Controllability Test for Fast-Oscillating Systems with Constrained Control. Application to Solar Sailing

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***Abstract*— For control systems whose uncontrolled solutions are periodic (or more generally recurrent), there are geometric tools, developed in the 1980s, that assess controllability based on a Lie algebra rank condition, under the assumption that the control set contains zero in its interior. Motivated by solar sails control, the present study explores the case where zero is rather on the boundary of the control set. More precisely, it investigates the controllability of fast-oscillating dynamical systems subject to positivity constraints on the control variable, *i.e.*, the control set is contained in a cone with vertex at the origin. A novel sufficient controllability condition is stated, and a constructive methodology is offered to check this condition, and to generate the controls, with values in the convex cone, that move, at first order, the slow state to an arbitrary direction of the tangent space. Controllability of a solar sail in orbit about a planet is analysed to illustrate the developments. It is shown that, given an initial orbit, a minimum cone angle parametrising the control set exists which satisfies the sufficient condition.**

I. INTRODUCTION

We are interested in studying controllability properties of fast-oscillating smooth dynamical systems of the form

$$\begin{cases} \dfrac{\mathrm{d}\,I}{\mathrm{d}\,t} = \varepsilon \displaystyle\sum_{i=1}^{m} u_i F_i(I,\varphi) \\ \dfrac{\mathrm{d}\,\varphi}{\mathrm{d}\,t} = \omega(I) \\ u = (u_1,\dots,u_m) \in U \end{cases} \tag{1}$$

with state (I,φ) in $M \times \mathbb{S}^1$ with M a real analytic n-dimensional manifold and control u constrained to the fixed bounded subset U of $\mathbb{R}^m$, typically of the shape represented in Fig. 1; $\varepsilon > 0$ is a small parameter, each F_i, $1 \le i \le m$, is a smooth map $M \times \mathbb{S}^1 \to TM$ and ω a smooth map $\omega : M \to \mathbb{R}$. This a particular case of an affine control system

$$\dot{x} = F_0(x) + u_1 F_1(x) + \ldots + u_m F_m(x),$$

with $x = (I,\varphi)$, $F_0 = \omega(I)\,\partial/\partial\varphi$ and we use the same notation F_i, $1 \le i \le m$, for both the above-mentioned smooth map $M \times \mathbb{S}^1 \to TM$ and for the vector field on $M \times \mathbb{S}^1$ whose projections on the first and second factor of the product are respectively that smooth map and zero.

A classical approach to study controllability of these general systems is to evaluate the rank of their Lie algebra. The so called Lie algebra rank condition (LARC) requires that this rank be equal to the dimension of the state space at all points. It is necessary for controllability, at least in the real analytic case, but sufficiency requires additional conditions. A well known condition (*e.g.*, [1], [2] or the textbook [3, Chapter 4, Section 6]) requires that the drift F_0 be recurrent; in our case, all solutions of $\dot{x} = F_0(x)$ are periodic, which is a special case of recurrence. The LARC plus this recurrence property imply controllability if the control $u = (u_1,\dots,u_m)$ is constrained to a subset U of $\mathbb{R}^m$ whose convex hull is a neighbourhood of the origin [3, Theorem 5, Chapter 4]. Here, we are interested in systems where the origin is rather *on the boundary* of U, see Fig. 1. To the best of our knowledge, controllability of such systems is not covered in the literature; it is surprising that this setting is rarely considered. We establish a new sufficient condition (Theorem 1) where an additional condition (2) is required.

This sufficient condition is however harder to check than the rank of a family of vector fields. We propose in Sections II-B and III an efficient test for this property. Such a test is performed by solving a convex optimisation problem, which leverages on the formalism of squared functional systems outlined in [4]. The proposed methodology does not require any initial guess to find admissible controls that allow the system to move in the desired direction.

The methodology is then applied to orbital control of a solar sail in orbit around a planet. This is a typical case of

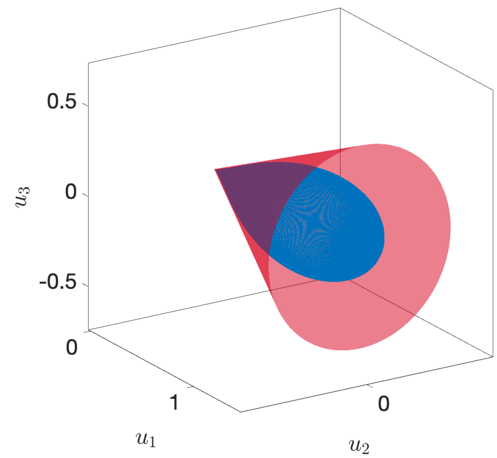

Fig. 1. Example of orbital control with solar sails. Equations are of the form of System (1), the control $u = (u_1, u_2, u_3)$ is homogeneous to a force, and the solar sail only allows forces contained in the set U figured in blue in the picture (for some characteristics of the sail). The minimal convex cone containing the control set U is depicted in red. Neither U nor this cone are neighbourhoods of the origin.

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systems of the type (1). Since [5], there has been a lot of literature on optimal transfers and locally-optimal feedback strategies for solar sails [6]–[8], but a thorough analysis of the controllability of solar sails is not available, yet. A preliminary result of ours assessed non-controllability in some situations [9]. Here, we show that a minimum cone angle parametrising the control set U exists which makes the system locally controllable. This result may serve as a mission design requirement.

In short, this paper brings a new sufficient controllability condition for systems with control constraints that do not allow to apply well known controllability criteria based on recurrence of drift; a process based on convex optimisation that allows one to check this criteria easily; and an idea of how this applies to control of solar sails. Section II introduces our controllability result and its proof, and also explains how to verify the property (2) by formulating a control problem. Section III offers a methodology to recast the previous control problem into a finite-dimensional convex optimisation problem. Finally, Section IV describes dynamics of solar sails and results from application of the proposed methodology to study the controllability of the system.

II. Controllability of fast-oscillating systems

A. A condition for controllability

Consider System (1) and the associated vector fields $F_0, \ldots, F_m$ on $M \times \mathbb{S}^1$ defined right after Eq. (1). Let ε be a small positive parameter, the drift vector field is F_0 and the control vector fields are $\varepsilon F_1, \ldots, \varepsilon F_m$. We consider the following conditions:
(i) the LARC holds, *i.e.* $\{F_0, F_1, \ldots, F_m\}$ is bracket generating, at all (I, φ) in $M \times \mathbb{S}^1$,
(ii) the control set U contains the origin, and
(iii) for all $I \in M$,

$$\operatorname{cone}\left\{\sum_{i=1}^{m} u_i F_i(I,\varphi),\ u \in U,\ \varphi \in \mathbb{S}^1\right\} = T_I M \qquad (2)$$

where cone indicates the conical hull (a convex cone). The following holds:

Theorem 1: Under assumptions (i) to (iii), System (1) is controllable, *i.e.* for any (I_0, φ_0) and (I_1, φ_1) in $M \times \mathbb{S}^1$, there is a time $T \geq 0$ and a measurable control $u(.) : [0, T] \to U$ that drives (I_0, φ_0) to (I_1, φ_1) for System (1).

Proof. As in [10, Chapter 8] or [3, Chapter 3], we associate to the vector fields $F_0, \ldots, F_m$, the family of vector fields

$$\mathcal{E} = \{\, F_0 + u_1 F_1 + \cdots + u_m F_m\,,\ (u_1, \ldots, u_m) \in U\}$$

made of all the vector fields obtained by fixing in (1) the control to a constant value that belongs to U. We denote by $\mathscr{A}_{\mathcal{E}}(I, \varphi)$ the accessible set from (I, φ) of this family of vector fields in all positive (unspecified) time, *i.e.*, the set of points that can be reached from (I, φ) by following successively the flow of a finite number of vector fields in $\mathcal{E}$, each for a certain positive time, which is the same as the set of points that can be reached, for the control System (1), with piecewise constant controls. Our goal is to show that $\mathscr{A}_{\mathcal{E}}(I, \varphi) = M \times \mathbb{S}^1$ for any (I, φ), which implies controllability, actually with the smaller class of piecewise constant controls.

Define new families $\mathcal{E} \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3$ as follows:

$$\mathcal{E}_1 = \mathcal{E} \cup \{-F_0\}\,, \quad \mathcal{E}_2 = \{\, \exp(t\,F_0)_\star X\,,\ X \in \mathcal{E}_1,\ t \in \mathbb{R}\}\,,$$
$$\mathcal{E}_3 = \operatorname{cone}(\mathcal{E}_2)\,,$$

where $\exp(t\,F_0)_\star X$ denotes the pullback of the vector field X by the difféomorphism $\exp(t\,F_0)$ and cone $(\mathcal{E}_2)$ denotes the family made of all vector fields that are finite combinations of the form $\sum_k \lambda_k X_k$ with each X_k in $\mathcal{E}_2$ and each λ_k a positive number (conic combination). One has, for all (I, φ),[1]

$$\mathscr{A}_{\mathcal{E}_1}(I, \varphi) = \mathscr{A}_{\mathcal{E}}(I, \varphi)$$

because on the one hand condition (ii) implies $F_0 \in \mathcal{E}$, and on the other hand, for any (I', φ'), $\exp(-t\,F_0)(I', \varphi') = \exp\big((-t + 2k\pi/\omega(I'))\,F_0\big)(I', \varphi')$ for all positive integers k, but for fixed t and I', $-t + 2k\pi/\omega(I')$ is positive for k large enough. Since F_0 and $-F_0$ now belong to $\mathcal{E}_1$, we have $\exp(t\,F_0)(I, \varphi) \in \mathscr{A}_{\mathcal{E}}(I, \varphi)$ for all (I, φ) in $M \times \mathbb{S}^1$ and all t in $\mathbb{R}$, hence $\exp(t\,F_0)$ is according to [3, Chapter 3, Definition 5 and next Lemma] a “normaliser” of the family $\mathcal{E}_1$ and, according to Theorem 9 in the same chapter of the same reference, this implies that[1]

$$\mathscr{A}_{\mathcal{E}_2}(I, \varphi) \subset \overline{\mathscr{A}_{\mathcal{E}_1}(I, \varphi)} \qquad (3)$$

where the overline denotes topological closure (for the natural topology on $M \times \mathbb{S}^1$). Now, [10, Corollary 8.2] or [3, Chapter 3, Theorem 8(b)] tell us that

$$\mathscr{A}_{\mathcal{E}_3}(I, \varphi) \subset \overline{\mathscr{A}_{\mathcal{E}_2}(I, \varphi)}\,. \qquad (4)$$

These inclusions are of interest because condition (iii) implies that $\mathscr{A}_{\mathcal{E}_3}(I, \varphi)$ is the whole manifold: indeed, (2) (written in terms of the I-directions only, but adding F_0 and $-F_0$ yields the whole tangent space to $M \times \mathbb{S}^1$) implies that $\mathscr{A}_{\mathcal{E}_3}(I, \varphi)$ is, for any (I, φ), a neighbourhood of (I, φ), obtained for small times, hence accessible sets are closed and open in the connected manifold). Together with (3)-(4), this implies $\overline{\mathscr{A}_{\mathcal{E}}(I, \varphi)} = M \times \mathbb{S}^1$, and finally $\mathscr{A}_{\mathcal{E}}(I, \varphi) = M \times \mathbb{S}^1$ from condition (i) and [10, Corollary 8.1]. This ends the proof of the theorem. □

Remark 2: If all vector fields are real analytic, (iii) implies (i). Indeed, if (i) does not hold, there is at least one point (I, φ) and a non zero one form ℓ of $T^*_{(I,\varphi)}(M \times \mathbb{S}^1)$ such that $\langle \ell, X(I, \varphi)\rangle = 0$ for any vector field X obtained as a Lie bracket of any order of $F_0, \ldots, F_m$. In particular,

(a) $\langle \ell, F_0(I, \varphi)\rangle = 0\,,$

(b) $\left\langle \ell, (\mathrm{ad}^j_{F_0} F_k)(I, \varphi)\right\rangle = 0$ for $j \in \mathbb{N},\ k = 1, \ldots, m\,.$

From (a), ℓ vanishes on the direction of $\partial/\partial\varphi$ and hence can be considered as a nonzero linear form on $T_I M$. Define the smooth maps $a_k : \mathbb{R} \to \mathbb{R}$ by $a_k(t) = \langle \ell, \exp(-tF_0)_\star F_k\,(I, \varphi)\rangle$ for $k = 1, \ldots, m$, where $\star$ denotes

[1] In the terminology of [10, Section 8.2], $-F_0$ is compatible with $\mathcal{E}$, the vector fields in $\mathcal{E}_2$ are compatible with $\mathcal{E}_1$, and the vector fields in $\mathcal{E}_3$ are compatible with $\mathcal{E}_2$.

the pullback by a diffeomorphism. Classical properties of the Lie bracket[2] imply

$$\frac{\mathrm{d}^j\, a_k}{\mathrm{d}t^j}(0) = \left\langle \ell\,,\ (\mathrm{ad}^j_{F_0} F_k)(I,\varphi) \right\rangle,$$

and the right-hand side is zero from (b); if all the vector fields are real analytic, so are the maps a_k, and they must be identically zero if all their derivatives at zero are zero. Seen the particular form of F_0, one has $(\exp(-tF_0)_\star F_k)(I,\varphi) = F_k(I,\varphi + t\omega(I))$; since $\omega(I) \neq 0$, one finally deduces that $\langle \ell, F_k(I,\varphi)\rangle = 0$ for all φ in $\mathbb{S}^1$, this contradicts point (iii).

Remark 3 (Localisation): Checking condition (iii) is not as simple as the rank condition (i). Assume that (i) holds everywhere but (iii) is only known to hold at *one* point $I \in M$. Then it also holds at all points in some neighborhood O of I, hence all the assumptions of Theorem 1 hold with $M \times \mathbb{S}^1$ replaced with $O \times \mathbb{S}^1$, hence controllability holds on $O \times \mathbb{S}^1$. According to Remark 2, it is even not necessary to assume that (i) holds everywhere in the analytic case. Localisation in general of theorems in the style of [3, Theorem 5, Chapter 4] would only hold on a set that is invariant under the flow of the drift vector field, which is structurally the case of $O \times \mathbb{S}^1$ here. Note that no additional requirement with respect to the control vector fields (in particular completeness) is needed.

Condition (i) can be checked via a finite number of differentiations, and (ii) by inspection. One goal of this paper is to give a verifiable check, relying on convex optimisation, of the property (iii) at a given point I. In view of Remark 3, it yields controllability on $O \times \mathbb{S}^1$ with O a neighbourhood of that point.

B. Accessory convex control problems

Fix I in M, and consider the evaluation of condition (iii) at this single point:

$$\mathrm{cone}\left\{ \sum_{i=1}^m u_i F_i(I,\varphi),\ u \in U,\ \varphi \in \mathbb{S}^1 \right\} = T_I M. \qquad (5)$$

Proposition 4: Let $e_0, \dots, e_n$ in $T_I M$ be the vertices of an n-simplex containing 0 in its interior; condition (5) holds if and only if, for all $k \in \{0,\dots,n\}$, the accessory convex control problem with state δI valued in $T_I M$

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\delta I(\varphi) = \sum_{i=1}^m u_i(\varphi) F_i(I,\varphi), \quad u(\varphi) \in \mathrm{cone}(U), \qquad (6)$$

$$\delta I(0) = 0, \quad \delta I(2\pi) = e_k, \qquad (7)$$

is feasible.

Proof. Negating (5) is equivalent to asserting the existence of p_I in $T_I^* M$, nonzero, such that

$$\left\langle p_I, \sum_{i=1}^m u_i F_i(I,\varphi) \right\rangle \leq 0, \quad \varphi \in \mathbb{S}^1, \quad u \in U.$$

In this inequality, one can replace U by its conical hull. Moreover, it is still equivalent that

$$\left\langle p_I, \int_0^{2\pi} \sum_{i=1}^m u_i(\varphi) F_i(I,\varphi)\,\mathrm{d}\varphi \right\rangle \leq 0 \qquad (8)$$

for all u in $\mathscr{L}^\infty(0,2\pi)$ valued in $\mathrm{cone}(U)$. Indeed, one implication is obvious by linearity and positivity of the integral, while the converse is true since the Dirac measure at any φ in $[0,2\pi]$ can be approximated by a sequence of $\mathscr{L}^\infty$ functions valued in $\mathrm{cone}(U)$. Finally, since the simplex generated by $e_0,\dots,e_n$ is a neighbourhood of the origin in $T_I M$, negating the existence of a nonzero p_I in $T_I^* M$ such that (8) holds takes us back to condition (5), and says the following: for all k in $\{0,\dots,n\}$, there is an essentially bounded control valued in the conical hull of U such that

$$\int_0^{2\pi} \sum_{i=1}^m u_i(\varphi) F_i(I,\varphi)\,\mathrm{d}\varphi = e_k,$$

which is the expected set of $n+1$ feasibility conditions. □

One way to check these conditions is to consider, for each k in $\{0,\dots,n\}$, the accessory convex optimal control problem

$$\frac{1}{2}\int_0^{2\pi} |u(\varphi)|^2\,\mathrm{d}\varphi \to \min$$

under constraints (6)-(7). We show in the next section that each of these problems can be accurately approximated by a convex mathematical program. These finite dimensional problems are obtained by approximating $K := \mathrm{cone}(U)$ by a polyhedral cone and truncating the Fourier series of the control.

III. Discretisation of the accessory problem

A conservative discretisation of the accessory control problems is achieved in two steps. First, K is approximated by the polyhedral cone $K_g \subset K$ generated by g vertices $G_1,\dots,G_g$ chosen in ∂K: admissible controls are given by a conical combination of the form

$$u(\varphi) = \sum_{j=1}^g \gamma_j(\varphi) G_j, \quad \gamma_j(\varphi) \geq 0, \quad \varphi \in \mathbb{S}^1, \quad j = 1,\dots,g.$$

Second, an N-dimensional basis of trigonometric polynomials, $\Phi(\varphi) = \left(1, e^{i\varphi}, e^{2i\varphi}, \dots, e^{(N-1)i\varphi}\right)$, is used to model functions γ_j as

$$\gamma_j(\varphi) = (\Phi(\varphi)|c_j)_H$$

where $c_j \in \mathbb{C}^N$ are complex-valued coefficients (serving as design variables of the finite-dimensional problem), and $(\cdot|\cdot)_H$ is the Hermitian product on $\mathbb{C}^N$. Positivity constraints on the functions γ_j define a semi-infinite optimisation problem; these constraints are enforced by leveraging on the formalism of squared functional systems outlined in [4] which allows to recast continuous positivity constraints into linear matrix inequalities (LMI). Specifically, given trigonometric polynomial $p(\varphi) = (\Phi(\varphi)|c)_H$ of degree at most $N-1$ and the linear operator $\Lambda^* : \mathbb{C}^{N\times N} \to \mathbb{C}^N$ associated to $\Phi(\varphi)$ (more details are provided in Appendix A), it holds that p

[2]For any two vector fields Y and Z, one has $\frac{\mathrm{d}}{\mathrm{d}t}\,(\exp(-tY)_\star Z)\,(I,\varphi) = (\exp(-tY)_\star [Y,Z])\,(I,\varphi)$.

is representable as an appropriate sum of squares (which is sufficient to ensure nonnegativeness for all φ in $\mathbb{S}^1$) if and only if

$$(\exists\, Y \geq 0) : \Lambda^*(Y) = c.$$

For an admissible control u valued in K_g, one has

$$\int_0^{2\pi} \sum_{i=1}^{m} u_i(\varphi) F_i(I, \varphi)\, \mathrm{d}\varphi = \sum_{j=1}^{g} \left(L_j c_j + \bar{L}_j \bar{c}_j\right)$$

with $L_j(I)$ in $\mathbb{C}^{n\times N}$ defined by

$$L_j(I) = \frac{1}{2} \sum_{i=1}^{m} \int_{\mathbb{S}^1} G_{ij} F_i(I, \varphi) \Phi^H(\varphi)\, \mathrm{d}\varphi,$$

where $\Phi^H(\varphi)$ denotes the Hermitian transpose of $\Phi(\varphi)$ and where $G_j = (G_{ij})_{i=1,\ldots,m}$. We note that the components of $L_j(I)$ are Fourier coefficients of the function $\sum_{i=1}^m G_{ij} F_i(I, \varphi)$. The discrete Fourier transform (DFT) can be used to approximate $L_j(I)$. Since vector fields F_i are smooth, truncation of the series is justified by the fast decrease of the coefficients. Finally, for a control u valued in K_g with coefficients γ_j that are truncated Fourier series of order $N-1$, the L^2 norm over $\mathbb{S}^1$ is easily expressed in terms the coefficients c_j using orthogonality of the family of complex exponentials:

$$\begin{aligned}\frac{1}{2} \int_{\mathbb{S}^1} |u(\varphi)|^2\ \mathrm{d}\varphi &= \frac{1}{2} \sum_{j,l=1}^{g} \sum_{k=0}^{N-1} G_l^T G_j \left(\bar{c}_{lk} c_{jk} + c_{lk} \bar{c}_{jk}\right) \\ &= \sum_{l,j=1}^{g} G_l^T G_j (c_j | c_l)_H.\end{aligned}$$

As a result, for every vertex e_k, the finite-dimensional convex programming approximation is

$$\begin{aligned} \min_{c_j \in \mathbb{C}^N,\ Y_j \in \mathbb{C}^{N\times N}} \ & \sum_{j,l=1}^{g} G_j^T G_l (c_j | c_l)_H \quad \text{subject to} \\ & \sum_{j=1}^{g} \left(L_j c_j + \bar{L}_j \bar{c}_j\right) = e_k \\ & Y_j \geq 0, \quad \Lambda^*\left(Y_j\right) = c_j, \quad j = 1, \ldots, g. \end{aligned} \tag{9}$$

Proposition 5: If, for all $k = 0, \ldots, n$, problem (9) admits a solution, then condition (5) holds.

Proof. Let $k = 0, \ldots, n$, and choose g vertices $G_1, \ldots, G_g$ in ∂K. Any solution of (9) generates a control valued in $K_g \subset K = \mathrm{cone}(U)$, that is a feasible control for constraints (6)-(7). Apply Proposition 4 to conclude. □

IV. Controllability of a non-ideal solar sail

A. Orbital dynamics

The equations of motion of a solar sail in orbit about a planet are now introduced. Consider a reference frame with origin at the center of the planet, x_1 toward the Sun-planet direction, x_2 toward an arbitrary direction orthogonal to x_1, and x_3 completes the right-hand frame. Slow variables consist of Euler angles denoted I_1, I_2, I_3 orienting the orbital plane and perigee via a $1-2-1$ rotation as shown in Fig. 2. Then,

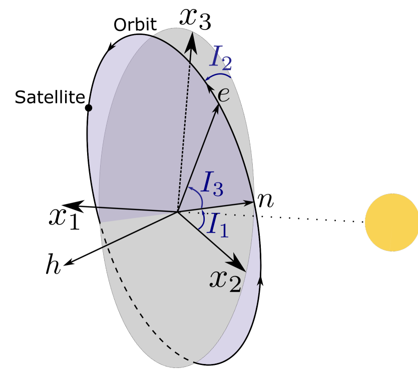


Fig. 2. Orbital orientation using Euler angles I_1, I_2, I_3. Here, h and e denote the angular momentum and eccentricity vectors of the orbit.

I_4 and I_5 are semi-major axis and eccentricity of the orbit, respectively. These coordinates define on an open set of $\mathbb{R}^5$ a standard local chart of the five-dimensional configuration manifold M [11]. The fast variable, $\varphi \in \mathbb{S}^1$, is the mean anomaly of the satellite. The motion of the sail is governed by Eq. (1). Vector fields $F_i(I, \varphi)$ are detailed in Appendix B, and $m = 3$. These fields are deduced by assuming that:

(i) Solar eclipses are neglected
(ii) solar radiation pressure (SRP) is the only perturbation
(iii) Orbit semi-major axis, I_4, is much smaller than the Sun-planet distance (so that radiation pressure has reasonably constant magnitude)
(iv) The period of the heliocentric orbit of the planet is much larger than the orbital period of the sail (so that motion of the reference frame is neglected).

We note that removing the first assumption may be problematic, since eclipses would introduce discontinuities (or very sharp variations) in the vector fields, which jeopardise the convergence of DFT coefficients. Other assumptions are only introduced to facilitate the presentation of the results and are not critical for the methodology.

B. Solar sail models

Solar sails are satellites that leverage on SRP to modify their orbit. Interaction between photons and sail's surface results in a thrust applied to the satellites. Its magnitude and direction depend on several variables, namely distance from Sun, orientation of the sail, cross-sectional area, optical properties (reflectivity and absorptivity coefficients of the surface) [12]. A realistic sail model combines both absorptive and reflective forces. Here, a simplified model is used by assuming that the sail is flat with surface A, and that only a portion ρ of the incoming radiation is reflected in a specular way ($\rho \in [0, 1]$ is referred to as reflectivity coefficient in the reminder). Hence, denoting n the unit vector orthogonal to the sail, δ the angle between n and x_1 (recall that x_1 is the direction of the Sun), $t = \sin^{-1}\delta\ x_1 \times (n \times x_1)$ a unit vector orthogonal to x_1 in the plane generated by n and x_1, the force per mass unit, m, of the sail is given by

$$F(n) = \frac{AP}{m} \cos\delta \left[(1 + \rho\cos 2\delta)\ x_1 + \rho \sin 2\delta\ t\right]$$

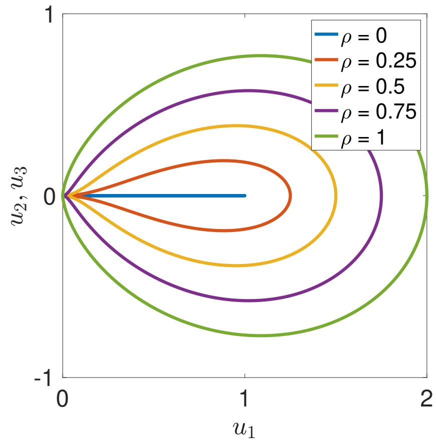


Fig. 3. Control sets for different reflectivity coefficients ρ

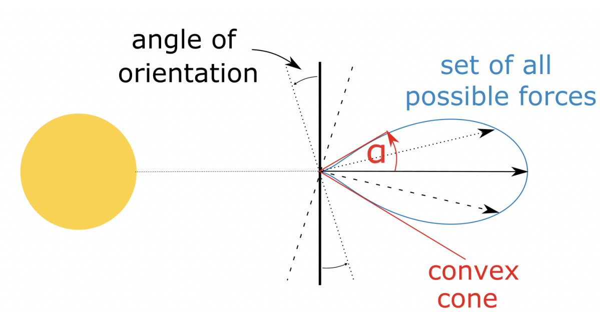


Fig. 4. Convexification of the control set

where P is the SRP magnitude and is a function of the Sun-sail distance. By virtue of assumption (iii), P is assumed to be constant, and the small parameter ε is set to $\varepsilon = AP/m$. Control set is thus given by

$$U = \left\{ \frac{F(n)}{\varepsilon},\ \forall\ n \in \mathbb{R}^3,\ |n| = 1,\ (n \,|\, x_1) \geq 0 \right\}$$

Fig. 3 shows control sets for different reflectivity coefficients of the sail. When $\rho = 0$ the sail is perfectly absorptive. In this particular case, Lie algebra of the system is not full rank. Conversely, $\rho = 1$ represents a perfectly-reflective sail, which is the ideal case. This set is symmetric with respect to x_1, and $K = \text{cone}(U)$ is a circular cone with angle obtained by solving

$$\tan\alpha = \min_{\delta\in[0,\pi/2]} \frac{(F(n)\,|\,t)}{(F(n)\,|\,x_1)} = \min_{\delta\in[0,\pi/2]} \frac{\rho\sin 2\delta}{(1+\rho\cos 2\delta)}$$

which yields

$$\alpha = \tan^{-1}\left(\frac{\rho}{\sqrt{1-\rho^2}}\right) \tag{10}$$

Fig. 1 and 4 show the minimal convex cone of angle α including the control set.

C. Simulation and results

Problem (9) is an SDP that can be efficiently solved in polynomial time, *e.g.* by interior point methods. (The complexity is polynomial wrt. the problem size, that is N and g here, and wrt. $\log\varepsilon$ where ε is a fixed additive error [13].) We use CVX, a package for specifying and solving convex programs [14], [15]. Table I lists initial conditions and parameters used for the simulations. Because of the symmetries of the problem, the results do not depend on I_1 (first Euler angle) or I_4 (semi-major axis), so we have not included them in the table. Figure 5 shows the magnitude of Fourier coefficients of vector fields. Polynomials are truncated at order 10. At this order, the magnitude of coefficients is reduced of a factor 10^3 with respect to zeroth-order terms. The possibility to truncate polynomials at low-order is convenient when multiple instances of Problem (9) need to be solved.

A major takeoff of the proposed methodology is the assessment of a minimum cone angle required to have local controllability of the system. To this purpose, Problem (9) is solved for various α between 0 and 90 deg, and for all e_k, vertices of a n-simplex of $T_I M$ $k = 1, \ldots, n$. The minimum cone angle necessary for local controllability is the smallest angle such that Problem (9) is feasible for all vertices. When it is the case, we define $\zeta^*(e_k)$ to be the inverse of the value function,

$$\zeta^*(e_k) = \frac{2}{\|u\|_2^2}\,,$$

TABLE I
SIMULATION PARAMETERS

Initial conditions	
I_2	20 deg
I_3	30 deg
I_5	0.5
Constants for Figs. 7 and 8	
DFT order, N	10
Number of generators, g	10
Direction of displacement	$\partial/\partial I_5$
Cone angle, α	80 deg

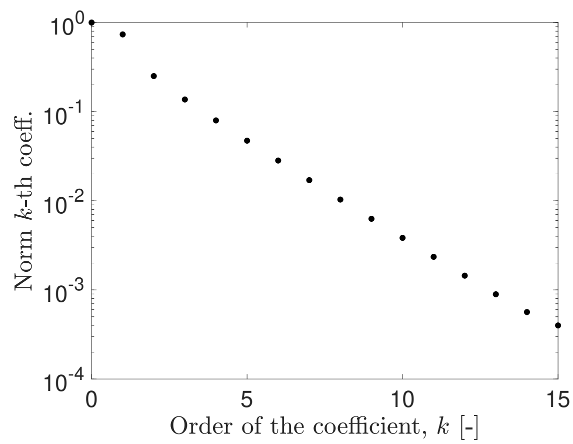


Fig. 5. Convergence of coefficients of the DFT. The norm is evaluated as $\sqrt{\sum_i \left|\int_{\mathbb{S}^1} F_i e^{ik\varphi} \mathrm{d}\varphi\right|^2} \Big/ \sqrt{\sum_i \left|\int_{\mathbb{S}^1} F_i \mathrm{d}\varphi\right|^2}$.

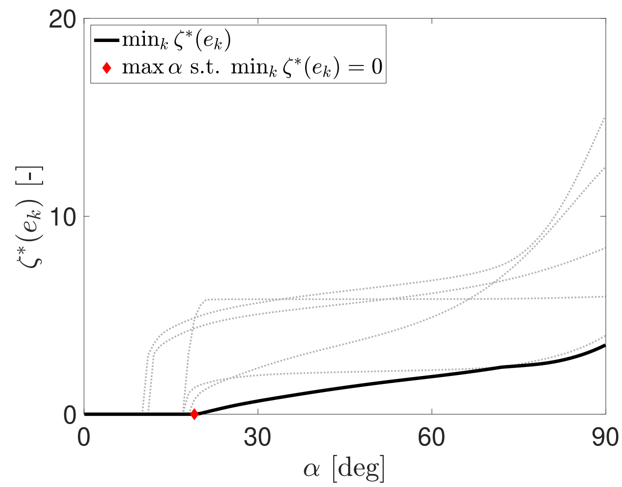


Fig. 6. Grey lines show the resulting displacement ζ^* toward all vectors of a n-simplex e_k, $k = 1, \ldots, n$. Black line shows the minimum of these curves. The minimal angle α ensuring local controllability is highlighted in red. One can notice that some curves do not strictly increase, but are constant instead. It means that the control is inside the cone, and increasing α does not change the result.

and set $\zeta^*(e_k) = 0$ when the problem is not feasible. For the orbit at hand, feasibility occurs for $\alpha = 19$ deg as depicted in Fig. 6. This angle may serve as a minimal requirement for the design of the sail. Specifically, the reflectivity coefficient associated to this cone angle can be evaluated by inverting Eq. (10), namely

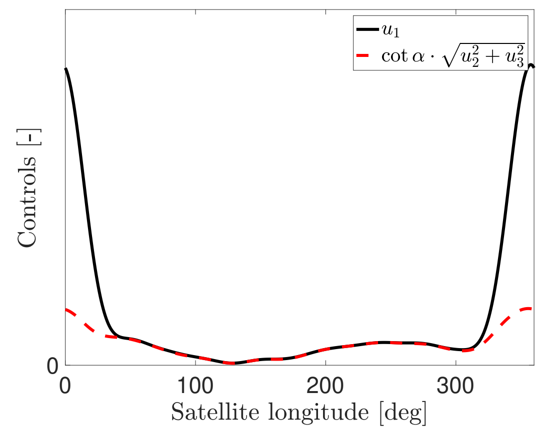
$$\rho = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}.$$

In the example at hand, $\rho \simeq 0.3$ is the minimum reflectivity that satisfies the controllability criterion. In addition, optical properties degrade in time [16], so that this result may be also used to investigate degradation of the controllability of a sail during its lifetime.

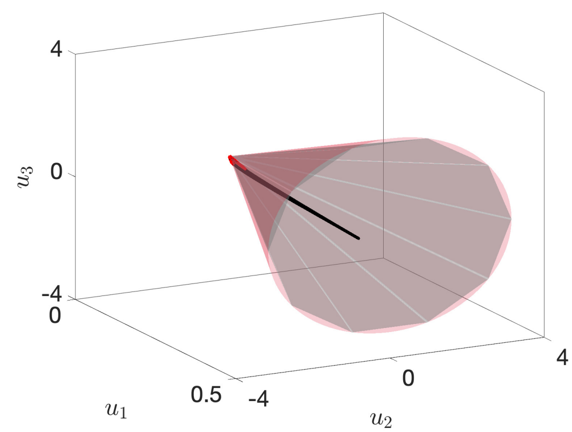
Let us now consider a scenario when the maneuver consists in changing only one orbital element. Figures 7 and 8 show controls and trajectory for the desired displacement direction $\partial/\partial I_5$ (*i.e.*, increase of orbital eccentricity) with $\alpha = 80$ deg. Periodic control obtained as solution of Problem (9) is applied for several orbits. The displacement of the averaged state is clearly toward the desired direction, namely all slow variables but I_5 exhibit periodic variations, while I_5 has a positive secular drift. The structure of the control arcs is such that control is on the surface of the cone in the middle of the orbit whereas it is at the interior at the beginning and end. We note that no initial guess is required to solve Problem (9). As such, *a priori* knowledge of this structure is not necessary.

V. Conclusions

A methodology to verify local controllability of a system with conical constraints on the control set was proposed. A convex optimisation problem needs to be solved to this purpose. Controllability of solar sails is investigated as case study, and it is shown that a minimum cone angle α exists that satisfies the proposed criterion. This angle yields a minimum



(a) Black line shows control in Sun direction, the red one combines the two other components. When they coincide, the control is on the cone's boundary.



(b) The polyhedral approximation K_g of K. Black line is the resulting control inside the cone, while the red one shows when the control is on the boundary.

Fig. 7. Control force solution of Problem (9).

requirement for the surface reflectivity of the sail.

VI. Acknowledgments

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Appendix

A. Positive polynomials

Consider the basis of trigonometric polynomials $\Phi = (1, e^{i\varphi}, e^{2i\varphi}, \ldots, e^{(N-1)i\varphi})$. Its corresponding squared functional system is $\mathcal{S}^2(\varphi) = \Phi(\varphi)\Phi^H(\varphi)$ where $\mathcal{S}^H(\varphi)$ denotes conjugate transpose of $\mathcal{S}(\varphi)$. Let $\Lambda_H : \mathbb{C}^N \to \mathbb{C}^{N \times N}$ be a linear operator mapping coefficients of polynomials in $\Phi(\varphi)$ to the squared base, so that application of Λ_H on $\Phi(\varphi)$ yields

$$\Lambda_H(\Phi(\varphi)) = \Phi(\varphi)\Phi^H(\varphi)$$

and define its adjoint operator $\Lambda_H^* : \mathbb{C}^{N \times N} \to \mathbb{C}^N$ as

$$(Y|\Lambda_H(c))_H \equiv (\Lambda_H^*(Y)|c)_H, \quad Y \in \mathbb{C}^{N \times N}, \quad c \in \mathbb{C}^N.$$

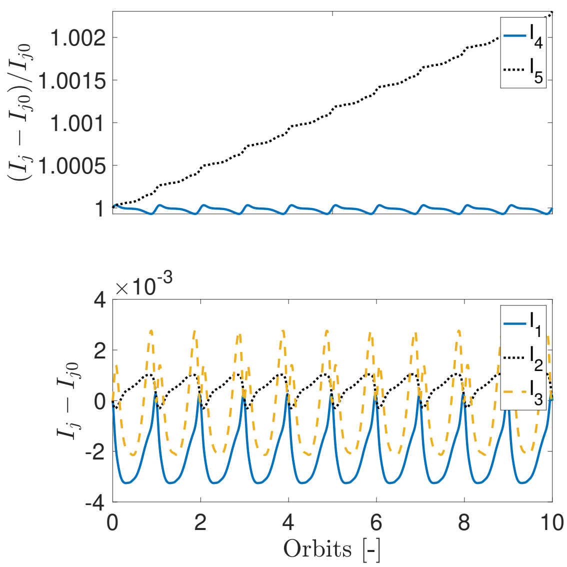

Fig. 8. For verification, controls resulting from the optimisation problem are injected into real dynamical equations. Plots of trajectories of slow variables correspond to the desired movement (increase of eccentricity, I_5). Moreover, this trajectory is stable over multiple orbits.

Theory of squared functional system postulated by Nesterov [4] proves that polynomial $(\Phi(\varphi)|c)_H$ is non-negative for all $\varphi \in \mathbb{S}^1$ if and only if there is a Hermitian positive semidefinite matrix Y such that $c = \Lambda_H^*(Y)$, namely

$$(\forall \varphi \in \mathbb{S}^1) : (\Phi(\varphi)|c)_H \geq 0 \iff (\exists Y \geq 0) : c = \Lambda_H^*(Y).$$

In fact in this case it holds

$$\begin{aligned}(\Phi(\varphi)|c)_H &= (\Phi(\varphi)|\Lambda_H^*(Y))_H = (\Lambda_H(\Phi(\varphi))|Y)_H, \\ &= (\Phi(\varphi)\Phi^H(\varphi)|Y)_H = \Phi^H(\varphi)Y\Phi(\varphi) \geq 0.\end{aligned}$$

For trigonometric polynomials Λ^* is given by

$$\Lambda^*(Y) = \begin{bmatrix} (Y|T_0) \\ \vdots \\ (Y|T_{N-1}) \end{bmatrix}$$

where $T_j \;\; j = 0, \ldots, N-1$ are Toeplitz matrices such that

$$T_0 = I, \quad T_j^{(k,l)} = \begin{cases} 2 & \text{if } k - l = j \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, \ldots, N-1.$$

B. Equations of motion of solar sails

Slow component of the state vector consists of three Euler angles, which position the orbital plane and perigee in space, I_1, I_2, I_3, and of the semi-major axis and eccentricity, I_4 and I_5, respectively. Mean anomaly is the fast variable, φ. Kepler's equation is used to relate φ to the eccentric anomaly, ψ, and then to the true anomaly, θ, as

$$\varphi = \psi - I_5 \sin\psi, \qquad \tan\frac{\theta}{2} = \sqrt{\frac{1+I_5}{1-I_5}} \tan\frac{\psi}{2}$$

Vector fields of the equations of motion are given by

$$F_i = \sum_{i=1}^{3} R_{ij} F_j^{(LVLH)}$$

where, R_{ij} are components of the rotation matrix from the reference to the local-vertical local-horizontal frames,

$$R = R_1(I_3 + \theta) R_2(I_2) R_1(I_1)$$

($R_i(x)$ denoting a rotation of angle x about the i-th axis), and vector fields $F_j^{(LVLH)}$ can be deduced from Gauss variational equations (GVE) expressed with classical orbital element [17] by replacing the right ascension of the ascending node, inclination, and argument of perigee with I_1, I_2, and I_3, respectively. Rescaling time such that the planetary constant equals 1, these fields are

$$F_1^{(LVLH)} = \sqrt{I_4\left(1 - I_5^2\right)} \begin{bmatrix} 0 \\ 0 \\ -\dfrac{\cos\theta}{I_5} \\ 2\dfrac{I_4 I_5}{1 - I_5^2}\sin\theta \\ \sin\theta \end{bmatrix}$$

$$F_2^{(LVLH)} = \sqrt{I_4\left(1 - I_5^2\right)} \begin{bmatrix} 0 \\ 0 \\ \dfrac{2 + I_5\cos\theta}{1 + I_5\cos\theta}\dfrac{\sin\theta}{I_5} \\ 2\dfrac{I_4 I_5}{1 - I_5^2}(1 + I_5\cos\theta) \\ \dfrac{I_5\cos^2\theta + 2\cos\theta + I_5}{1 + I_5\cos\theta} \end{bmatrix}$$

$$F_3^{(LVLH)} = \frac{\sqrt{I_4\left(1 - I_5^2\right)}}{1 + I_5\cos\theta} \begin{bmatrix} \dfrac{\sin(I_3 + \theta)}{\sin I_2} \\ \cos(I_3 + \theta) \\ -\dfrac{\sin(I_3 + \theta)\;\cos I_2}{\sin I_2} \\ 0 \\ 0 \end{bmatrix}$$

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