a complex torus.

It is an important problem to find more examples.

Note the contrast with Calabi-Yau manifolds: the physicists have constructed more than 30 000 families of Calabi-Yau threefolds! Whether the number of such families is finite is an open problem.

The period map

X IHS:

$$H^2(X,\mathbb{C}) = \underset{\displaystyle\mathbb{C}\varphi}{\underset{\|}{H^{2,0}}} \oplus H^{1,1} \oplus \underset{\displaystyle\mathbb{C}\bar{\varphi}}{\underset{\|}{H^{0,2}}}$$

Put $H = H^2(X,\mathbb{C})$. Let $\mathscr{X} \to \mathrm{Def}_X$ be a local universal deformation. The groups $H^2(\mathscr{X}_b,\mathbb{C})$ form a local system on Def_X; we can assume that it is constant $\rightsquigarrow$ $H^2(\mathscr{X}_b,\mathbb{C}) \overset{can}{=\!=} H$.

The **period map** $\wp : B \to \mathbb{P}(H)$ is defined by $\wp(b) = [\varphi_b]$.

Theorem

$\wp$ is a local isomorphism onto a quadric $Q \subset \mathbb{P}(H)$.

Sketch of proof: 1) Write $\varphi_b = a\varphi + \omega + b\bar{\varphi}$, with $\omega \in H^{1,1}(X)$.

$$0 = \varphi_b^{r+1} = (a\varphi + \omega + b\bar{\varphi})^{r+1} = (r+1)(a\varphi)^r b\bar{\varphi} + \binom{r+1}{2}(a\varphi)^{r-1}\omega^2$$
$$(\in H^{2r,2}) + \text{terms in } H^{p,q},\ q \geqslant 3.$$

Multiply by $\bar{\varphi}^{r-1}$ and integrate: $0 = ab\int_X (\varphi\bar{\varphi})^r + \frac{r}{2}\int_X \omega^2(\varphi\bar{\varphi})^{r-1}$

$=$ a quadratic form q on H, such that $\wp(B) \subset Q : \{q = 0\}$.

2) $T_{\mathrm{o}}(\wp) : H^1(X, T_X) \to \mathrm{Hom}(H^{2,0}, H^{1,1} + H^{0,2})$ deduced from

$$H^1(X, T_X) \otimes H^0(X, \Omega_X^2) \xrightarrow{\ \cup\ } H^1(X, \Omega_X^1) \qquad \text{(Griffiths)}$$

$\Rightarrow$ $T_{\mathrm{o}}(\wp)$ isomorphism onto codimension 1 subspace of $T_{\wp(\mathrm{o})}(\mathbb{P}(H))$, thus $= T_{\wp(\mathrm{o})}(Q)$. ∎

The quadratic form

Recall: $q(a\varphi + \omega + b\bar{\varphi}) = ab\int_X (\varphi\bar{\varphi})^r + \int_X \omega^2 (\varphi\bar{\varphi})^{r-1}$.

Thus $q(\varphi) = q(\bar{\varphi}) = 0$, $q(\varphi, \bar{\varphi}) > 0$, $H^{1,1} \perp_q (H^{2,0} \oplus H^{0,2})$.

Theorem

1. *Q is defined by an integral quadratic form $q : H^2(X, \mathbb{Z}) \to \mathbb{Z}$, non-degenerate, of signature $(3, b_2 - 3)$.*
2. *$\wp$ is a local isomorphism $B \to \Omega$, where $\Omega =$ open subset of Q defined by $q(\varphi, \bar{\varphi}) > 0$.*
3. $\exists\ f_X \in \mathbb{N}$ (*the* **Fujiki constant**) *such that* $\int_X \alpha^{2r} = f_X\, q(\alpha)^r$.

The global period map

Fix a lattice L and an integer r.

Marked IHS manifold: (X,σ) where $\sigma : H^2(X,\mathbb{Z}) \xrightarrow{\sim} L$.

$\mathcal{M}_L$:=$\{$iso. classes of (X,σ), $\dim X = 2r\}$ – complex manifold, but **non-Hausdorff**.

Period domain $\Omega_L = \{[v] \in \mathbb{P}(L_{\mathbb{C}}) \mid v^2 = 0\,,\ v.\bar{v} > 0\}$.

Period map $\wp : \mathcal{M}_L \to \Omega_L$: $\wp(X,\sigma) = \sigma_{\mathbb{C}}(H^{2,0}) \subset L_{\mathbb{C}}$.

Theorem

1. (*Huybrechts*) $\wp$ *is surjective.*
2. (*Verbitsky*) *Let* $\mathcal{M}_L^{\circ}$ *be a connected component of* $\mathcal{M}_L$. *Then* $\wp : \mathcal{M}_L^{\circ} \to \Omega_L$ *is generically injective, and identifies* Ω_L *with the Hausdorff reduction of* $\mathcal{M}_L^{\circ}$ (**"Torelli theorem"**).

Remark: Many examples where $\mathcal{M}_L$ is not connected.

Projective IHS

Proposition

The projective *IHS form a countable union of hypersurfaces, everywhere dense in $\mathcal{M}_L$.*

Sketch of proof: A IHS X is projective iff $H^2(X,\mathbb{Z}) \ni \alpha$ with $q(\alpha) > 0$ and $q(\alpha,\varphi) = 0$ (Huybrechts). Thus the locus $\mathcal{M}_L^{\mathrm{alg}}$ of projective IHS in $\mathcal{M}_L$ is $\bigcup \wp^{-1}(\alpha^\perp)$, for $\alpha \in L$ with $q(\alpha) > 0$. One checks that $\bigcup(\alpha^\perp \cap Q)$ is everywhere dense in Q. ■

Example: • For K3, $\dim \mathcal{M}_L = 20$, $\dim \mathcal{M}_L^{\mathrm{alg}} = 19$.

• For X of type $S^{[r]}$, $\dim \mathcal{M}_L = 21$, $\dim \mathcal{M}_L^{\mathrm{alg}} = 20$. Thus the IHS $S^{[r]}$ with S projective form only a hypersurface in $\mathcal{M}_L^{\mathrm{alg}}$.

Problem: find complete families ($=$ dim 20) in $\mathcal{M}_L^{\mathrm{alg}}$.

Complete projective families

- (AB-Donagi) $V \subset \mathbb{P}^5$ smooth cubic, depends on 20 moduli. $F(V) :=$ variety of lines $\subset V$ is an IHS, of type $S^{[2]}$.
- (Debarre-Voisin) ψ general in $\mathrm{Alt}^3(V_{10})$ depends on 20 moduli; $X_\psi := \{L \in \mathbb{G}(6, V_{10}) \mid \psi_{|L} = 0\}$ is an IHS of type $S^{[2]}$.
- (O'Grady) The "EPW-sextics" $V \subset \mathbb{P}^4$ depend on 20 moduli. X double cover of V is an IHS of type $S^{[2]}$.
- (Lehn-Lehn-Sorger-Van Straten) $V \subset \mathbb{P}^5$ cubic, $H = \{$twisted cubic curves $\subset V\}$; $\exists\, H \to X$ with X IHS of type $S^{[4]}$.
- (Bayer, Lahoz, Macrì, Nuer, Perry, Stellari, 2019) $V \subset \mathbb{P}^5$ cubic. For $a, b \in \mathbb{N}$ coprime and $r = a^2 - ab + b^2 + 1$, X_r moduli space of certain stable objects in $D(V)$ is an IHS of type $S^{[r]}$.
- A few other examples... But **no** example known for type $K_r(T)$.

THE END