CONTROLLABILITY OF SATELLITES ON PERIODIC ORBITS WITH CONE-CONSTRAINTS ON THE THRUST DIRECTION

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Many observation satellites are subject to attitude constrains arising from peculiar mission requirements or environmental conditions. These obstructions often constrain the direction of the thrust vector to remain within a cone. In this study, we investigate the local controllability of station-keeping maneuvers of satellites with low thrust capabilities or small chemical impulsions on nominal periodic orbits subject to such constraints. We offer a numerical methodology based on convex optimization to identify the minimum cone angle guaranteeing local controllability for a specific orbit. An illustrative example inspired by the James Webb Space Telescope is proposed. Specifically, we consider a satellite is on a Halo orbit around L_2 in the Sun-Earth circular restricted three-body problem.

INTRODUCTION

Due to specific mission goals, many satellites are subject to cone constraints on the thrust direction. For example, James Webb Space Telescope, launched on December 25, 2021 toward a Halo orbit around the Sun-Earth L_2 libration point, has a thermal shield that must prevent the telescope and other instruments from overheating.[1] Therefore, it is constrained to always keep its attitude such that the angle between the normal to the shield and the Sun direction is smaller than 53 deg. It results in conical constraints for the propulsion directions. Using chemical propulsion to perform small impulsive corrections of the trajectory or a low-thrust satellite with very specific constraints on the control does not always allow to do any desirable maneuver, as we showed in,[2] where the controllability of non-ideal solar sails in orbit about a planet was investigated.

In,[2] we considered elliptic Keplerian orbits, and we formulated a convex optimization problem aimed at assessing whether some functions of the integrals of motion could not be decreased after one orbital period. Existence of such functions implies that there is a half-space of the neighborhood orbit's coordinates (orbital elements) where motion is locally forbidden.[3] In that paper, we strongly relied on the super-integrability of the Kepler problem. Here, we extend the methodology to infer local controllability of station-keeping satellites for *any* periodic orbit, regardless the dynamical system at hand. Given the projection of the nominal orbit on a surface of section, the methodology aims at verifying if a half space of such projection exists where the motion is forbidden after one orbital period. Variation of parameters is used to achieve a convex optimization problem that investigates the existence of obstructions to variations of local integrals of motion. Conical constraints are enforced by leveraging on the formalism of positive polynomials postulated by Nesterov,[4] so

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that a finite-dimensional formulation of the convex program is achieved. Halo orbit in the circular restricted three-body problem (CRTBP) is eventually considered in the case study, but we emphasize again that the methodology is developed for a generic locally-integrable system.

EQUATIONS OF MOTION

Consider the equations of motion of a control-affine dynamical system of dimension n with m controls subject to cone constraint on the control, namely

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = f(x) + B(x)u, \qquad x \in M \subseteq \mathbb{R}^n, \qquad u \in K_\alpha \subset \mathbb{R}^m,\ \|u\| \leq \varepsilon \tag{1}$$

Here, K_α is a cone of revolution characterized by an opening angle α, ε is thrust magnitude, which is assumed to be small, and $f(x)$ denotes a generic drift, e.g., for the CRTBP we have

$$f(x) = \begin{pmatrix} v_x \\ v_y \\ v_z \\ \left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 1\right) r_x - \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right)(1-\mu)\mu + 2v_y \\ \left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + 1\right) r_y - 2v_x \\ \left(-\frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right) r_z \end{pmatrix}, \qquad B(X) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2}$$

where μ is the mass ratio of the system, $x = (r_x, r_y, r_z, v_x, v_y, v_z)$ are position and velocity coordinates in the classical synodic frame, and r_1 and r_2 are distances between the satellite and the two main bodies:

$$r_1 = \sqrt{(r_x + \mu)^2 + r_y^2 + r_z^2},$$
$$r_2 = \sqrt{(r_x - 1 + \mu)^2 + r_y^2 + r_z^2}.$$

NECESSARY CONDITION FOR LOCAL CONTROLLABILITY

Given the conical constraint on the thrust vector, $u \in K_\alpha$, we are interested in determining if System (1) is locally controllable. Specifically, given a periodical (uncontrolled) reference orbit $y(t)$ of period T and a surface of section $S(x)$, and denoting x_0 the coordinates of the orbit at the crossing of $S(x)$, namely

$$\begin{cases} \frac{\mathrm{d}\,y}{\mathrm{d}\,t} = f(y) \\ y(0) = y(T) = x_0 \\ S(x_0) = 0 \end{cases} \tag{3}$$

we are interested in determining if controls in K_α are capable of moving the crossing point on $S(x)$ in an open neighborhood of x_0 after a period T, as shown in Fig. 1. To this purpose, *we introduce a necessary condition on α for the given orbit in order to have local controllability under the constraint $u \in K_\alpha$.*

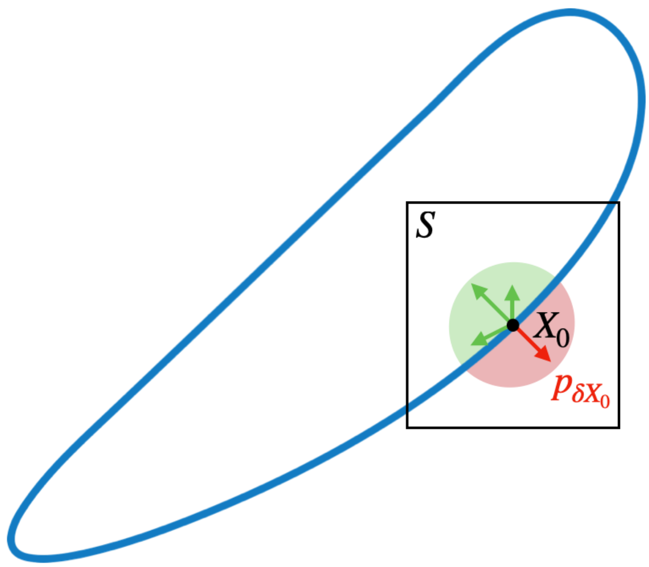

Figure 1: Forbidden half-space of δx_0 generated by $p_{\delta x_0}$

Denoting by $\Phi(t, x_0)$ the state transition matrix of the system, and by $\delta x_0 \in T_{x_0}S$ a perturbation of the initial state x_0, uncontrolled linearized motion in proximity of the periodic orbit is governed by

$$\delta x(t) = \Phi(t, x_0)\delta x_0. \tag{4}$$

Linearization of Eq. (1) gives:

$$\frac{\mathrm{d}\,\delta x}{\mathrm{d}\,t} = \left.\frac{\partial\, f}{\partial\, x}\right|_y \delta x + B(y)u. \tag{5}$$

Recalling that $\frac{\mathrm{d}\,\Phi}{\mathrm{d}\,t} = \frac{\partial\, f}{\partial\, x}\,\Phi$, differentiation of Eq. (4) and substitution in Eq. (5) yields the classical variation of parameters

$$\frac{\mathrm{d}\,\delta x_0}{\mathrm{d}\,t} = \Phi^{-1}(t, x_0)\,B(y(t))\,u, \quad \delta x_0 \in T_{x_0}S, \quad u \in K_\alpha. \tag{6}$$

The necessary condition for local controllability of the satellite is written in terms of possible displacements of the system on the Poincaré map, *i.e.* by verifying if the system can be moved everywhere in the tangent space $T_{x_0}S$ after one orbital period. For mathematical proof of the necessary condition please refer to.[3] Negation of this condition implies the existence of a not accessible half-space in the neighborhood of x_0, as shown in Fig. 1. Since the interior thrust directions of K_α can be approximated by combinations of vectors on the boundary of the cone, ∂K_α, we propose to solve the following problem in order to verify the necessary condition:

if $\exists\, p_{\delta x_0} \in T^*_{x_0}S,\ p_{\delta x_0} \neq 0$ such that

$$\left\langle p_{\delta x_0}, \frac{\mathrm{d}\,\delta x_0}{\mathrm{d}\,t} \right\rangle \geq 0, \qquad \forall\, u \in \partial K_\alpha, \|u\| = 1,\ t \in [0, T) \tag{7}$$

then System (1) is not locally controllable in one orbit.

If $p_{\delta x_0}$ solution of Problem (7) exists, then the linear functional

$$V(t,u) = \left\langle p_{\delta x_0},\ \Phi^{-1}(t,x_0)\, B(y(t))\, u \right\rangle$$

cannot be decreased for any $u \in K_\alpha$ and $t \in [0,T)$, hence motion is forbidden in the half-space with normal $p_{\delta x_0}$, and the satellite cannot move in any direction pointing inside this half-space after one orbital period. Absence of forbidden directions for control of satellites is crucial for station-keeping.

CONVEX OPTIMIZATION PROBLEM TO VERIFY THE NECESSARY CONDITION

A practical check of the necessary condition is carried out by solving

$$\begin{aligned} &\max_{J,\ \|p_{\delta x_0}\| \le 1} \quad J \quad \text{s.t.} \\ &\qquad \left\langle p_{\delta x_0},\ \Phi^{-1}(t,x_0) B(y(t)) u \right\rangle \ge J, \qquad \forall\, u \in \partial K_\alpha,\ \|u\| = 1,\ t \in [0,T]. \end{aligned} \tag{8}$$

Problem (8) is convex and semi-infinite, because inequality constraints need to be enforced for all u on the surface of the cone and for all time between 0 and the period T. Evaluating inequalities in the interior of the cone is not necessary because dynamics is affine in u. If J^*, solution of Problem (8), is strictly positive, then the necessary condition is not satisfied and the system is not locally controllable for the given α and x_0. The constraint $\|p_{\delta x_0}\| \le 1$ is preferred to the equality condition $\|p_{\delta x_0}\| = 1$ to preserve the convexity properties of Problem (8).

For mission design purposes, it is interesting to know which is the minimum α angle of the thrust cone satisfying the necessary condition. This angle can be identified by solving

$$\begin{aligned} \min_{\alpha}\ \alpha \qquad & \text{s.t.} \\ & J^*(\alpha) = 0 \end{aligned} \tag{9}$$

where $J^*(\alpha)$ denotes solution of Problem (8) for a given α. Problem (9) can be efficiently solved by means of a simple bisection method.

Discretization of the optimization problem

Numerical solution of Problem (8) is achieved by:

1. Parametrizing K_α by means of an angle δ, as shown in Fig. 2, to avoid discretization of the cone by using, for example, a polyhedral cone with a finite number of generators. Thus, vectors of $\boldsymbol{u}$ on the surface of the cone can be expressed as:

$$\boldsymbol{u} = \begin{bmatrix} \cos\alpha \\ \cos\delta\ \sin\alpha \\ \sin\delta\ \sin\alpha \end{bmatrix} \tag{10}$$

   with $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\delta \in [0, 2\pi]$;

2. Given that u is trigonometric in δ, using Fourier transform for Eq. (6):

$$\Phi^{-1}(t,x_0)\, B(y(t))\, u = \sum_{l=-1}^{1} \sum_{k=-d}^{d} C^{(k,l)} \mathrm{e}^{ikt} \mathrm{e}^{il\delta}$$

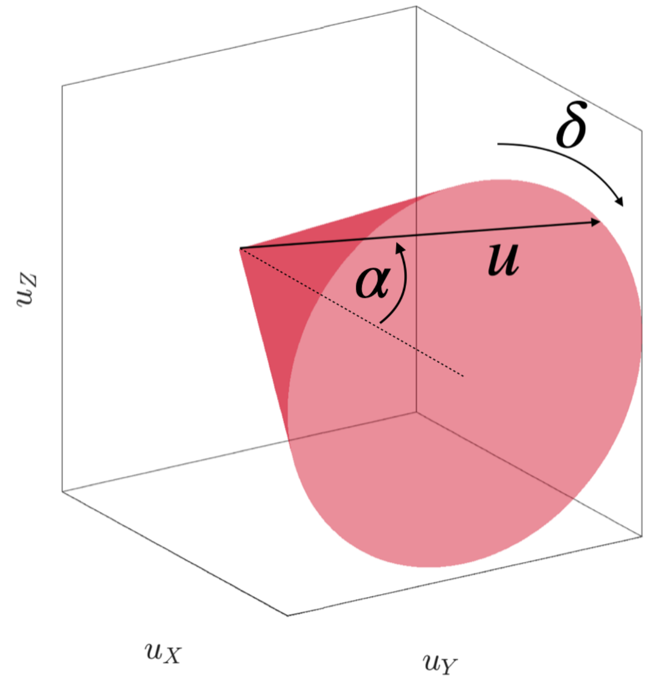

Figure 2: Parametrization of the control vector.

where $C^{(k,l)}$ is the kl-th coefficient of the Fourier transform of $\Phi^{-1}(t,x_0)\,B(y(t))\,u$ and d the degree of truncation of the series in t. Note that u is already an exact trigonometric polynomial of degree 1 in δ. Thus, the inequality from Eq. (8) becomes:

$$\left\langle p_{\delta x_0},\ \Phi^{-1}(t,x_0)B(y(t))u\right\rangle \geq J \iff p_{\delta x_0}^T\left(\sum_{l=-1}^{1}\sum_{k=-d}^{d} C^{(k,l)}\mathrm{e}^{ikt}\mathrm{e}^{il\delta}\right) - J \geq 0 \tag{11}$$

In the example of a Halo orbit given in this paper, we decide to truncate the Fourier series at d = 30, as the convergence of the coefficients is enough to find the minimum cone angle, as shown in Fig. 3;

3. Using the formalism of positive polynomials[4,5] to enforce positivity constraints.

Consider the basis of bivariate trigonometric polynomials of degree d in t and 1 in δ: $\mathcal{P}(t,\delta) = \left[1,\ \mathrm{e}^{i\delta}\right]^T \otimes \left[1,\ \mathrm{e}^{it},\ \mathrm{e}^{2it},\ldots,\ \mathrm{e}^{dit}\right]^T = \left[1,\ \mathrm{e}^{it},\ \mathrm{e}^{2it},\ldots,\ \mathrm{e}^{dit},\ \mathrm{e}^{i\delta},\ \mathrm{e}^{it}\mathrm{e}^{i\delta},\ \mathrm{e}^{2it}\mathrm{e}^{i\delta},\ldots,\ \mathrm{e}^{dit}\mathrm{e}^{i\delta}\right]^T$ and C vector of coordinates of the polynomial in the basis. Its corresponding squared functional system is $\mathcal{S}^2(t,\delta) = \mathcal{P}(t,\delta)\mathcal{P}^H(t,\delta)$, where $\mathcal{P}^H(t,\delta)$ denotes conjugate transpose of $\mathcal{P}(t,\delta)$. Let N be the dimension of $\mathcal{P}(t,\delta)$ ($N = 2\times(d+1)$ in our application) and $\Lambda_H : \mathbb{C}^N \rightarrow \mathbb{C}^{N\times N}$ a linear operator mapping coefficients of polynomials in $\mathcal{P}(t,\delta)$ to the squared base, so that application of Λ_H on $\mathcal{P}(t,\delta)$ yields

$$\Lambda_H(\mathcal{P}(t,\delta)) = \mathcal{P}(t,\delta)\mathcal{P}^H(t,\delta) \tag{12}$$

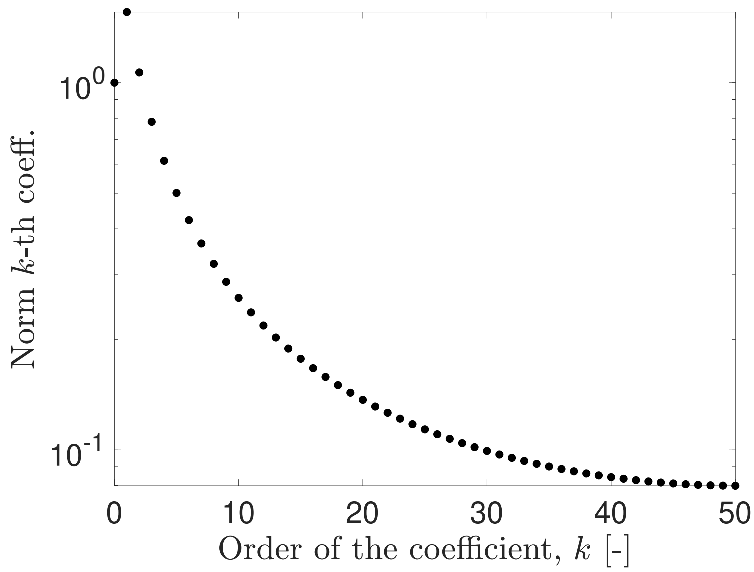

Figure 3: Convergence of Fourier coefficients

and define its adjoint operator $\Lambda_H^* : \mathbb{C}^{N\times N} \to \mathbb{C}^N$ as

$$\langle Y,\ \Lambda_H(C)\rangle_H \equiv \langle \Lambda_H^*(Y),\ C\rangle_H\,, \quad Y \in \mathbb{C}^{N\times N}, \quad C \in \mathbb{C}^N. \tag{13}$$

Theory of squared functional systems postulated by Nesterov[4] proves that trigonometric polynomial is non-negative if and only if a Hermitian positive semidefinite matrix Y exists such that $C = \Lambda_H^*(Y)$. Dumitrescu extends this theory for multivariate trigonometric polynomials in[5] and shows that all nonnegative bivariate trigonometric polynomials can be written as sum-of-squares. This equivalence is false for three or more variables.

Thus, $\left\langle \mathcal{P}(t,\delta), C\right\rangle_H$ is non-negative for all $t \in [0,T)$ and for all $\boldsymbol{u} \in K_\alpha$ if and only if a Hermitian positive semidefinite matrix Y exists such that $C = \Lambda_H^*(Y)$, namely

$$\langle \mathcal{P}(t,\delta),\ C\rangle_H \geq 0,\ t \in [0,T),\ \boldsymbol{u} \in K_\alpha \quad \Longleftrightarrow \quad \exists Y \succeq 0 :\ C = \Lambda_H^*(Y). \tag{14}$$

In fact, it holds in this case that

$$\begin{aligned} \left\langle \mathcal{P}(f,\delta),\ C\right\rangle_H &= \left\langle \mathcal{P}(f,\delta),\ \Lambda_H^*(Y)\right\rangle_H = \left\langle \Lambda_H(\mathcal{P}(f,\delta)),\ Y\right\rangle_H, \\ &= \left\langle \mathcal{P}(f,\delta)\mathcal{P}^H(f,\delta),\ Y\right\rangle_H = \mathcal{P}^H(f,\delta)\, Y\, \mathcal{P}(f,\delta) \geq 0. \end{aligned} \tag{15}$$

For trigonometric polynomials Λ^* is given by

$$\Lambda_H^*(Y) = \begin{bmatrix} \mathrm{tr}(\langle Y,\ T_{00}\rangle) \\ \vdots \\ \mathrm{tr}(\langle Y,\ T_{kl}\rangle) \\ \vdots \\ \mathrm{tr}(\langle Y,\ T_{21}\rangle) \end{bmatrix} \quad k = 0,1,2,\ l = 0,1. \tag{16}$$

where T_j $j = 0, 1, 2$ are the elementary Toeplitz matrices with ones on the j-th diagonal and zeros elsewhere and T_{kl} are obtained from a Kronecker product of such matrices, *e.g.*,

$$T_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$T_{10} = T_0 \otimes T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{17}$$

Finally, the inequality in Eq. (8) is rewritten as an linear matrix inequalities (LMI):

$$\begin{aligned} & \left\langle p_{\delta x_0},\, \Phi^{-1}(t, x_0) B u \right\rangle - J \geq 0,\ t \in [0, T),\ \boldsymbol{u} \in \partial K_\alpha \\ \iff \quad & \exists\, Y \succeq 0 \ \text{ such that } \quad C\, p_{\delta x_0} - \boldsymbol{e_1} J = \Lambda_H^*(Y) \end{aligned} \tag{18}$$

where $Y \in \mathbb{C}^{N \times N}$ is a Hermitian matrix to be determined, with $N = 2 \times (d+1) = 62$, and $\boldsymbol{e_1}$ is a vector of dimension N with 1 in the first position and zeros elsewhere. Hence, the finite-dimensional counterpart of Problem (8) is

$$\begin{aligned} \min_{J,\, \left\| p_{\delta x_0} \right\| \leq 1,\, Y \in \mathbb{C}^{62 \times 62}} \quad J \quad & \text{s.t.:} \\ & Y \succeq 0 \\ & \Lambda_H^*(Y) = C\, p_{\delta x_0} - \boldsymbol{e_1} J \end{aligned} \tag{19}$$

Solution of Problem (9) is carried out by means of a simple bisection algorithm, which does not require the evaluation of derivatives of the non-smooth function $J^*(\alpha)$ (we note that Problem (8) has trivial solution $J = 0,\ p_{\delta x_0} = 0$ for $\alpha > \alpha_{min}$). The CVX software[6,7] is used to solve the convex Problem (19). Fourier coefficients of $\Phi^{-1}(t, x_0)\, B(y(t))\, u$ are evaluated by means of the fast Fourier transform (FFT) algorithm. The only relaxation of Problem (19) with respect to Problem (8) is truncation of the Fourier series. Remarkably, no discretization was done to approximate $\boldsymbol{u}$ on the surface of a cone.

CASE STUDY

Let us consider a periodical Halo orbit situated around Sun-Earth L2 point, as shown in Fig. 4. It is the same point where James Webb Space Telescope was sent. We suppose that a satellite has to perform station-keeping around this orbit. The given satellite can produce either small impulsions using chemical propulsion or low-thrust engines, and has a conical constraint on the directions of the thrust. Our goal is to determine what is the maximum conical constraint that can be imposed on the propulsion, *i.e.* what is the minimum cone angle for thrust directions that allows local controllability after one orbital period. To find out the minimum requirement, we apply the proposed methodology on the given periodical Halo orbit. Initial data of the orbit is $x_0 = (1.0083, 5.15 \times 10^{-19}, 0.0010, 1.3714 \times 10^{-16}, 0.0102, -4.1015 \times 10^{-17})$ in AU according to the Sun-centered reference frame.

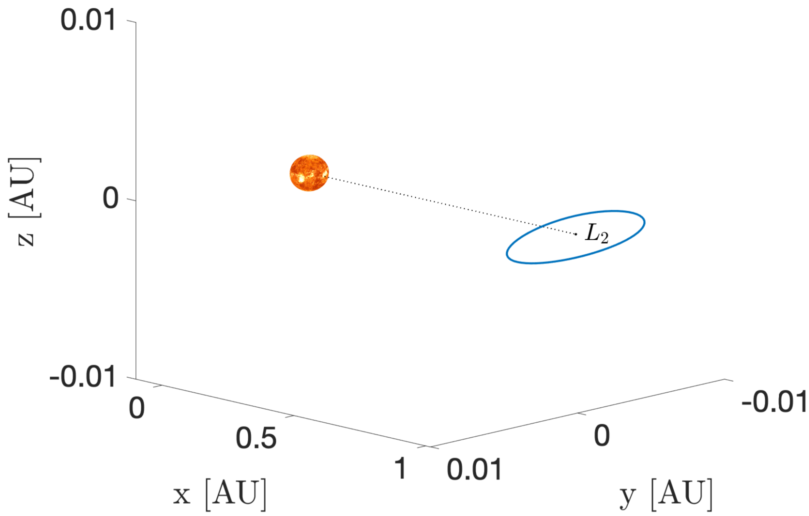

Figure 4: Halo orbit used for the simulation. Size of the Sun is schematical.

The results given by Fig. 5 show that the minimum thrust cone angle $\alpha = 43$ deg exists, and is a necessary requirement for local controllability of a station-keeping satellite using low-thrust or small chemical impulsions. The results mean that a satellite with thrust directions limited by a cone of less than 43 degrees is not capable of moving anywhere around the neighborhood of an intial position after one orbital period. Moreover, it indicates that there exists a half-space of the initial configuration neighborhood which includes all forbidden directions. For example, the satellite might not be capable of raising its velocity in y-direction, or decreasing its z-position after one orbital period, therefore it is not locally controllable in one orbital period. Global controllability can still hold, but in this case the satellite has to move away from the initial orbit to perform the necessary maneuver and then to come back. Nevertheless, it would require an important amount of propellant or it is probably not feasible by the low-thrust engines.

In[8] the authors looked at the controllability and the impact of limitations of the thrust direction on the station-keeping from a dynamical point of view. They use the Floquet Mode reference frame to describe the motion of the satellite in a close proximity to the orbit, and study the cost of station-keeping by projecting the thurst direction on the saddle plane. Their results show that, for a satellite that is escaping away from the Sun, a delta-v maneuver pointing towards the Sun is required and this one is only possible for a cone angle $\alpha > 50$ deg, which is consistent with the results presented in this cases study. However, a more detailed analyses is required to compare both approaches.

CONCLUSION

In this paper we propose a methodology to find the minimum requirement for station-keeping of the satellites with cone-constrained thrust. Our analysis is inspired by the James Webb Space Telescope, which has to maintain the imposed attitude towards the Sun because of the solar shield protecting its instruments. We formulate a convex optimization problem giving a solution in terms of a minimum cone angle of the thrust directions allowing local controllability. In other words, the

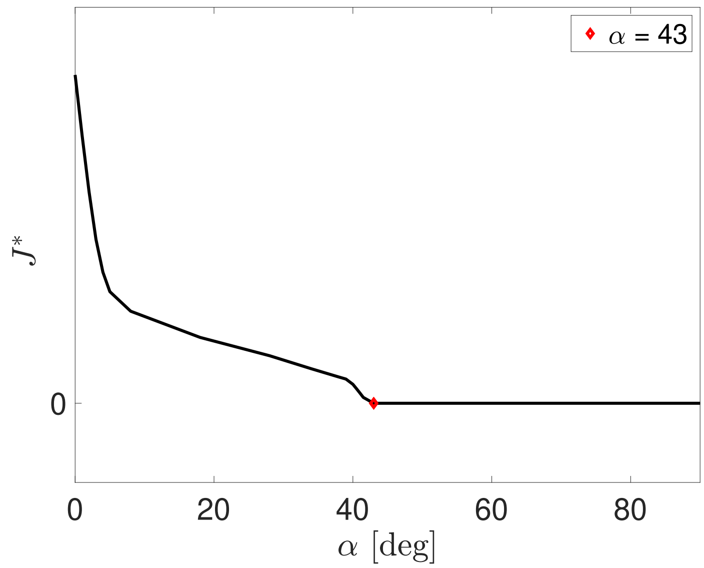

Figure 5: Solution of Problems (8) (black curve) and (9) (red dot).

minimum condition is a necessary condition for the satellite to be capable of moving anywhere to maintain its position on the orbit. The proposed methodology verifies when the condition does not hold and consists in finding a forbidden half-space in the neighborhood of the initial configuration of the satellite on the Poincaré map where it cannot move after one orbital period. The optimization problem is solved using convex programming and theory of positive bivariate trigonometric polynomials. The minimum requirement that we propose can be used for a design of space missions around any periodic orbit for satellites that have specific constraints on the thrust directions. It can be applied to low-thrust or even chemical propulsion under condition of using small impulses to maintain the satellite on the orbit.

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