Symmetry-preserving observers for water tank problems: theory and application to an oceanography data assimilation example

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29 sept 2009

Outline

Symmetries have been used in control for feedback design for non-linear systems but much less for the design of non-linear observers to our knowledge.

In control theory, a state observer is a system that uses

- a model of the real system
- *noisy* measurements of the input and output of the real system
- Goal : provide a *real-time* estimate of the internal state

It is typically a computer-implemented mathematical model (preferably low cost of computation). Exemple : Kalman filter

Linear observers

Consider the linear system

$$\frac{d}{dt}x = Ax + Bu, \qquad y = Cx$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input, and $y \in \mathbb{R}^p$ the output; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$.
Luenberger Observer or Kalman filter

$$\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(C\hat{x} - y)$$

Copy + correction term $L(C\hat{x} - y)$ equal to 0 when $\hat{x} = x$.

L is the gain matrix

Non-linear observers

$\frac{d}{dt}x = f(x,u), \quad y = h(x,u) \qquad y, u$ known signals

Estimator, observer, filter, etc:

$$\frac{d}{dt}\hat{x} = f(\hat{x}, u(t)) - L(\hat{x}, y(t)) \cdot \big(h(\hat{x}, u(t)) - y(t)\big)$$

- Luenberger observer, gain scheduling, high gains, ...
- Extended Kalman Filter (M, N “tuning" matrices)

$$A = \frac{\partial f}{\partial x}(\hat{x}, u) \qquad L = -PC^T N$$
$$C = \frac{\partial h}{\partial x}(\hat{x}, u) \qquad \frac{d}{dt}P = AP + PA^T + M^{-1} - PC^T NCP$$

- Tuning? Domain of convergence? Computational cost?

Non-linear symmetry-preserving observers

What can we do when the model

$$\frac{d}{dt}x = f(x, u), \quad y = h(x, u)$$

admits symmetries ? [1]

- The linear system $\frac{d}{dt}x = Ax + Bu$ is invariant by scaling

$$\forall \lambda > 0 \quad \frac{d}{dt}(\lambda x) = A(\lambda x) + B(\lambda u)$$

- so is the correction term in $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + L(C\hat{x} - y)$

When $f(x, u)$ is not linear why should the correction term be linear ??

- Redemption by geometry $\rightarrow$ make $L(\hat{x}, y(t)) \cdot (h(\hat{x}, u(t)) - y(t))$ respect the symmetries.

[1] Main results in the paper "Symmetry-Preserving Observers"', S. Bonnabel, P. Martin, P. Rouchon, IEEE Trans. on. Automatic Control, 2008. See also "Non-linear symmetry-preserving observers on Lie groups", IEEE Trans. on. Automatic Control, 2009.

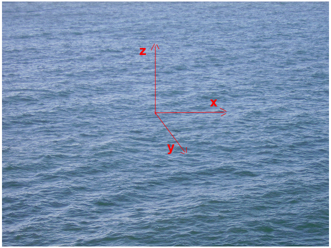
System considered and motivations of the talk

The amount of available data has drastically increased in the last years.

- Measurement : height of the ocean
- Goal : Estimate the marine streams.

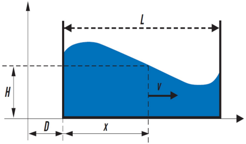
Symmetries ?

- Invariance by SE(2) (2D rotations and translations).



Water tank systems, symmetries, and observer design

Saint-Venant system



The Saint-Venant equations write on the rectangular domain

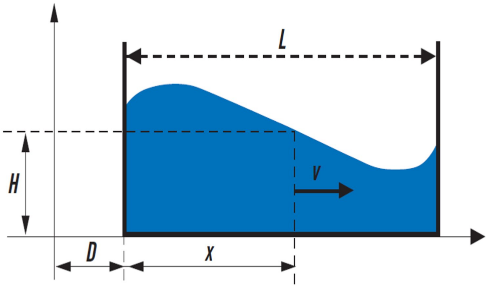
$$\frac{\partial}{\partial t}h = -\nabla(hv), \qquad \frac{\partial}{\partial t}v = -v\nabla v - g\nabla h$$

where $hv = h(v_x\mathbf{i} + v_y\mathbf{j})$ is the horizontal transport.

There has been theoretical work on motion planning and feedback for this system but much less on observers.[2]

[2] See J. M. Coron, B. DAndrea-Novel and G. Bastin, “A Lyapunov approach to control irrigation canals modeled by Saint-Venant equations", Proc. European Control Conference, Karlsruhe, 1999.
N. Petit and P. Rouchon. “Dynamics and solutions to some control problems for water-tank systems" IEEE Trans. Automatic Control, 2002.

Saint-Venant system



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where $hv = h(v_x \mathbf{i} + v_y \mathbf{j})$ is the horizontal transport.

- Assumption : $h(x, y, t)$ is measured (with noise) for all x, y, t.
- Goal : Estimate all the state variables $v(x, y)$ and $h(x, y)$ at any point $(x, y) \in [0, L]^2$ of the domain.

Model symmetries : SE(2)-invariance

Take R_θ rotation of angle θ, then the transformations

$$\begin{aligned} (X, Y) &= R_\theta(x, y) + (x_0, y_0) \\ H(X, Y) &= h(x, y) \\ V(X, Y) &= R_\theta v(x, y) \end{aligned}$$

leave the system unchanged

$$\begin{aligned} \frac{\partial}{\partial t} V &= -V \nabla V - g \nabla H \\ \frac{\partial}{\partial t} H &= -\nabla \cdot (HV) \end{aligned}$$

This results from : $\nabla h(x, y) = R_\theta \nabla H(X, Y)$.

Note that the domain becomes $(R_\theta D + (x_0, y_0)) \subset \mathbb{R}^2$.

Observer design

We consider asymptotic observer of the form

$$\frac{\partial}{\partial t}\hat{v} = -\hat{v}\nabla\hat{v} - g\nabla\hat{h} + N_v(h, \hat{v}, \hat{h})$$

$$\frac{\partial\hat{h}}{\partial t} = -\nabla\cdot(\hat{h}\hat{v}) + N_h(h, \hat{v}, \hat{h})$$

where N_h and N_v are operators (versus space variables) such that

$$N_v(h, \hat{v}, h) = 0, \quad N_h(h, \hat{v}, h) = 0$$

$N_v(h, \hat{v}, \hat{h})$ is a vector and $N_h(h, \hat{v}, \hat{h})$ a scalar.

How to preserve $SE(2)$ invariance in the choice of N_h and N_v ?

Symmetry-preserving observer: scalar differential terms

- Classical result: any differential scalar operator, $SE(2)$ invariant, is polynomial in Δ.
- Scalar invariant correction N_h:

$$N_h = Q_1(\Delta, h, \hat{v}^2, \hat{h} - h) + \nabla \left(Q_2(\Delta, h, \hat{v}^2, \hat{h} - h) \right) \cdot \hat{v}$$

where Q_1 and Q_2 of the form

$$Q_i(\Delta, h, \hat{v}^2, \hat{h} - h) = \sum_{k=0}^{N} a_k^i(h, \hat{v}^2, \hat{h} - h)\, \Delta^k \left(b_k^i(h, \hat{v}^2, \hat{h} - h) \right)$$

the functions a_k^i and b_k^i being smooth functions of their arguments such that

$$a_k^i(h, \hat{v}^2, 0) = b_k^i(h, \hat{v}^2, 0) = 0.$$

Symmetry-preserving observer: vectorial differential terms

- Vector invariant correction terms N_v:

$$N_v = P_1(\Delta, h, \hat{v}^2, \hat{h} - h)\hat{v} + \nabla P_2(\Delta, h, \hat{v}^2, \hat{h} - h)$$

where P_1 and P_2 are similar to the Q_i used for N_h:

$$P_i(\Delta, h, \hat{v}^2, \hat{h} - h) = \sum_{k=0}^{N} c_k^i(h, \hat{v}^2, \hat{h} - h)\, \Delta^k \left(d_k^i(h, \hat{v}^2, \hat{h} - h) \right)$$

symmetry-preserving observer: integral terms

$$N_v(x,y,t) = \int\int \left[R_1(\Delta, h, \hat{v}^2, \hat{h}-h)\hat{v} + \nabla R_2(\Delta, h, \hat{v}^2, \hat{h}-h)\right]_{(x-\xi, y-\zeta, t)} \phi_v(\xi^2+\zeta^2)\, d\xi d\zeta$$

$$N_h(x,y,t) = \int\int \left[S_1(\Delta, h, \hat{v}^2, \hat{h}-h) + \nabla S_2(\Delta, h, \hat{v}^2, \hat{h}-h)\cdot\hat{v}\right]_{(x-\xi, y-\zeta, t)} \phi_h(\xi^2+\zeta^2)\, d\xi d\zeta$$

where ϕ_v and ϕ_h are convolution kernels and the R_i's and S_i's are polynomials versus Δ.

Chosen observer

$$\frac{\partial}{\partial t} h = -\nabla(hv), \qquad \frac{\partial}{\partial t} v = -v\nabla v - g\nabla h$$

Simplest symmetry-preserving observer (with integral corrections):

$$\begin{aligned}
\frac{\partial}{dt}\hat{h} &= -\nabla(\hat{h}\hat{v}) + \int\int \phi_h(\xi^2+\zeta^2)(h-\hat{h})_{(x-\xi,y-\zeta,t)}\, d\xi d\zeta \\
&= -\nabla(\hat{h}\hat{v}) + \varphi_h * (h-\hat{h}) \\
\frac{\partial}{dt}\hat{v} &= -\hat{v}\nabla\hat{v} - g\nabla\hat{h} + \int\int \phi_v(\xi^2+\zeta^2)\nabla(h-\hat{h})_{(x-\xi,y-\zeta,t)}\, d\xi d\zeta \\
&= -\hat{v}\nabla\hat{v} - g\nabla\hat{h} + \varphi_v * \nabla(h-\hat{h})
\end{aligned}$$

where ϕ_h and ϕ_v have to be designed to ensure convergence

Comparison with Nudging

Here the 2D image $(h - \hat{h})$ is filtered with an isotropic smooth kernel (heat equation filtering) before being fed in the observer.

$$\frac{\partial}{dt}\hat{h} = -\nabla(\hat{h}\hat{v}) + \varphi_h * (h - \hat{h})$$
$$\frac{\partial}{dt}\hat{v} = -\hat{v}\nabla\hat{v} - g\nabla\hat{h} + \varphi_v * \nabla(h - \hat{h})$$

Convergence study on the linearized Saint-Venant system

Let us linearize the system around the steady-state $(h, v) = (\bar{h}, 0)$, where the equilibrium height $\bar{h}$ is constant.

It means we only consider small velocities $\delta v = v - \bar{v} \ll \sqrt{g\bar{h}}$ and heights $\delta h = h - \bar{h} \ll \bar{h}$.

Design of ϕ_h and ϕ_v

The estimation errors, $\tilde{h} = \delta\hat{h} - \delta h$ and $\tilde{v} = \delta\hat{v} - \delta v$, obey the following linearized equations:

$$\frac{\partial}{dt}\tilde{h} = -\bar{h}\nabla\tilde{v} - \varphi_h * \tilde{h}, \qquad \frac{\partial}{dt}\tilde{v} = -g\nabla\tilde{h} - \varphi_v * \nabla\tilde{h}.$$

Eliminating $\tilde{v}$ leads to the following modified damped wave equation with external viscous damping

$$\frac{\partial^2}{\partial t^2}\tilde{h} = g\bar{h}\Delta\tilde{h} + \varphi_v * \Delta\tilde{h} - \varphi_h * \frac{\partial}{\partial t}\tilde{h}$$

since $\nabla(\varphi_v * \nabla h) = \varphi_v * \Delta h$

where $\tilde{h} = \hat{h} - h$ and $\tilde{v} = \hat{v} - v$ are the estimation errors.

Theorem

If φ_v and φ_h are defined by

$$\begin{aligned} \varphi_v(x,y) &= \beta_v \exp(-\alpha_v(x^2+y^2)), & (1)\\ \varphi_h(x,y) &= \beta_h \exp(-\alpha_h(x^2+y^2)), & (2) \end{aligned}$$

with $\beta_v, \beta_h, \alpha_v, \alpha_h > 0$, then the first order approximation of the error system around the equilibrium $(h, v) = (\bar{h}, 0)$ is strongly asymptotically convergent. Indeed if we consider the following Hilbert space and norm:

$$\mathcal{H} = H^1(\Omega) \times L^2(\Omega), \quad \|(u,w)\|_{\mathcal{H}} = \left(\int_\Omega \|\nabla u\|^2 + |w|^2 \right)^{1/2}, \quad (3)$$

then, for every $\tilde{h}$ solution of the error equation,

$$\lim_{t\to\infty} \left\| \left(\tilde{h}(t), \frac{\partial \tilde{h}}{\partial t}(t) \right) \right\|_{\mathcal{H}} = 0\,. \quad (4)$$

Convergence study

We have the equation in 2D (where $\psi_v := g\bar{h}\delta_0 + \varphi_v$)

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u &= \psi_v * \Delta u - \varphi_h * \frac{\partial}{\partial t} u && \text{in } \mathbb{R}^+ \times \Omega = \mathbb{R}^+ \times [0,\pi]^2, \\ u &= 0 && \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(0) &= u_0, \quad u_t(0) = u_1 && \text{in } \Omega, \end{aligned} \tag{5}$$

where $u(t,x,y) = \tilde{h}$.

Dirichlet boundary condition $\leftrightarrow$ we set $\hat{h} = h$ on the boundary.

Let (e_{pq}) be the orthonormal basis of $H_0^1(\Omega)$ composed of eigenfunctions of the unbounded operator Δ:

$$e_{pq} = \frac{2}{\pi} \sin(px)\sin(qy). \tag{6}$$

Convergence study - Gain design

For the kernels φ_v and φ_h we choose:

$$\begin{aligned} \varphi_v(x,y) &= (f(x) * f(x))\,(f(y) * f(y)), &(7)\\ \varphi_h(x,y) &= (g(x) * g(x))\,(g(y) * g(y)), &(8) \end{aligned}$$

where f and g are smooth even functions.

To respect the symmetries $\varphi_v(x,y)$ and $\varphi_h(x,y)$ must be functions of $x^2 + y^2$.

Take for instance

$$\begin{aligned} \varphi_v(x,y) &= \beta_v \exp(-\alpha_v(x^2 + y^2)), &(9)\\ \varphi_h(x,y) &= \beta_h \exp(-\alpha_h(x^2 + y^2)), &(10) \end{aligned}$$

Convergence study - Gain design

We have

$$\varphi_v(x,y) = \psi_v(x^2+y^2) = (f(x)*f(x))\,(f(y)*f(y)) \qquad (11)$$

$$\varphi_h(x,y) = \psi_h(x^2+y^2) = (g(x)*g(x))\,(g(y)*g(y)) \qquad (12)$$

Take f and g smooth even functions. The Fourier coefficients are real

$$\hat{c}_{pq} = \hat{f}_p^2\hat{f}_q^2 \quad \text{and} \quad \hat{g}_p^2\hat{g}_q^2$$

We have

$$(\varphi_v * \Delta)e_{pq} = -(p^2+q^2)\hat{f}_p^2\hat{f}_q^2\; e_{pq} = -(p^2+q^2)f_{pq}^2\; e_{pq}$$

So the convolution products lead to a frequency-modified damped wave equation,

e_{pq} are still eigenvectors of the modified Laplacian operator.

Convergence study - back to convergence

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2} u &= \psi_v * \Delta u - \varphi_h * \frac{\partial}{\partial t} u && \text{in } \mathbb{R}^+ \times \Omega = \mathbb{R}^+ \times [0,\pi]^2, \\
u &= 0 && \text{on } \mathbb{R}^+ \times \partial\Omega, \\
u(0) &= u_0, \quad u_t(0) = u_1 && \text{in } \Omega,
\end{aligned}
\tag{13}
$$

rewrites

$$
\frac{d}{dt} U = \mathcal{A} U
$$

where $U = (u, u_t) = (u, v)$ and $\mathcal{A}$ is the unbounded linear operator $\mathcal{A}(u, v) = (v, \varphi_h * \Delta u - \varphi_v * v)$

$$
E_{pq} = \begin{pmatrix} 1 \\ \lambda_{\pm pq} \end{pmatrix} e_{pq} \tag{14}
$$

form a Riesz basis of $\mathcal{H}$ and are eigenvectors of $\mathcal{A}$ associated to the eigenvalues $\lambda_{\pm pq}$, solutions of

$$
\lambda_{\pm pq}^2 + g_{pq}^2 \lambda_{\pm pq} + f_{pq}^2 (p^2 + q^2) = 0. \tag{15}
$$

Convergence study - Form of the solution

The solution u is given by the series

$$u(t,x,y) = \frac{2}{\pi} \sum_{1 \leq p,q} u_{pq}(t) \sin(px) \sin(qy),$$

with either

$$u_{pq}(t) = e^{\frac{-g_{pq}^2}{2}t}(A_{pq} \cos(\omega_{pq} t) + B_{pq} \sin(\omega_{pq} t)),$$

or

$$u_{pq}(t) = e^{\frac{-g_{pq}^2}{2}t}(A_{pq} \cosh(\tilde{\omega}_{pq} t) + B_{pq} \sinh(\tilde{\omega}_{pq} t)). \qquad (16)$$

with

$$\omega_{pq} = \sqrt{4(p^2+q^2) f_{pq}^2 - g_{pq}^4}$$

$$\tilde{\omega}_{pq} = \sqrt{g_{pq}^4 - 4(p^2+q^2) f_{pq}^2}$$

but

- g_{pq}^2 are the Fourier coefficients of $\beta_h \exp(-\alpha_v(x^2+y^2))$
- f_{pq}^2 coefficients of $g\bar{h}\delta_0 + \beta_v \exp(-\alpha_v(x^2+y^2))$

Convergence study

Finally, the coefficients can be found using the Fourier series of the initial condition. We have

$$A_{pq} = \frac{4}{\pi^2} \int_{[0,\pi]^2} u(0,x,y) \sin(px) \sin(qy) dxdy,$$

$$B_{pq} = \frac{4}{\omega_{pq}\pi^2} \int_{[0,\pi]^2} \left(u_t(0,x,y) + \frac{g_{pq}^2}{2} u(0,x,y) \right) \sin(px) \sin(qy) dxdy.$$

Convergence study

Let

$$u_N(t,x,y) = \frac{2}{\pi} \sum_{p+q\leq N} e^{\frac{-g_{pq}^2}{2}t}(A_{pq}\cos(\omega_{pq}t) + B_{pq}\sin(\omega_{pq}t))\sin(px)\sin(qy)$$

- $\|u_N, \frac{d}{dt}u_N\|_{\mathcal{H}} \to 0$ exponentially (cf numerical simus).
- $\|u-u_N, \frac{d}{dt}(u-u_N)\|_{\mathcal{H}}$ can be arbitrarily small for N large enough because of Parseval's lemma ($u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$)

We proved the strong convergence of the linearized error system $u = \tilde{h}$:

$$\lim_{t\to\infty} \|\tilde{h}, \frac{d}{dt}\tilde{h}\|_{\mathcal{H}} = 0$$

Application to an oceanography exemple

Single layer model model[3]

$$\frac{\partial(hv)}{\partial t} + (\nabla \cdot (hv) + (hv) \cdot \nabla) v = \dots$$

$$\dots - g' h \nabla h - \mathbf{k} \times f(hv) + (\alpha_A A \nabla^2 - R)(hv) + \alpha_{tau} \tilde{\tau} \mathbf{i} / \rho$$

$$\frac{\partial h}{\partial t} = -\nabla \cdot (hv)$$

where

- density ρ, layer height $h(x, y, t)$, fluid velocity $v(x, y, t)$, rectangular domain $0 < x < L$, $0 < y < L$ where x and y point east and north
- f represents Coriolis effect, $\mathbf{k}$ points upward, g' is the reduced gravity
- $\tilde{\tau}\mathbf{i}$ wind term of intensity $\tilde{\tau}$
- R and A known damping coefficients.

The Goal: to estimate $v(x, y, t)$ from the satellite data $h(x, y, t)$.

[3] Jiang et al. *Tracking Nonlinear Solutions with Simulated Altimetric Data in a Shallow-Water Model*. J. of Physical Oceanography, vol 27, p.72, 1997

$SE(2)$ invariance

i and **j** point respectively towards East and North...

Take R_θ rotation of angle θ, then the transformations

$$\begin{aligned} (X, Y) &= R_\theta(x, y) + (x_0, y_0) \\ H(X, Y) &= h(x, y) \\ V(X, Y) &= R_\theta v(x, y) \end{aligned}$$

leave the dynamics unchanged

$$\frac{\partial (HV)}{\partial t} + (\nabla \cdot (HV) + (HV) \cdot \nabla) V = -g' H \nabla H - \mathbf{K} \times F(HV) + (\alpha_A A \nabla^2 - R)(HV) + \alpha_{tau} \tilde{\tau} \mathbf{I} / \rho$$

$$\frac{\partial H}{\partial t} = -\nabla \cdot (HV)$$

This results from : $\nabla h(x, y) = R_\theta \nabla H(X, Y)$, $\mathbf{K} = \mathbf{k}$, $\mathbf{I} = R_\theta \mathbf{i}$....

Symmetry-preserving observers

As in the case of Saint-Venant equations, we take

$$\frac{\partial(\hat{h}\hat{v})}{\partial t} + (\nabla \cdot (\hat{h}\hat{v}) + (\hat{h}\hat{v}) \cdot \nabla)\hat{v} = -g'\hat{h}\nabla\hat{h} - \mathbf{k} \times f(\hat{h}\hat{v})$$
$$+ (\alpha_A A \nabla^2 - R)(\hat{h}\hat{v}) + \alpha_{tau}\tilde{\tau}\mathbf{i}/\rho + \varphi_v * \left(\nabla(h - \hat{h})\right)$$
$$\frac{\partial \hat{h}}{\partial t} = -\nabla \cdot (\hat{h}\hat{v}) + \phi_h * (h - \hat{h})$$

and we use a heuristic gain tuning on the linearized simplified system.

Heuristic gain tuning on the linearized simplified model

Reminding

$$\varphi_h(x,y) = \beta_h \exp(-\alpha_h(x^2+y^2))$$
$$\varphi_v(x,y) = g\bar{h}\delta_0 + \beta_v \exp(-\alpha_v(x^2+y^2))$$

The error system can be approximated by the following system ($\alpha = +\infty$):

$$\frac{\partial^2 \tilde{h}}{\partial t^2} + 2\xi_0\omega_0 \frac{\partial \tilde{h}}{\partial t} = (L_0\omega_0)^2 \Delta \tilde{h}.$$

where $L_0^2\omega_0^2 = g\bar{h} + \bar{h}\beta_v$, $2\xi_0\omega_0 = \beta_h$,

A dimensional analysis allows to choose :

- ω_0 and L_0 the characteristic pulsation and length
- $\alpha_h^{-2} = \alpha_v^{-2}$ is the size of the region of influence.

Numerical simulations

1. Saint-Venant system (water tank)
2. Full non-linear shallow water model (ocean)

Model parameters and gain tuning

- Domain = square box, of dimension 2000 km $\times$2000 km.
- equilibrium height $\bar{h} = 500$ m,
- regular spatial discretization with 81 $\times$ 81 gridpoints $\rightarrow$ space step of 25 km. The time step 30 mn, time periods of 1 to 4 months (1440 to 5760 time steps).
- height varies between 497.7 and 501.9 m, transversal velocity in $\pm 0.008\ m.s^{-1}$.
- $\alpha = 1\ m^{-2}$.
- frequency for the error system $\omega_0\sqrt{1-\xi_0^2}$ chosen close to the natural frequency $\sqrt{g\bar{h}}/L_0$ of the physical system
- Truncation of the Gaussian at 10 pixels away from center

1) Saint-Venant system

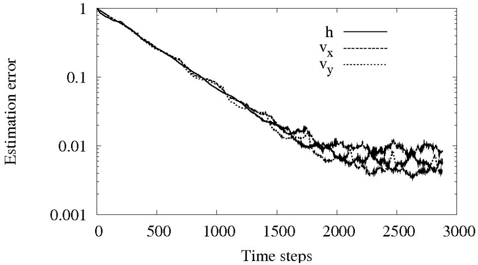
we consider the Saint-Venant system with small velocities $\delta v = v - \bar{v} \ll \sqrt{g\bar{h}}$ and heights $\delta h = h - \bar{h} \ll \bar{h}$

$$e_h = \frac{\|(\hat{h} - \bar{h}) - (h - \bar{h})\|}{\|h - \bar{h}\|}, \qquad e_v = \frac{\|(\hat{v} - \bar{v}) - (v - \bar{v})\|}{\|v - \bar{v}\|} \qquad (17)$$

where $\| \, . \, \|$ is the standard L^2 norm. We observe

$$e_h(t) = e_h(0)\exp(-c_h t), \qquad e_v(t) = e_v(0)\exp(-c_v t) \qquad (18)$$

With a 20 % noise:



1) Comparison with the standard Nudging technique

The nudging algorithm (Luenberger observer) writes

$$\frac{\partial \hat{h}}{\partial t} = -\nabla(\hat{h}\hat{v}) + K_h(h - \hat{h}), \quad (19)$$

$$\frac{\partial \hat{v}}{\partial t} = -\hat{v}\nabla\hat{v} - g\nabla\hat{h} + K_v\nabla(h - \hat{h}). \quad (20)$$

It corresponds to $\varphi_v = \varphi_h = \delta_0$, i.e. $\alpha = +\infty$.

Size of the Gaussian kernel	Decrease rate (h, v_x, v_y)	Estimation error at convergence (h, v_x, v_y)
$\alpha_h = \alpha_v = 0.5$	1.49×10^{-6} 1.40×10^{-6} 1.42×10^{-6}	4.43×10^{-3} 7.51×10^{-3} 4.06×10^{-3}
$\alpha_h = \alpha_v = 1$	7.55×10^{-7} 7.44×10^{-7} 7.44×10^{-7}	5.92×10^{-3} 1.04×10^{-2} 5.53×10^{-3}
$\alpha_h = \alpha_v = 10^3$	2.45×10^{-7} 2.49×10^{-7} 2.48×10^{-7}	1.70×10^{-2} 3.02×10^{-2} 1.59×10^{-2}

2) Full non-linear oceanographic shallow water model

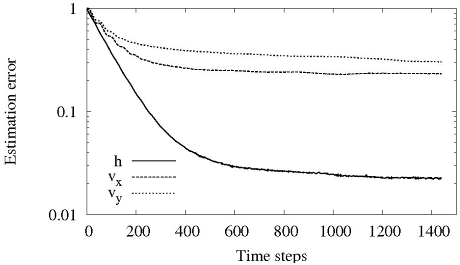
$$e_h = \frac{\|(\hat{h} - \bar{h}) - (h - \bar{h})\|}{\|h - \bar{h}\|}, \qquad e_v = \frac{\|(\hat{v} - \bar{v}) - (v - \bar{v})\|}{\|v - \bar{v}\|} \quad (21)$$

where $\| \, . \, \|$ is the standard L^2 norm. We observe

$$e_h(t) = e_h(0) \exp(-c_h t), \qquad e_v(t) = e_v(0) \exp(-c_v t) \quad (22)$$

only at the beginning.

With a 20 % noise:



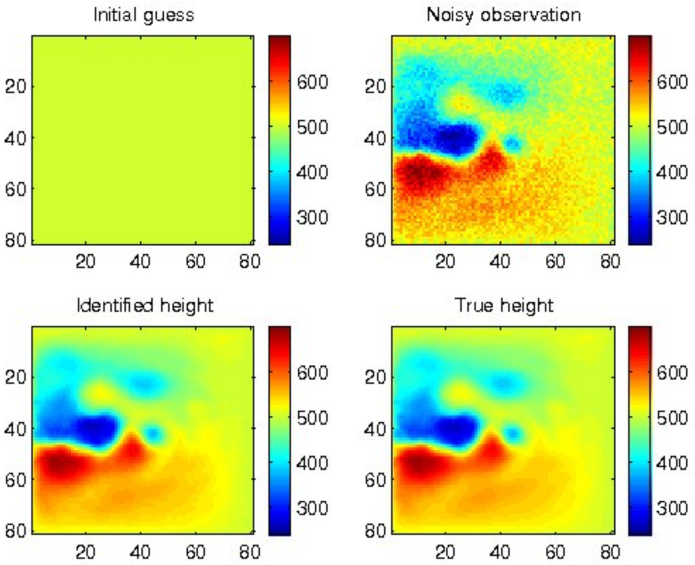
2) Comparison with the standard Nudging technique

Size of the Gaussian kernel	Decrease rate (h, v_x, v_y)	Estimation error at convergence (h, v_x, v_y)
$\alpha_h = \alpha_v = 0.5$	2.74×10^{-6} 1.87×10^{-6} 1.62×10^{-6}	1.71×10^{-2} 1.72×10^{-1} 2.21×10^{-1}
$\alpha_h = \alpha_v = 1$	1.36×10^{-6} 9.65×10^{-7} 8.38×10^{-7}	1.57×10^{-2} 1.30×10^{-1} 1.59×10^{-1}
$\alpha_h = \alpha_v = 10^3$	4.42×10^{-7} 2.98×10^{-7} 2.55×10^{-7}	2.26×10^{-2} 2.25×10^{-1} 3.04×10^{-1}

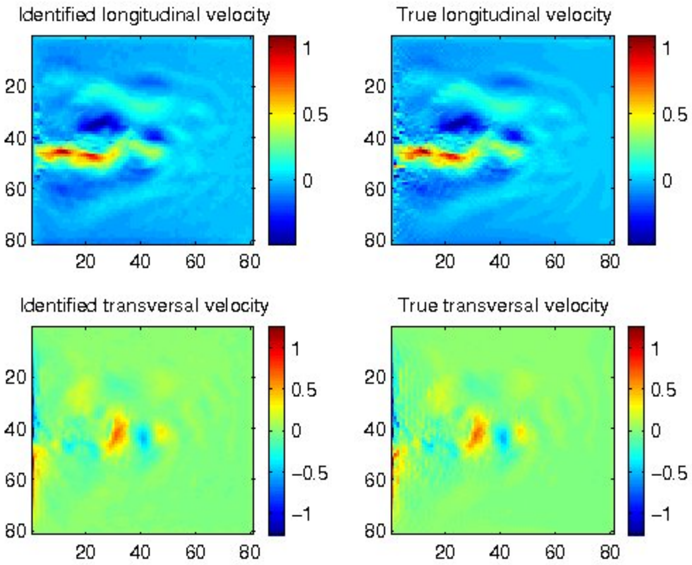
Table: Full non-linear model: decrease rate and value at convergence of the estimation error, for the three variables h, v_x and v_y, in the case of noisy observations (20% noise).

In the next slide we will see that the observer allows to identify very well the main currents in a realistic setting (e.g. gulf stream).

2) Results with the chosen observer



2) Results with the chosen observer



Conclusion

We designed an observer

- Much more economical computationally than EKF or variational methods.
- Gives better results than the nudging (Luenberger). Especially much more robust to gaussian noise.
- Gain design based on heuristic arguments on the linear first order system (easy to tune).
- Practical gain design in two steps:
  1. Convergence analysis easy around a steady state: linear and local gain design.
  2. gain extrapolations to the nonlinear regime becomes natural via invariance.

Here we consider space-continuous and time-continuous measurements: other situations of practical interest exist: boundary measures, discrete-time and/or discrete-space measurements.

Other possibilities

$$\frac{\partial}{dt}\hat{h} = -\nabla(\hat{h}\hat{v}) + \phi_h * (h - \hat{h}) + \phi_h^\Delta * \Delta(\hat{h} - h)$$
$$\frac{\partial}{dt}\hat{v} = -\hat{v}\nabla\hat{v} - g\nabla\hat{h} + \phi_v * \nabla(h - \hat{h})$$

Then the first variation around $h = \bar{h}$ and $v = 0$, can be identified to the wave equation

$$\frac{\partial^2}{\partial t^2}\tilde{h} = (g\bar{h} + \beta_v)\Delta\tilde{h} - \beta_h\frac{\partial}{\partial t}\tilde{h} + \beta_h^\Delta\Delta\left(\frac{\partial}{\partial t}\tilde{h}\right)$$

An additional structural damping changes drastically the spectrum.
Dimensional analysis design:

$$\beta_h = \omega_0, \quad K\beta_v = \max(0, (L_0\omega_0)^2 - g\bar{h}), \quad \beta_h^\Delta = L_0^2\omega_0.$$

Back to the single layer system

Forward Nudging

$$\frac{\partial(\hat{h}\hat{v})}{\partial t} + (\nabla \cdot (\hat{h}\hat{v}) + (\hat{h}\hat{v}) \cdot \nabla)\hat{v} = -g'\hat{h}\nabla\hat{h} - \mathbf{k} \times f(\hat{h}\hat{v})$$
$$+ (\alpha_A A\nabla^2 - R)(\hat{h}\hat{v}) + \alpha_{tau}\tilde{\tau}\mathbf{i}/\rho + \phi_v * \left(h\nabla(h - \hat{h})\right)$$
$$\frac{\partial\hat{h}}{\partial t} = -\nabla \cdot (\hat{h}\hat{v}) + \phi_h * (h - \hat{h}) + \phi_h^\Delta * \Delta(\hat{h} - h)$$

For backward nudging: $\phi_h \mapsto -\phi_h$, $K_h^\Delta \mapsto -\phi_h^\Delta$ and ϕ_v unchanged.