Polygonal surfaces in the pseudo-hyperbolic space

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Moving to higher rank: from hyperbolic to Anosov

The pseudo-hyperbolic space

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Projective model: open set in $\mathbb{P}(\mathbb{R}^{n+3})$.
Boundary: $\partial\mathbf{H}^{2,n}$ is the frontier of $\mathbf{H}^{2,n}$ in $\mathbb{P}(\mathbb{R}^{n+3})$.

Higher rank Teichmüller theory

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- a ρ-equivariant *positive* boundary loop Λ in $\partial\mathbf{H}^{2,n}$,
- a ρ-equivariant *maximal* surface in $\mathbf{H}^{2,n}$, bounded by Λ.

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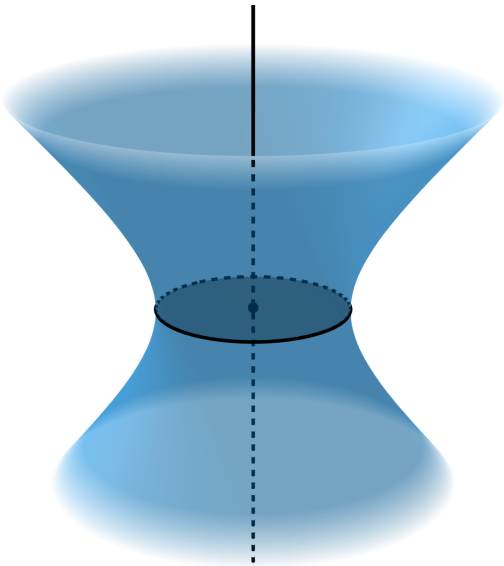
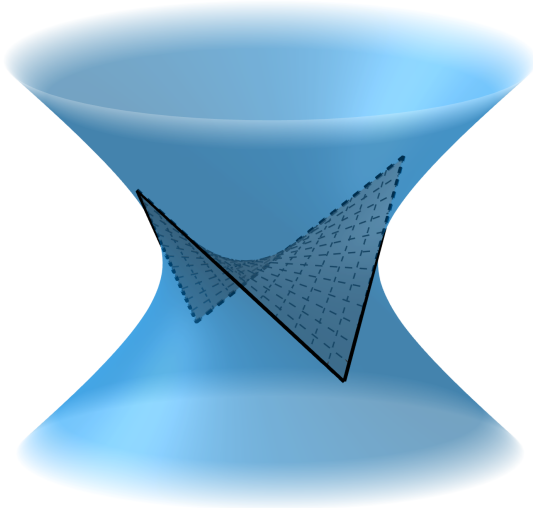
Semi-positive loops in $\partial\mathbf{H}^{2,n} \Leftrightarrow$ maximal surfaces in $\mathbf{H}^{2,n}$.

Two examples





Left : the maximal surface bounded by a regular circle. Totally geodesic copy of the **hyperbolic plane**.
Right : the maximal surface bounded by a polygon with 4 vertices. The surface is isometric to the **euclidean plane**.

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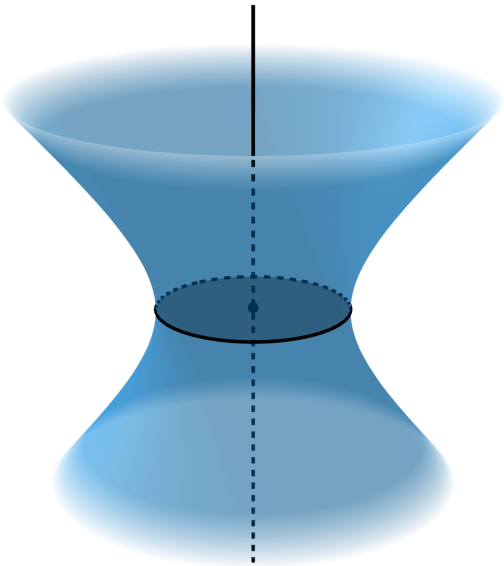

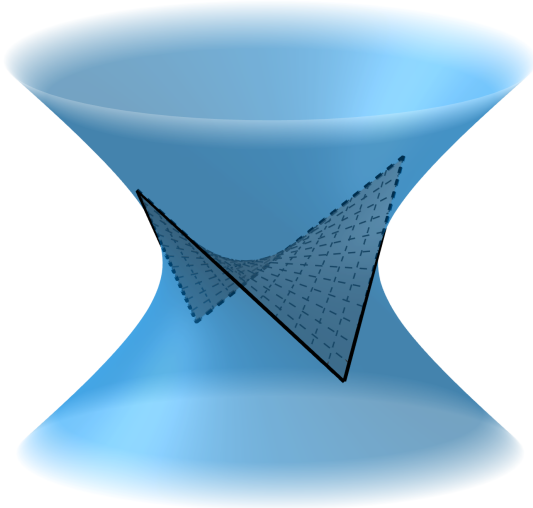

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These examples are extreme because of the following [Cheng]:
A maximal surface in $\mathbf{H}^{2,n}$ has sectional curvature $-1 \leq K \leq 0$.

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3. *Σ is asymptotically flat.*

Moreover, $(1), (2), (3)$ imply that $(\Sigma, g) \simeq \mathbb{C}$ and $q_4 \in \mathbb{C}[X]$.

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- Horofunctions and horoballs,
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- Tits distance on the ideal boundary.

Two difficulties

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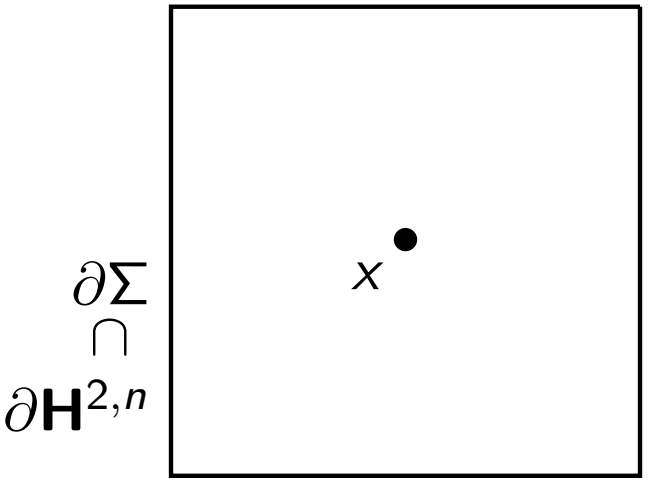
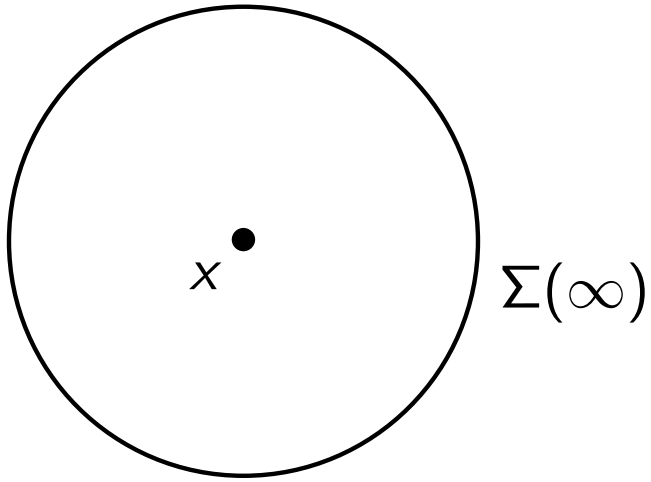
The boundaries are not easy to compare.

Barbot surface (polygon with 4 vertices)





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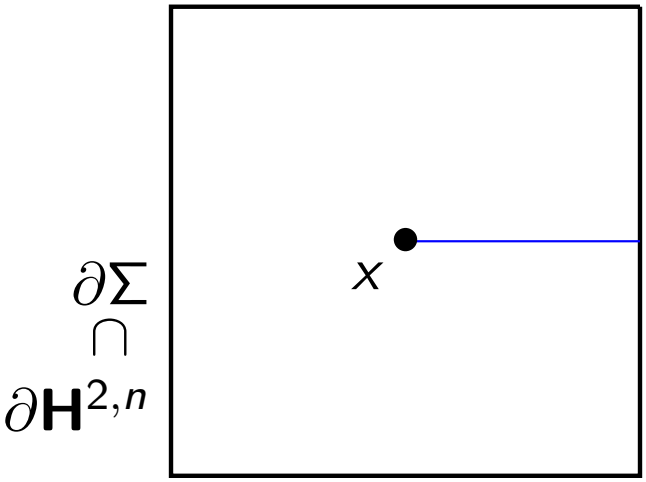

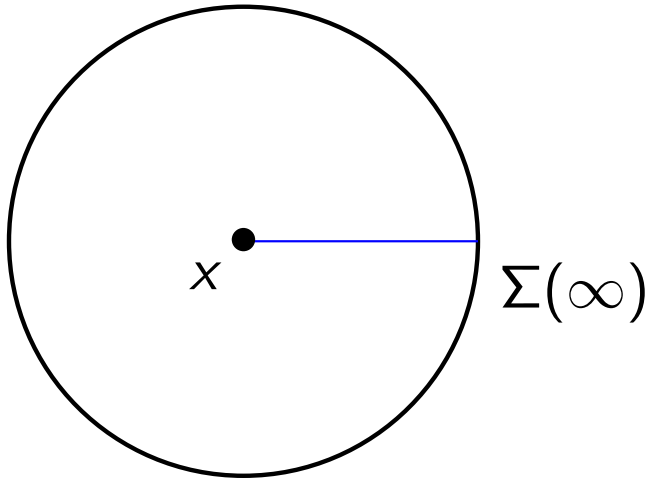

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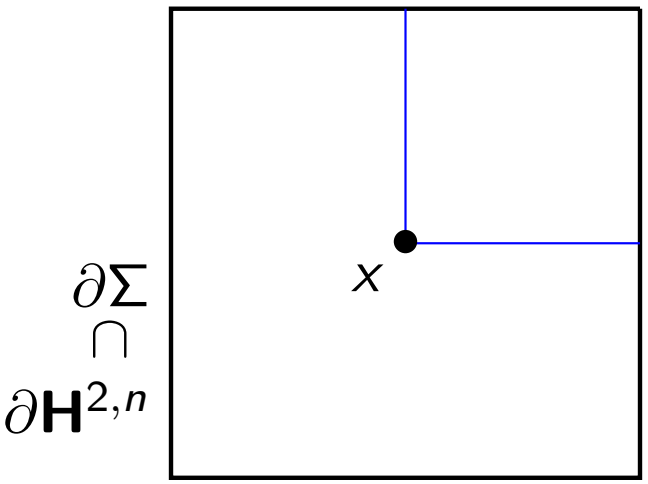

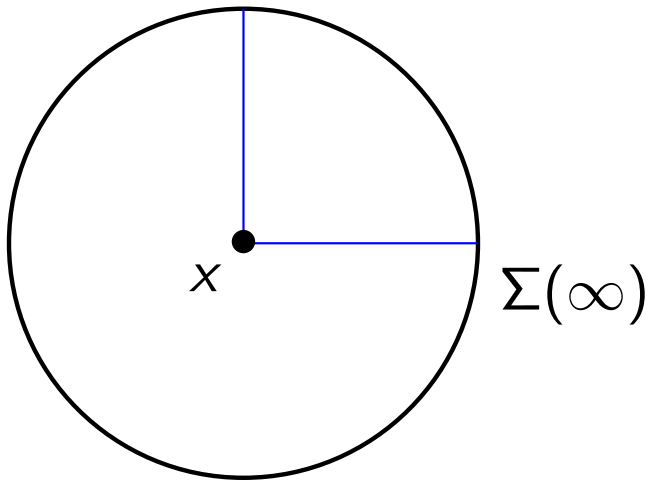

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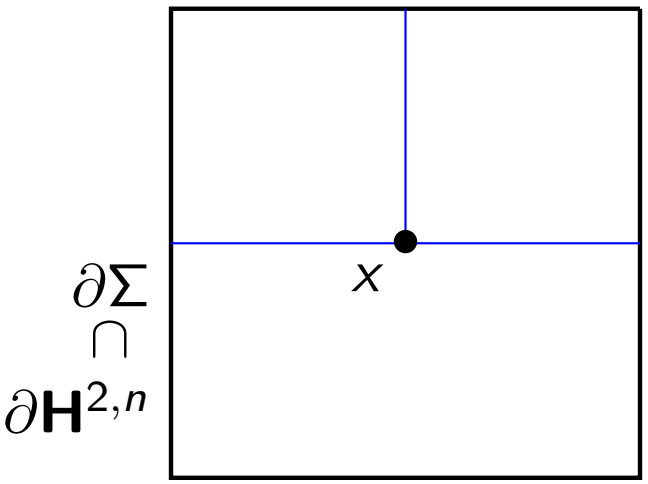

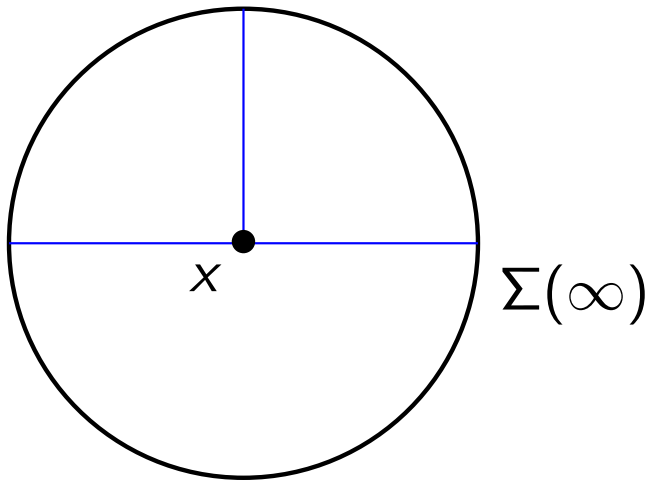

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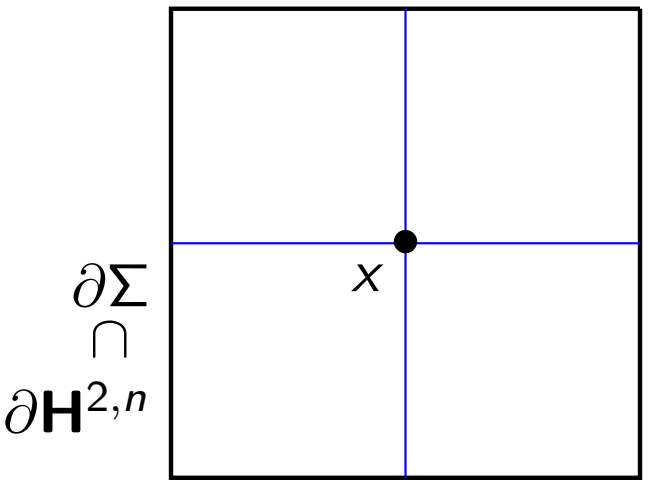

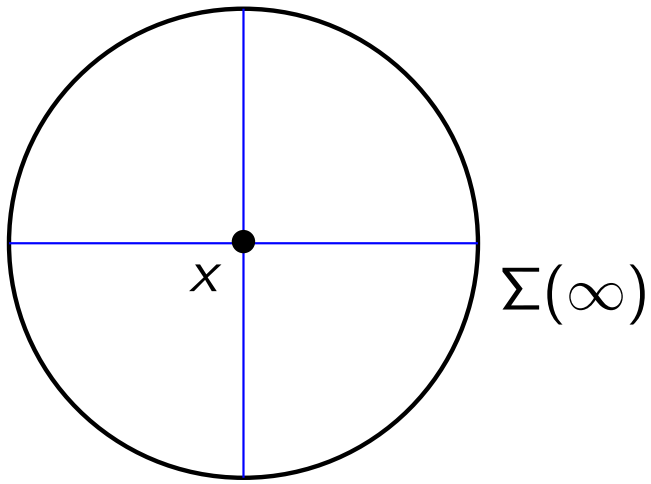

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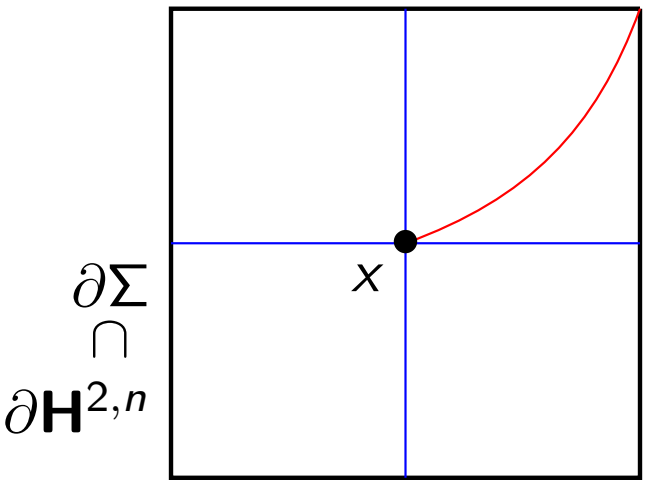

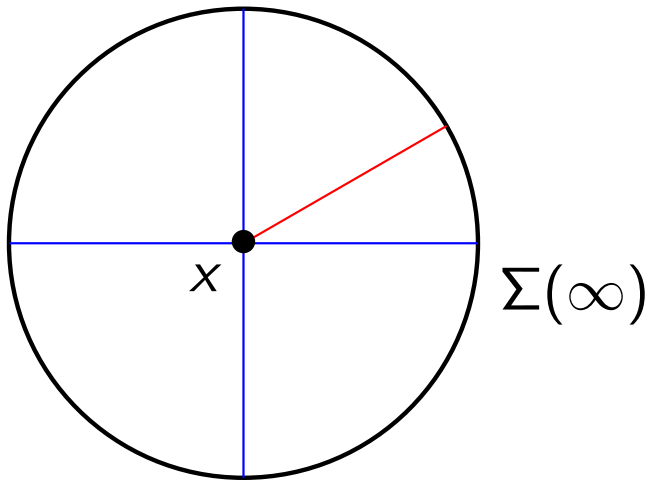

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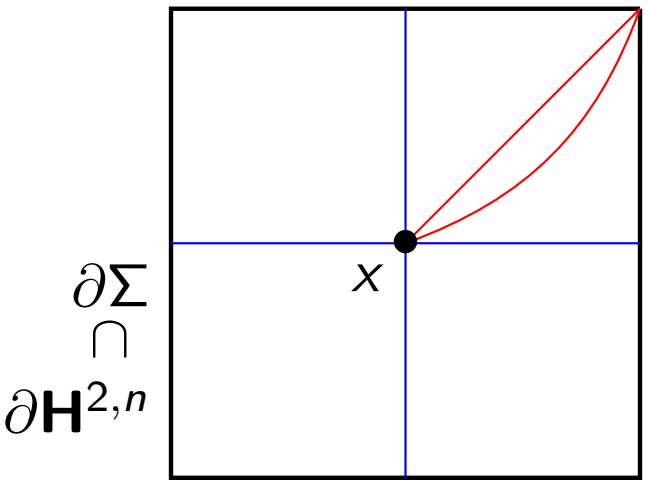

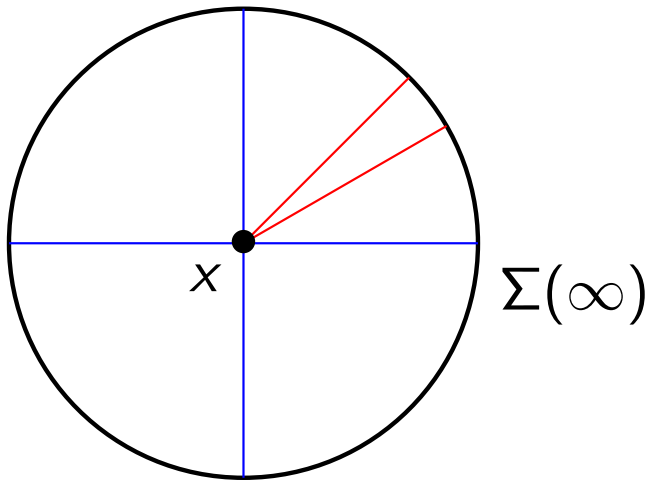

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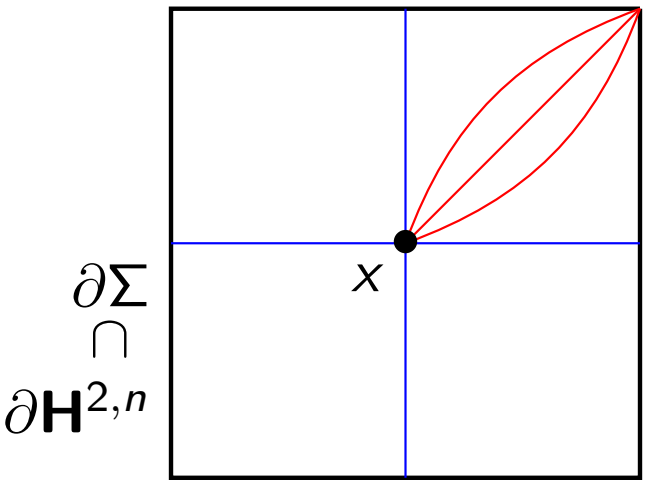

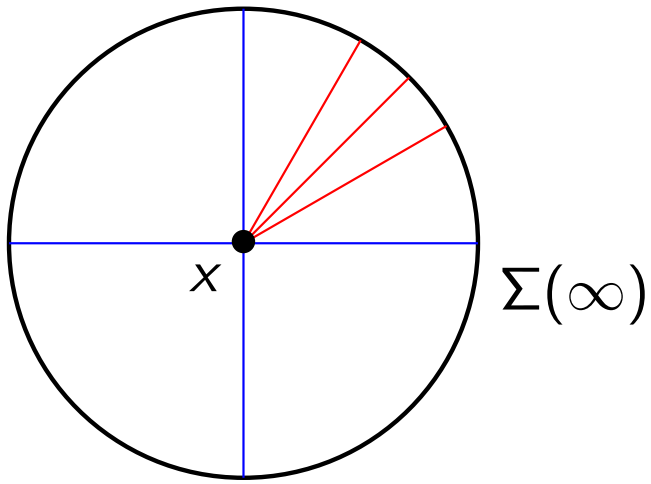

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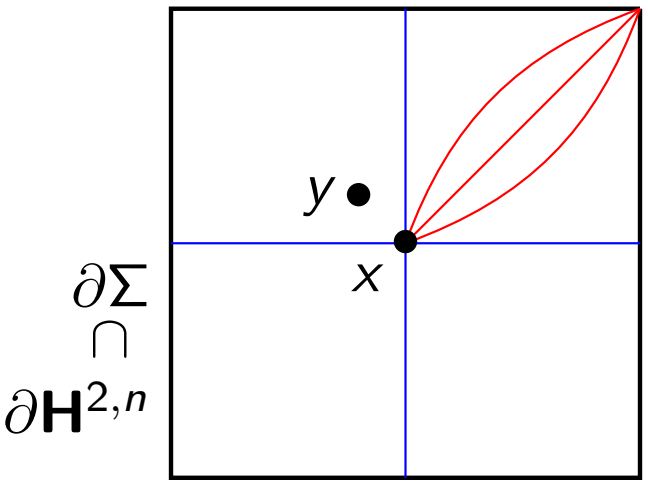

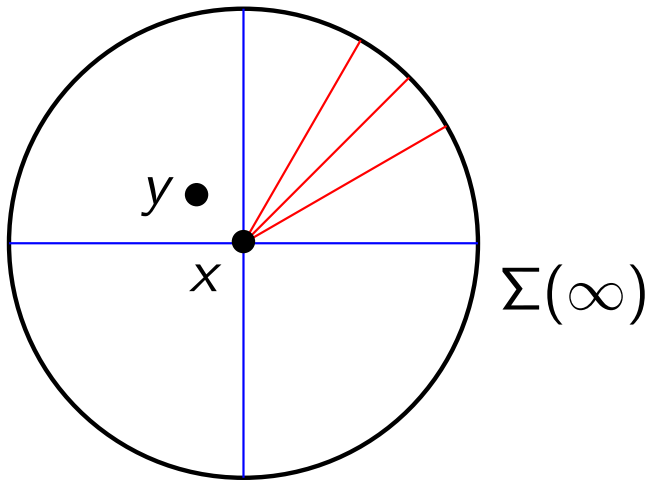

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