Pseudo-hyperbolic spaces

Definition: $\mathbb{R}^{p,q}$ is the vector space $\mathbb{R}^{p+q}$ with the quad. form

$$Q(x) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$$

$\langle \cdot , \cdot \rangle$ polar form of Q.

$$\operatorname{sign}(Q) = (p, q, 0).$$

Definition: The pseudo-hyperbolic space $\widehat{\mathbb{H}}^{p,q}$ is the quadric

$$\{ x \in \mathbb{R}^{p,q+1} \mid Q(x) = -1 \}.$$

Q is a submersion on $\widehat{\mathbb{H}}^{p,q}$:

$$d_x Q = 2\langle x, \cdot \rangle : \mathbb{R}^{p,q+1} \longrightarrow \mathbb{R}$$

Hence $\hat{\mathbb{H}}^{p,q}$ is a submanifold of $\mathbb{R}^{p,q+1}$, of dimension $p+q$, with

$$T_x \hat{\mathbb{H}}^{p,q} = \ker(d_x Q) = x^{\perp_Q}$$

Since $\mathbb{R}x$ is a negative definite line in $\mathbb{R}^{p,q+1}$, the quadratic form Q restricts to a signature (p,q) pseudo-Riemannian metric on $\hat{\mathbb{H}}^{p,q}$!

$\parallel$

(smoothly varying non-degenerate quadratic form on the tangent space of the manifold.)

Remarks:

- One could look at

$$\{x \in \mathbb{R}^{p+1,q} \mid Q(x) = +1\}$$

these are the "pseudo-spheres" and $\forall p,q \geq 0$,

$$S^{p,q} \underbrace{=}_{\text{antiisometric}} -\mathbb{H}^{q,p}$$

- $\mathbb{H}^{0,q}$ is a sphere $S^q \subset \mathbb{R}^{q+1}$ with negative metric.

- $\mathbb{H}^{p,0}$ are the hyperbolic spaces (in the hyperboloid model).

Topology: Let $\mathbb{R}^{p,q+1} = P \overset{\perp}{\oplus} N$ be an orthogonal decomposition, with P positive def & N negative def.

Denote by $\mathbb{S}^q$ the sphere

$$\mathbb{S}^q := \{ x \in N \mid Q(x) = -1 \}.$$

Denote by $\mathbb{B}^p$ the unit ball in P. The map

$$\mathbb{B}^p \times \mathbb{S}^q \xrightarrow{\phi} \hat{\mathbb{H}}^{p,q}$$

$$(x, y) \longmapsto \left(\frac{1+Q(x)}{1-Q(x)}\right)\left(\frac{2}{1+Q(x)} x + y\right)$$

is a diffeo onto

and

$$\phi^*\langle \cdot, \cdot \rangle_{(x,y)} = \left(\frac{1+Q(x)}{1-Q(x)}\right)^2 \left(\frac{4}{(1+Q(x))^2} g_{eucl} \oplus - g_{\mathbb{S}^q}\right)$$

$$= \left(\frac{1+Q(x)}{1-Q(x)}\right)^2 \left(g_{\mathbb{S}^p_+} \oplus - g_{\mathbb{S}^q}\right)$$

$\mathbb{S}^p_+$ open hemisphere

$$= g_{hyp} \oplus - \left(\frac{1+Q(x)}{1-Q(x)}\right)^2 g_{\mathbb{S}^q}$$

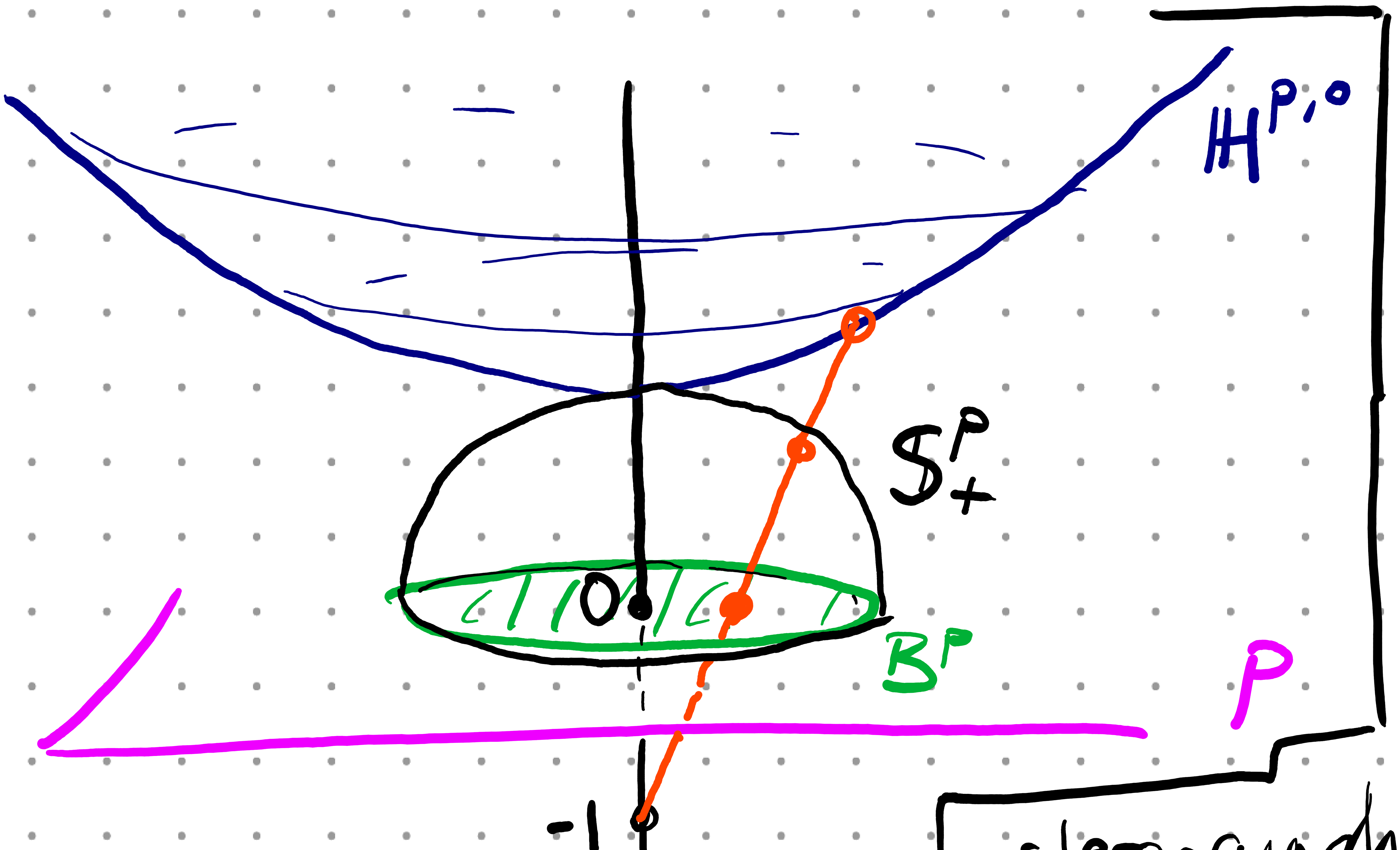
Idea of the diffeo:

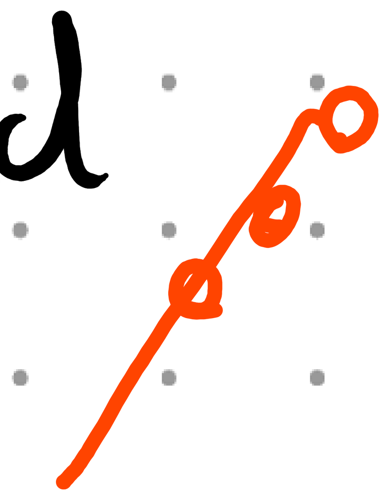
$$P \times S^q \xrightarrow{\;C^\infty\;} \mathbb{H}^{p,q} \qquad \text{diffeo.}$$
$$(x,y) \longmapsto x + (1+Q(x))^{1/2}\, y$$

Then, identify P with $\mathbb{B}^p \subset P$ via

$$\mathbb{B}^p \longrightarrow P$$
$$x \longmapsto \frac{1}{1-Q(x)}\, x \,.$$

You can now play between the metrics hyperbolic, spherical (in S^p_+), euclidean, via the maps in red = stereographic proj.





Important subspaces:

For every $y_0 \in \mathbb{S}^q$,

$\phi(\mathbb{B}^p \times \{y_0\})$ is a tot. geod. copy of $\hat{\mathbb{H}}^{p,0}$:

$$\phi(\mathbb{B}^p \times \{y_0\}) = \underbrace{P \oplus \mathbb{R} y_0}_{\text{sign} = (p,1)} \cap \hat{\mathbb{H}}^{p,q}$$

For $x_0 \in P$, $\phi(\{x_0\} \times \mathbb{S}^q)$ is diffeomorphic to $\mathbb{S}^q$, and is tot. geod. iff $x_0 = 0$:

$$\phi(\{0\} \times \mathbb{S}^q) = \mathbb{S}^q = \hat{\mathbb{H}}^{0,q} \subset \hat{\mathbb{H}}^{p,q}$$

$\phi(\{x_0\} \times \mathbb{S}^q)$ changes the metric of $\mathbb{S}^q$ by a homothety of factor $\left(\frac{1+Q(x)}{1-Q(x)}\right)^2$.

<u>Isometries</u>: $G = O(p,q+1) = \mathrm{Isom}(Q) \curvearrowright \mathbb{R}^{p,q+1}$ and preserves $\hat{\mathbb{H}}^{p,q}$.

- It acts by isometries in $\hat{\mathbb{H}}^{p,q}$:

$$g \in G, \quad d_x g\big|_{x^\perp} = g\big|_{x^\perp} \text{ isometry.}$$

- $G \curvearrowright \hat{\mathbb{H}}^{p,q}$ transitively & faithfull.

- $\mathrm{Stab}_G(x) \simeq O(p,q)$.

Consequence (1) $\hat{\mathbb{H}}^{p,q} = O(p,q+1)/O(p,q)$

$\rightsquigarrow$ Symmetric space.

(2) $\mathrm{Isom}(\hat{\mathbb{H}}^{p,q}) = O(p,q+1)$

Because $O(p,q+1)$ gives all the possible isometries (see the following proposition).

Prop: A local isometry is determined by its value + differential at a point

Proof: If ϕ, ψ are local isometries, $\{p \in M \mid d_p\phi = d_p\psi\}$ is closed, and open:

For a local isometry, you have

$$\begin{tikzcd}
(0, T_pM) \arrow[r, "d_p\phi"] \arrow[d, "\exp_p"'] \arrow[dr, phantom, "\circlearrowright"] & (0, T_{\phi(p)}M) \arrow[d, "\exp_{\phi(p)}"] \\
(p, M) \arrow[r, "\phi"] & (\phi(p), M)
\end{tikzcd}$$

hence ϕ is determined in a small neighb. around p by $d_p\phi$.

Now if $d_q\phi = d_q\psi$ (in particular $\phi(q) = \psi(q)$)

then $\{p \in M \mid d_p\phi = d_p\psi\} = M$ $\square$

Geodesics:

Prop: The geodesics of $\hat{\mathbb{H}}^{p,q}$ are the intersection of planes with $\hat{\mathbb{H}}^{p,q}$. Moreover, $P \cap \hat{\mathbb{H}}^{p,q}$ is $\begin{cases} 2 \text{ hyperbolas} \\ \text{a line} \\ \text{a circle} \end{cases}$ iff $\operatorname{sign}(P) = \begin{cases} (1,1) \\ (0,1) \\ (0,2) \end{cases}$

iff the geodesic is $\begin{cases} \text{spacelike} \\ \text{lightlike} \\ \text{timelike} \end{cases}$

Proof: Take $x \in \hat{\mathbb{H}}^{p,q}$, and $X \in T_x \hat{\mathbb{H}}^{p,q} = x^\perp$.

If $Q(X) > 0$, or $Q(X) < 0$, then

$$P = \mathbb{R}x \oplus \mathbb{R}X \quad \text{is non degenerate.}$$

Hence the orthog. reflexion wrt. P is an isometry of $\mathbb{R}^{p,q+1}$, hence descends as an isometry of $\hat{\mathbb{H}}^{p,q}$.

This isometry fixes only 1 curve: $P \cap \hat{\mathbb{H}}^{p,q}$. By the next two properties, this curve is the geodesic $\exp_x(tX)$.

If $Q(X) = 0$, then $x + \mathbb{R}X$ is included in $\hat{\mathbb{H}}^{p,q}$. Since it's a geod. of $\mathbb{R}^{p,q+1}$, it's a geod of $\hat{\mathbb{H}}^{p,q}$. $\square$

Prop: In a pseudo-R. manifold, an isometry sends geodesics to geodesics. $\square$

Prop: for every $x \in \hat{\mathbb{H}}^{p,q}$, $X \in T_x \hat{\mathbb{H}}^{p,q}$, there is a unique geodesic starting at x with initial direction X, maximal for this property. $\square$

Def: $\exp_x(X)$ = geodesic at time 1 with $\dot{\gamma}(0) = X$, $\gamma(0) = x$.

Parametrization: $x \in \hat{\mathbb{H}}^{p,q}$, $X \in T_x \hat{\mathbb{H}}^{p,q}$,

$$\gamma_X(t) = \begin{cases} \cosh(t)x + \sinh(t)X \\ x + tX \\ \cos(t)\, x + \sin(t)\, X \end{cases} \quad \text{if} \quad \begin{cases} Q(X) = 1 \\ Q(X) = 0 \\ Q(X) = -1 \end{cases}$$

Question: When are two points x, y linked by a geodesic?

$$\langle x, \gamma_X(t) \rangle = \begin{cases} -\cosh(t)\, x \\ -1 \\ -\cos(t) \end{cases} \quad \text{gives:}$$

Characterization: Let $x, y \in \hat{\mathbb{H}}^{p,q}$, $\begin{cases} x \neq y \\ x \neq -y \end{cases}$.

$\langle x, y \rangle < -1 \iff x \,\&\, y$ are linked by a unique spacelike geodesic

$\langle x, y \rangle = -1 \iff x \,\&\, y$ are linkd by a unique lightlike geodesic

$|\langle x, y \rangle| < 1 \iff x \,\&\, y$ are linked by a unique minimizing timelike geodesic

⚠ In pseudo-R. geometry you don't have Hopf-Rinow:

geodesic completeness (geod defined on $\mathbb{R}$) $\not\Longleftrightarrow$ $\forall x$, $\exp_x$ is surjective

In $\widehat{\mathbb{H}}^{p,q}$, geodesics are defined on $\mathbb{R}$ <u>but</u> if $x, y \in \widehat{\mathbb{H}}^{p,q}$ with $\langle x, y\rangle > 1$, there are no geodesics from x to y.

Projective model:

$$\mathbb{H}^{p,q} := \widehat{\mathbb{H}}^{p,q} / \pm 1 = \{\text{negative lines in } \mathbb{R}^{p,q+1}\}$$

Since $-id$ is an isom of $\mathbb{R}^{p,q+1}$, it induces an isometry of $\widehat{\mathbb{H}}^{p,q}$. Hence the quotient $\mathbb{H}^{p,q}$ has a well def. pseudo-R. metric coming from above.

Topology: $\mathbb{H}^{p,q}$ is an open set in $\mathbb{P}(\mathbb{R}^{p+q+1})$, diffeomorphic to

$$\mathbb{B}^p \times \mathbb{S}^q / \{\pm 1\}$$

Geodesics: A geodesic in $\mathbb{H}^{p,q}$ is [a projective line] $\cap\, \mathbb{H}^{p,q}$. $\mathbb{P}(P)$ is

$$\begin{cases} \text{spacelike} \\ \text{lightlike} \\ \text{timelike} \end{cases} \longleftrightarrow \operatorname{sign}(P) = \begin{cases} (1,1) \\ (0,1) \\ (0,2) \end{cases} \qquad \square$$

In particular, in the projective model, there are geodesics between every pair of points.

Boundary:

$$\partial\hat{\mathbb{H}}^{p,q} = \{\underbrace{[z]_+}_{\text{half-line}} \mid Q(z)=0\}$$

$$\partial\mathbb{H}^{p,q} = \{[z] \mid Q(z)=0\}$$
$$= \{\text{isotropic lines}\}$$

Conformal struct:

In projective spaces $\mathbb{RP}^n$, $[x]\in\mathbb{RP}^n$,

$$T_{[x]}\mathbb{RP}^n \simeq \mathrm{Hom}\left([x], \mathbb{R}^n/_{[x]}\right)$$

For the cone $\partial\mathbb{H}^{p,q}$, $[z]\in\partial\mathbb{H}^{p,q}$,

$$T_{[z]}\partial\mathbb{H}^{p,q} \simeq \mathrm{Hom}\Big([z], \underbrace{z^\perp/_{[z]}}_{\text{sign }(p-1,q)\text{ from }Q}\Big)$$

hence $\partial \mathbb{H}^{p,q}$ is endowed with a conformal class of pseudo.-R. metrics. Same for $\partial \hat{\mathbb{H}}^{p,q}$.

The diffeos $\phi : B^p \times S^q \to \hat{\mathbb{H}}^{p,q}$ extend to diffeos

$$\partial\phi : S^{p-1} \times S^q \to \partial \hat{\mathbb{H}}^{p,q}$$

that are conformal for $g_{S^{p-1}} \oplus -g_{S^q}$.

Also $\qquad \partial \mathbb{H}^{p,q} \underset{\text{diff}}{\simeq} \mathbb{R}P^{p-1} \times \mathbb{R}P^q$.

⚠ In general, two conformal metrics do not define the same unparametrized geodesics, but for lightlike geodesics.

Hence an unparametrized lightlike geodesic makes sense in $\partial \mathbb{H}^{p,q}$ ($\partial \hat{\mathbb{H}}^{p,q}$), and is called photon.

Prop: Photons are projectivization of isotropic planes, i.e. projective lines included in $\partial \mathbb{H}^{p,q}$.

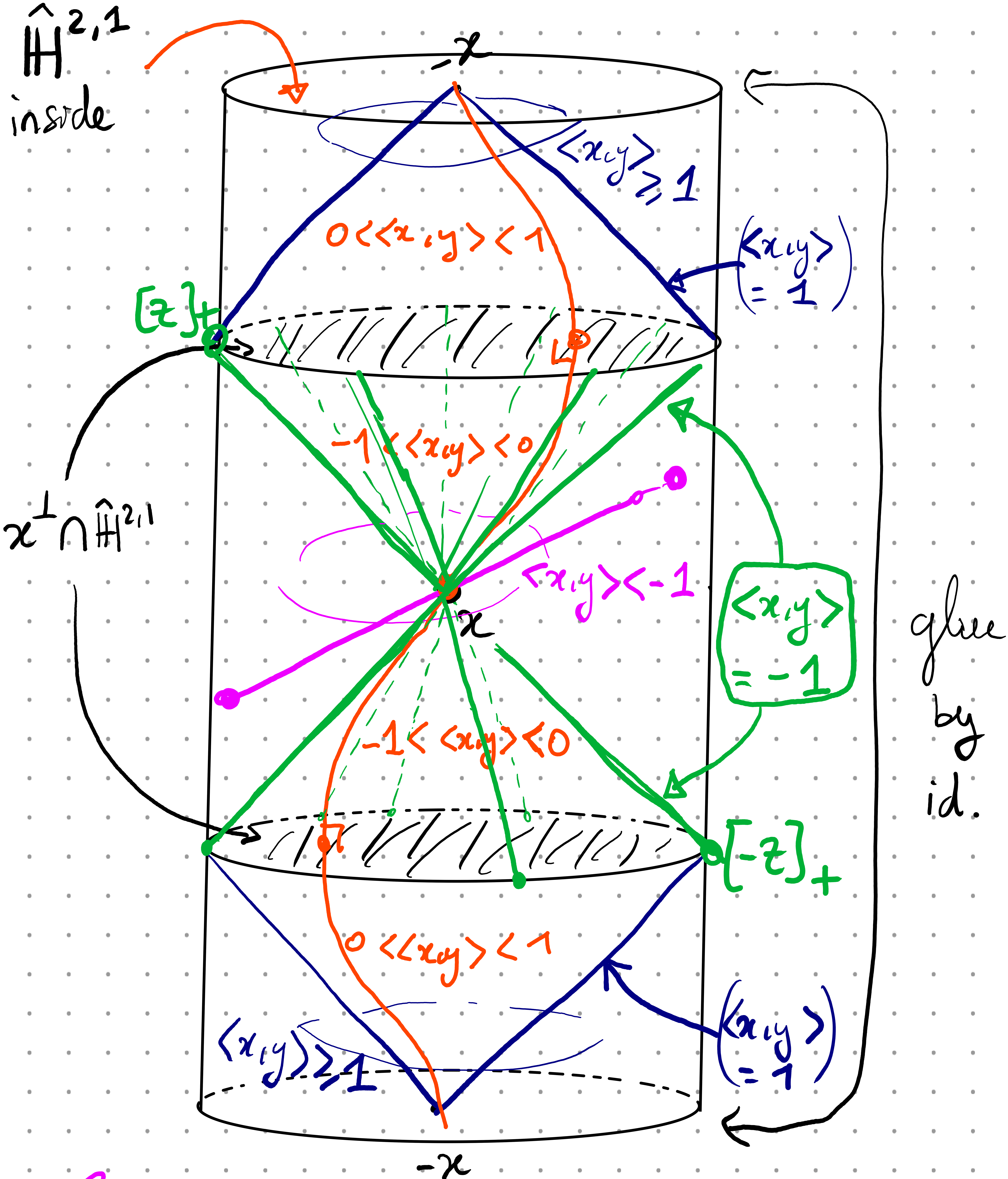
Proof: In a product $\partial \hat{\mathbb{H}}^{p,q} \simeq S^{p-1} \times S^q$, with the conformal metric $g_{S^{p-1}} \oplus -g_{S^q}$, a lightlike geodesic is $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ where $\gamma_i(t)$ is a unit speed geodesic in each sphere $\square$

Take $[x]_+, [y]_+$ in $\partial \hat{\mathbb{H}}^{p,q}$, $x \notin [y]$.

$\langle x, y \rangle = 0$ iff they span a photon

$\langle x, y \rangle < 0$ iff they bound a spacelike geodesic in $\hat{\mathbb{H}}^{p,q}$.

⚠ There are <u>no</u> "spacelike" or "timelike" pairs of ph.



Causal structure

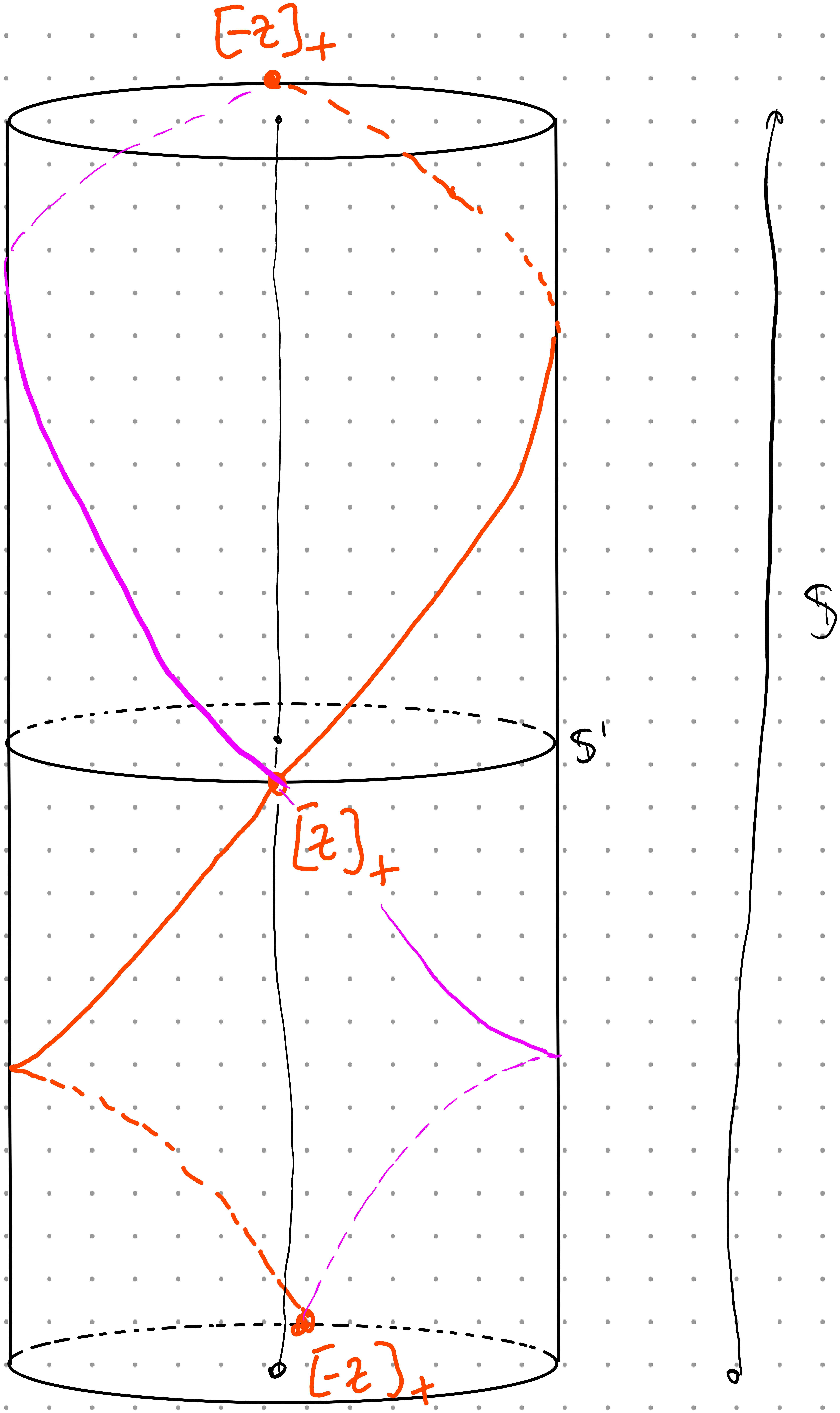
Photons:

in

$\partial \hat{\mathbb{H}}^{2,1}$

is

$S^1 \times S^1$



Triples of points in $\partial \mathbb{H}^{p,q}$:

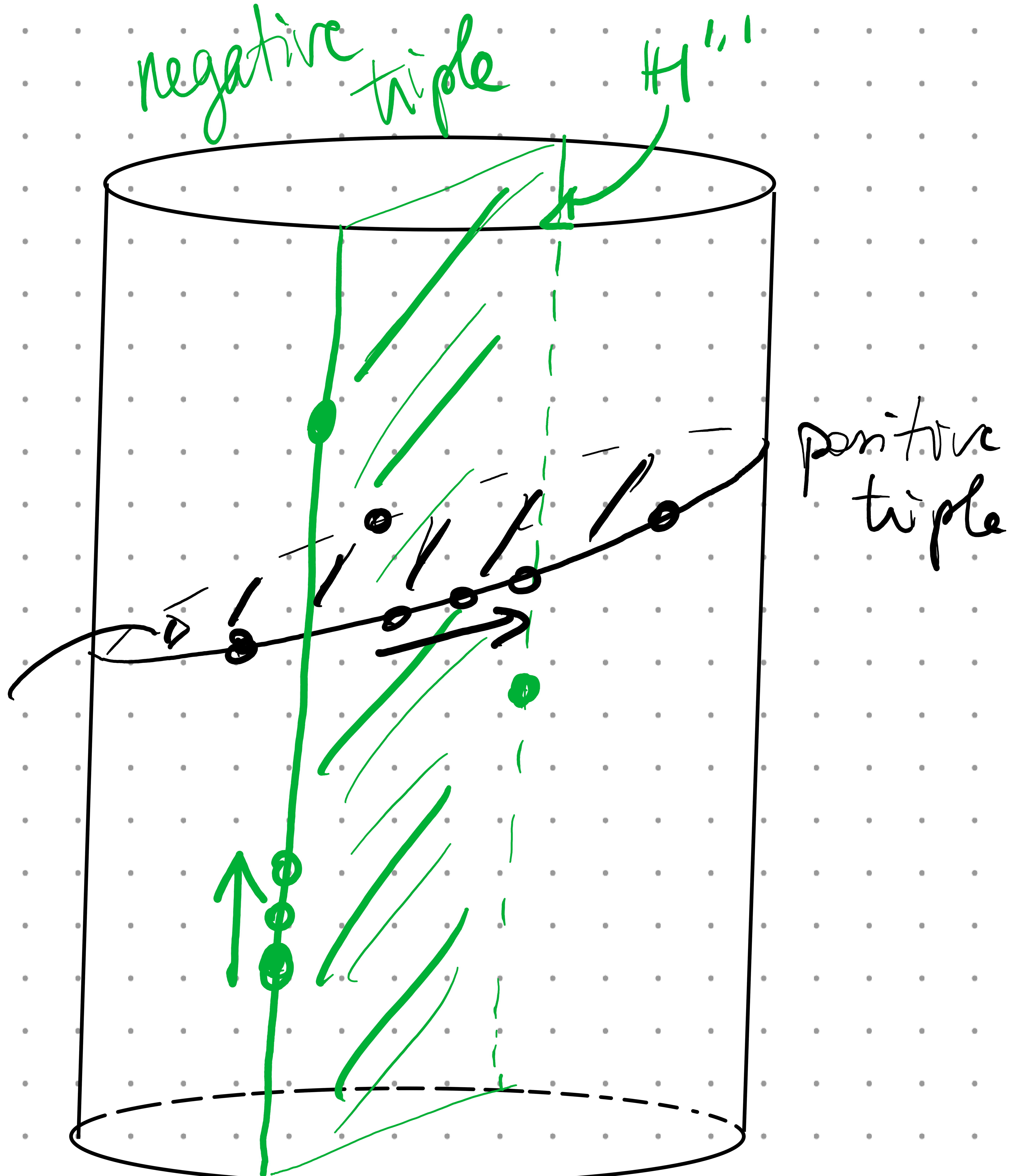
let $[x], [y], [z]$ in $\partial \mathbb{H}^{p,q}$, that span a 3 space in $\mathbb{R}^{p,q+1}$

$$Q|_{x,y,z} = \begin{pmatrix} 0 & \langle x,y\rangle & \langle x,z\rangle \\ & 0 & \langle y,z\rangle \\ & & 0 \end{pmatrix}$$

$$\det = \boxed{2\langle x,y\rangle\langle y,z\rangle\langle x,z\rangle}$$

3 cases:
- if $\det = 0$, then two points are in a photon and the third is not.
- if $\det > 0$, then $\operatorname{sign}([x]\oplus[y]\oplus[z]) = (1,2)$ and $\boxed{[x]\oplus[y]\oplus[z]} \cap \mathbb{H}^{p,q} \simeq \mathbb{H}^{1,1}$ the triple is called negative
- if $\det < 0$, $\operatorname{sign} = (2,1)$, bounds a copy of $\mathbb{H}^{2,0}$: positive triple

$\mathbb{H}^{2,1}$



Question of Jacques: You really do not have a notion of spacelike/timelike pair of points ??

Answer: • The complementary of a lightlike cone in $\partial\mathbb{H}^{p,q}$ is connected.

set of photons passing through a point p.

• Take $[x],[y] \in \partial\mathbb{H}^{p,q}$ such that

$$P = \mathbb{R}x + \mathbb{R}y \text{ is non-degenerate.}$$

Then $\mathrm{sign}(P) = (1,1)$. Hence $[x]$ & $[y]$ bound a spacelike geodesic of $\mathbb{H}^{p,q}$.

Only 1 case! For triples, two cases:

• Take $[x],[y],[z] \in \partial\mathbb{H}^{p,q}$ such that

$$E = \mathbb{R}x \oplus \mathbb{R}y \oplus \mathbb{R}z \text{ is non-dege.}$$

Then $\mathrm{sign}(E) = \begin{cases} (1,2) \\ (2,1) \end{cases}$ iff $E \cap \mathbb{H}^{p,q} = \begin{cases} \mathbb{H}^{1,1} \\ \mathbb{H}^{2,0} \end{cases}$

iff $\begin{cases} \text{negative triple} \\ \text{positive triple} \end{cases}$

<u>Curvature</u>: As in R. geometry, you have a Levi-Civita connexion ∇.

$$R_{XY} Z := \left(\nabla^2_{Y,X} - \nabla^2_{X,Y}\right) Z$$

$$= [\nabla_X, \nabla_Y] Z - \nabla_{[X,Y]} Z .$$

The sectional curvature is defined by

$$\sec(X,Y) = \frac{\langle R_{XY} Y, X\rangle}{Q(X)\,Q(Y) - \langle X,Y\rangle^2}$$

<u>Prop</u>: $\mathbb{H}^{p,q}$ & $\hat{\mathbb{H}}^{p,q}$ have constant curvature -1

<u>Remark</u>: Take X,Y that span a negative definite plane in $T_x \mathbb{H}^{p,q}$. Then

$$\underbrace{\langle R_{XY} Y, X\rangle}_{>0} = -\underbrace{\left(Q(X)\,Q(Y) - \langle X,Y\rangle^2\right)}_{\text{negative}}$$

hence, the sectional curvature is kind of opposite in timelike directions.

We recover a previous observation:

- If $P \subset T_x \mathbb{H}^{p,q}$ is a negative definite plane, then
$$\exp_x(P) \text{ is a sphere } \mathbb{H}^{0,2} = -\mathbb{S}^{2,0}$$
- If $P \subset T_x \mathbb{H}^{p,q}$ is a positive definite plane, then
$$\exp_x(P) \text{ is a hyperbolic plane } \mathbb{H}^{2,0}.$$

Sketch of proof of $\sec \equiv -1$:

Take X, Y in $T_x \hat{\mathbb{H}}^{p,q}$ st $\begin{cases} \langle X, Y \rangle = 0 \\ \langle X, X \rangle = \pm 1 \\ \langle Y, Y \rangle = \pm 1 \end{cases}$

Gauss equation:

$$\sec(X,Y) = \underbrace{\sec^{\mathbb{R}^{p,q+1}}(X,Y)}_{0} + \langle \mathrm{II}(X,X), \mathrm{II}(Y,Y) \rangle - \langle \mathrm{II}(X,Y), \mathrm{II}(X,Y) \rangle$$

By looking at the curves

$$\gamma_X = \exp(tX) \ \& \ \gamma_Y = \exp(tY)$$

you can show that $\mathrm{II}(X,Y) = 0$

$$\& \ \langle \mathrm{II}(X,X), \mathrm{II}(Y,Y) \rangle$$
$$= \langle x, x \rangle$$
$$= -1$$

$\square$