TOWARDS A NOTION OF TRUTH FOR LINEAR LOGIC

ANDRÉ HIRSCHOWITZ AND MICHEL HIRSCHOWITZ

ABSTRACT. We define a naive game for cut-free multiplicative additive linear logic with constants (but without variables). In this game two players fight for proving a formula respectively its linear negation. If the formula has a proof, the player has a winning strategy. A position in our game roughly consists of a single set of formulas arranged by the two players in two "orthogonal" families of sequents. An "active" move in our game closely corresponds to the application of an introduction rule for a positive formula, while a "passive" move "requests" from the other player the application of such a rule. We upgrade our game as a categorical game, which is the game-theoretic way to account for the cut-rule.

Keywords: Game semantics, linear logic, categorical models.

CEA LIST - Laboratoire MeASI - LIX
Boîte courrier 94, F91191 Gif-sur-Yvette Cedex France

and UMR 6621, CNRS, Université de Nice Sophia-Antipolis
ParcValrose, 06108 Nice Cedex
.

1. Introduction

1.1. **Logic as a game.** The idea that logic could be better understood as a game goes back to Gentzen and is much alive for instance through the theories proposed recently by Girard [7, 6] and Japaridze [11]. Our bet is that the next notion of logic will be based on games and we explore which kind of game should be linear logic. Accordingly, we search for a notion of truth in (linear) logic. The rough idea in order to have obvious (built-in) consistency, cut-rule and excluded-middle is to define truth through a two-player game, where Player tries to prove a formula, while Opponent tries to disprove it. Excluded-middle could then follow from some "copycat" strategy, while the cut-rule could follow if our game is sufficiently categorical (compositional).

1.2. **Introduction rules as moves.** The first main feature of our game corresponds to our main objective: we want moves which reflect introduction rules in the straightforward way. For this, our positions cannot be formulas as they are in [2, 10, 3, 12]; they must be some kind of families of sequents. Let us explain which kind of families emerge. Suppose for instance that Player faces $\vdash 1 \otimes \perp$ so that Opponent faces $1 \otimes \perp \vdash$. After one or two moves (according to who starts), Opponent will face the single sequent $1, \perp \vdash$ while Player will face the two sequents $\vdash 1$ and $\vdash \perp$. What we see is that players share formulas, but arrange them in two different ("orthogonal") families of sequents. That is why, roughly speaking, our positions are families of formulas dispatched by Player on the right-hand side of her family of sequents, and dispatched "orthogonally" by Opponent on the left-hand side of her (different) family of sequents. Our approach assigns a specific status to the multiplicative connectors, which appear as fully responsible for the multiplication of formulas in sequents, and for the multiplication of sequents, in other words for the fact that we have to manage families of sequents.

1.3. **No matter who starts.** The second main feature of our game corresponds to the following challenge: the fact that one player has to start seems to cause trouble by leading to two potentially conflicting notions of truth. Our solution is based on polarity: in our game, the player which faces a negative formula can only "pass the token" to the other player. So that who starts does not make any difference. More precisely, Player can only apply an introduction rule to a positive formula or "pass the token" along a negative formula. While dually, Opponent can only apply an introduction rule to a negative formula (which is on her LHS) or "pass the token" along a positive formula.

1.4. **The token.** The third main feature of our game is the token: our positions are families of formulas arranged as explained above, where a sequent (the "token") is selected, meaning which player is to play, and where. Thus the player which holds the token may either apply an introduction rule for a positive formula in the selected sequent, or pass the token, along a negative formula in the selected sequent, selecting the corresponding sequent of the opponent.

1.5. **Acyclicity.** The fourth main feature of our game is acyclicity: our positions have natural interpretations as trees. This acyclicity has the following expected consequence: when the token leaves a sequent through a (negative) formula, it will not come back to this sequent except by applying an introduction rule to the original formula.

1.6. **Compositionality.** The final main feature of our game is compositionality. For this, we have a notion of morphism of positions. Roughly speaking, such a morphism concerns three players, P, O and O' dealing with two families I and J of formulas: these formulas are dispatched "by P" respectively on the LHS and the RHS of her sequents, while they are dispatched "respectively by O' and by O" among their sequents respectively on the RHS and on the LHS. Such a morphism has a source, based on the family I, a target, based on the family J and a total position, based on the family $I \amalg J^{\perp}$. These morphisms turn positions into a category which is "one-way" in the terminology of [9] and allow us to upgrade our game into a categorical game in the sense introduced there. In this context, the cut-rule is derived from the general statement saying that if $m : M \to N$ and $n : N \to P$ are two composable morphisms, and if s and t are winning strategies respectively for m and n, then $t \circ s$ is a winning strategy for $n \circ m$. Our approach also opens new room to express symmetries of linear logic. For instance, in our setting, linear negation upgrades into a contravariant involutive endofunctor of our categorical game, exchanging forward and backward moves.

1.7. **Related works.** Game-theoretic interpretations of fragments of linear logic have been already given by many authors, notably [2, 10, 3]. These works aim at a "fully complete" correspondence between a suitable notion of proof (proof-nets) and a suitable notion of winning strategy. This approach is revisited in [12, 13], where the limitations of the previous solutions are highlighted and a new solution is proposed. We depart radically from this line of research by looking for a game where positions are not formulas, but suitable families of sequents, and accordingly where instances of the introduction rules may be understood as moves.

As mentioned above, the idea of understanding logic as a game has been already pushed (much further) by Girard (see e.g. [7, 6]) and on a different track by Japaridze (see e. g. [11]).

1.8. **Organisation of the paper.** Our paper is organised is follows. We start by describing our new category of trees (section 2) and our category of contests, which are trees weighted by logical formulas with a selected vertex (section 3). Then we describe the moves of our naive game (section 4) and briefly discuss this game, which we call the half game (section 5). In section 6, we describe the ("full") game where Player sits between two Opponents and accordingly tries to prove sequents with formulas on both sides, while in section 7 we present our categorical game. We conclude in section 8 by discussing some future work.

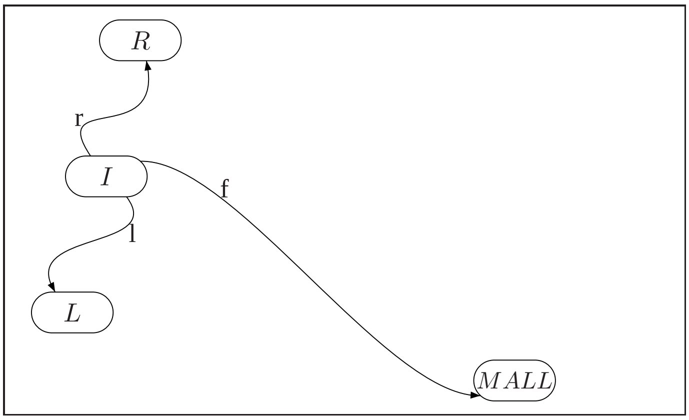
2. The category of trees

In this work, we use a new category T of trees, which we present in this section.

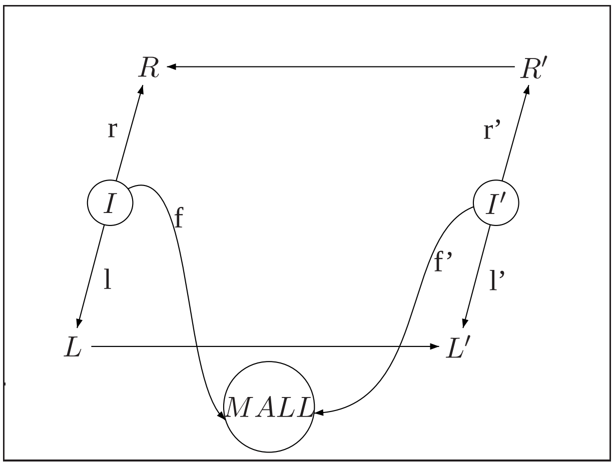
2.1. **The 0-trees.** An object of T, called a 0-tree, is a tuple (I, L, R, ℓ, r) where I, L, R are finite sets, and $\ell : I \to L$ and $r : I \to R$ are arbitrary maps, subject to the following acyclicity condition:

The directed graph with $L \amalg R$ as set of vertices, I as set of edges, and ℓ and r as source and target maps is a (non-empty) tree.

If I is empty, this means that $L \amalg R$ is reduced to the token, while if I is not empty, this just means that ℓ and r are surjective and define so-called orthogonal partitions on I. Remember that two partitions p and q are orthogonal [4] when the infimum and supremum partitions $p \wedge q$ and $p \vee q$ are respectively the coarsest and finest partitions.



2.2. **The premorphisms.** A premorphism m from (I, L, R, ℓ, r) to (I', L', R', ℓ', r') is a pair (m_*, m^*) consisting of a map from L to L' and a map from R' to R. These premorphisms compose in an obvious way, yielding a category PT.



2.3. **The heart and the graph of a premorphism.** In the first reading, the reader may want to skip this technical subsection.

Given such a premorphism m from (I, L, R, ℓ, r) to (I', L', R', ℓ', r'), we define the only natural map $q : I \amalg I' \to R \times L'$ as follows:

from I to R and from I' to L' we take r and ℓ'; while from I to L' we take $m_* \circ \ell$ and from I' to R, we take $m^* \circ r'$. If $I \amalg I'$ is not empty, we denote by $H := H(m)$ the image of q, while otherwise we take $H := R \times L'$ (which is a singleton).

We say that H is the heart of m.

We then build the bipartite graph $G := G(m)$ with

- H as set of LHS vertices,
- $L \amalg R'$ as set of RHS vertices,
- $I \amalg I'$ as set of edges,
- q as source map,
- $\ell \amalg r'$ astarget map.

2.4. **The** 1**-trees.** A morphism, also called 1-tree, from $t := (I, L, R, \ell, r)$ to $t' := (I', L', R', \ell', r')$ is a premorphism $m : t \to t'$ such that $G(m)$ is a tree, hence a 0-tree. In this case, we say that $G(m)$ is the 0-tree obtained by folding m.

The fact that these morphisms turn T into a category follows from the lemma:

Lemma. The composite of two morphisms m from (I, L, R, ℓ, r) to (I', L', R', ℓ', r') and m' from (I', L', R', ℓ', r') to $(I'', L'', R'', \ell'', r'')$ is a morphism again.

3. The groupoid of contests

In this section, we describe the positions of our naive game which we call contests.

3.1. **The set of formulas.** We first build the set of usual $MALL$-formulas, without variables [1, 5] :

$$F := \ 1 \mid 0 \mid \perp \mid \top \mid F \parr F \mid F \otimes F \mid F \oplus F \mid F \,\&\, F \ .$$

We denote this set by $MALL$. The positive formulas are

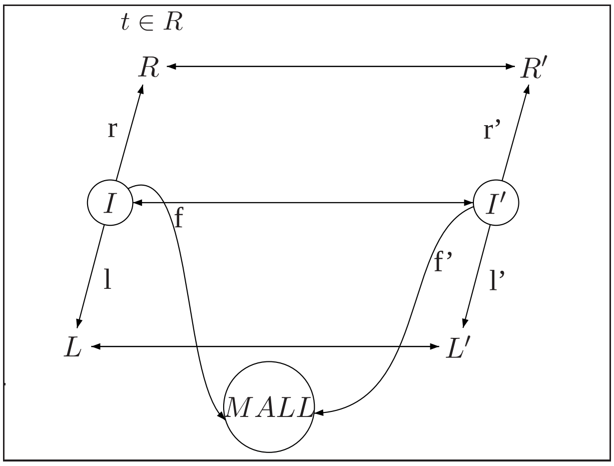
$$P := \ 1 \mid 0 \mid F \otimes F \mid F \oplus F,$$

the other ones being the negative ones. The linear negation $F \mapsto F^\perp$ sends respectively $1, 0, A \otimes B$ and $C \oplus D$ to $\perp, \top, A^\perp \parr B^\perp, C^\perp \,\&\, D^\perp$ and vice-versa.

3.2. **The groupoid of** 0**-contests.** A 0-contest, or simply contest, is 0-tree (I, L, R, ℓ, r) equipped with a map f from I to the set $MALL$ of formulas, and a selected vertex t (the *token*). The token singles out, in $L \amalg R$, the sequent where the player has to play. We say that I is the set of indices, L is the set of left sequents, and R is the set of right sequents.

We have an obvious notion of isomorphism between contests: an isomorphism from the contest (I, L, R, ℓ, r, f, t) to the contest $(I', L', R', \ell', r', f', t')$ consists of three bijections from I, L, R to I', L', R' commuting to ℓ and ℓ', to r and r', to f and f', and sending t to t'. In other words, it is an isomorphism of 0-trees compatible with weights and tokens.

These isomorphisms turn the set C of contests into a groupoid.



3.3. **The sign of 0-contests.** The set C of contests is the disjoint union of C_L, where the token is on the left-hand side and C_R, where the token is on the right-hand side. Isomorphisms of contests respect this sign.

4. The moves

In order to describe our game, we now describe our moves, which connect odd pairs of contests.

4.1. **Passive moves.** The simplest moves are passive ones, reflecting the idea that a player may just pass the token along a negative formula:

Up to isomorphism, a passive $\mathcal{P}$-move goes from the contest $(I, L, R, f, \ell, r, \ell(i))$ to the contest $(I, L, R, f, \ell, r, r(i)$ provided $f(i)$ is negative.

Dually, up to isomorphism, a passive $\mathcal{O}$-move goes from the contest $(I, L, R, f, \ell, r, r(i))$ to the contest $(I, L, R, f, \ell, r, \ell(i)$ provided $f(i)$ is negative.

4.2. **Additive moves.** The reader will easily convince himself that an additive $\mathcal{P}$-move as defined below corresponds exactly to the application of one of the (right) introduction rules for $\oplus$ in the sequent marked by the token.

Up to isomorphism, an additive $\mathcal{P}$-move goes from a contest $(I, L, R, \ell, r, f, \ell(i))$ where $f(i)$ is of the form $A \oplus B$ to the contest $(I, L, R, \ell, r, g, r(i))$, where g equals f except at i, where its value is A or B. Note that in the categorical game, we will carefully distinguish the two moves in case $A = B$.

Dually, up to isomorphism, an additive $\mathcal{O}$-move goes from a contest $(I, L, R, \ell, r, f, r(i))$ where $f(i)$ is of the form $A \& B$ to the contest $(I, L, R, \ell, r, g, \ell(i))$, where g equals f except at i, where its value is A or B.

4.3. **Multiplicative moves.** The reader will easily convince himself that a multiplicative $\mathcal{P}$-move as defined below corresponds exactly to the application of the (right) introduction rule for $\otimes$ in the sequent marked by the token.

A multiplicative $\mathcal{P}$-move from the contest (I, L, R, ℓ, r, f, t) to the contest $(I', L', R', \ell', r', f', t')$ is as follows (up to isomorphism): we have two indices i' and i" in I' with the same image by r, and I is obtained by collapsing i' and i" in a single index i; meanwhile f equals f' outside i while $f(i)$ is $f'(i') \otimes f'(i")$. The set of right sequents does not change ($R = R'$, and r is the map induced by r'), while L is obtained from L' by collapsing $\ell'(i')$ and $\ell'(i")$ (ℓ is the map induced by ℓ'.

The reader will easily deduce the dual definition of a multiplicative $\mathcal{O}$-move.

4.4. **Constant moves.** The reader will easily convince himself that a constant $\mathcal{P}$-move as defined below corresponds exactly to the application of the (right) introduction rule for 1 in the sequent marked by the token.

A constant $\mathcal{P}$-move starts from an odd contest $c := (I, L, R, f, \ell, r, t)$ where $\ell^{-1}(t)$ consists of a single element i such that $f(i) = 1$. This move to the contest $c' := (I', L', R', f', \ell', r', t')$ consists of

- **–:** an injection $s : I' \to I$ identifying I' with $I - \{i\}$
- **–:** an injection $s_L : L' \to L$ identifying L' with $L - \{t\}$ and satisfying $s_L \circ \ell' = \ell \circ s$
- **–:** a bijection $s_R : R' \to R$ satisfying $s_R \circ r' = s \circ r$ and $s_R(t') = r(i)$.

Dually, the reader will easily convince himself that a constant $\mathcal{O}$-move as defined below corresponds exactly to the application of the (left) introduction rule for $\perp$ in the sequent marked by the token.

A constant $\mathcal{O}$-move starts from an even contest $c := (I, L, R, f, \ell, r, t)$ where $r^{-1}(t)$ consists of a single element i such that $f(i) = \perp$. This move to the contest $c' := (I', L', R', f', \ell', r', t')$ consists of

- **–:** an injection $s : I' \to I$ identifying I' with $I - \{i\}$
- **–:** an injection $s_R : R' \to R$ identifying R' with $R - \{t\}$ and satisfying $s_R \circ r' = r \circ s$
- **–:** a bijection $s_L : L' \to L$ satisfying $s_L \circ \ell' = s \circ \ell$ and $s_L(t') = s(i)$.

5. Some meta-theory for the half game

So far, we have defined a game, which we call the half game. In this section, we explain which sort of game it is, which notion of truth it allows, and why this notion is adequate.

5.1. **Bipartite graphoids.** The kind of naive game we just have built is slightly less naive than what is associated with a signed (or bipartite) graph. Indeed, here we have to take isomorphisms into account. Moves compose with isomorphisms while two moves do not compose. This is not a category but this gap is fixed by the following trick. We have even objects, those where the token is on the RHS and odd objects, those where the token is on the LHS. The trick is to decide that moves from an odd position to an even one are taken backwards. So that all moves correspond to morphisms from an even position to an odd position. In this way, we obtain a signed category where there is no morphism from an odd to an even position, what we called a one-way category [9]. So odd morphisms all go from an even to an odd position, and they are split into two classes (forward and backward). While even morphisms are all isomorphisms. Summarising, we call bipartite graphoid a one-way category where the subcategory of even morphisms is a groupoid, and equipped with a splitting of odd morphisms compatible with the groupoid action.

Note that our graphoid is equipped with a contravariant involution corresponding to linear negation.

5.2. **Truth.** In a bipartite graphoid, we have a natural notion of path: a path from p to q is an anti-composable sequence of odd morphisms $(u_1, \cdots, u_n)$ where p is the end of u_1 which is not shared with u_2 and q is the end of u_n which is not shared with u_{n-1}. Our graphoid is strongly noetherian in the sense that for any position, the length of paths starting at this vertex is bounded.

Given a position in such a strongly noetherian bipartite graph, we have the usual notion of (deterministic) P-strategy and of O-strategy at v. We also have the notion of winning strategy: with any path ending up at a vertex where X is to play, a winning X-strategy contains a longer path, in other words the strategy has an answer. It is easily checked that at each vertex, there is either a winning P-strategy (in which case we say that the position is winning) or a winning O-strategy (in which case we say that the position is losing), and never both. Now given a formula in MALL, we have two ways to upgrade it as a position in our game, by putting to token either on the RHS or on the LHS. Since according to the polarity of the formula, on one of this position, the only possible move it to shift to the other one, both positions have simultaneously either a winning P-strategy, in which case we say that the formula is true, or a winning O-strategy, in which case we say that the formula is false.

5.3. **Adequacy of our half game.** The adequacy of our half-game to MALL reads as follows:

Theorem. Given a contest c, if the left sequents of c are provable, then there is a winning P-strategy at c, and similarly, if the right sequents of c are provable, then there is a winning O-strategy at c.

The proof relies on the fact that if a sequent S has a proof, then any sequent obtained from S by applying an introduction rule for a negative formula has also a proof.

Remark: as we already mentioned, there are some contests where P has a winning strategy while the left sequents are not all provable. This is the case for the single formula $\perp \otimes \perp$. Those who "prefer" $1 \parr 1$ will have to change the game by finding a way to add for instance the MIX rule.

6. The point of view of a player

We have seen a game where P tries to prove sequents with formulas only on the RHS while O tries to prove sequents with (the same) formulas only on the LHS. In fact, in the full-game which we will describe now, O is the right neighbour of

P, hence P is the left neighbour of O, and P, for instance, has also formulas on the LHS of her sequents, and a left neighbour, say O', with which she interacts through these formulas. What happens between P and O' doesn't affect O. So we may think of a line of players handling sequents with formulas on both sides, shared respectively with neighbour players on both sides. In order to understand the game, it is sufficient to consider a small line of three players O', P, O and forget about possible LHS formulas of O' and RHS formulas of O. In this section, we describe this "full" game, where P plays against O and O'.

6.1. **The groupoid of 1-contests.** A 1-contest is a pair (c, c') of 0-contests equipped with a morphism m among the underlying 0-trees compatible with the tokens (t, t') in the following sense:

- if both tokens are on the LHS, we require $t' = m_*(t)$
- if both tokens are on the RHS, we require $t = m^*(t')$
- if t is on the RHS and t' on the LHS, we require (t, t') to be in the heart of m,
- finally we forbid the case where t is on the LHS while t' is on the RHS.

We have a obvious notion of isomorphism of 1-contests: an isomorphism from C to C' is an isomorphism among the underlying morphisms of 0-trees compatible with formulas and tokens.

6.2. **Folding.** To a 1-contest $C := (c, c', m)$, with $c = (I, L, R, \ell, r, f, t)$ and $c' = (I', L', R', \ell', r', f', t')$, we assign its *folding*: it is a 0-contest denoted $fold(C)$ obtained from the folding of the underlying 1-tree by pushing there the formulas and the token in the natural way: for the formulas we take $f^{\perp} ⅋ f'$, while for the token, we take

- t if t and t' are on the LHS
- t' if t and t' are on the RHS
- (t, t') if t is in the RHS and t' in the LHS.

Note that the folding of a 1-contest comes equipped with a distinguished subset of its set of right sequents (the nonfolded part).

The folding assignment upgrades obviously into a functor from the groupoid of 1-contests to the one of 0-contests.

Conversely, if we want to "unfold" a 0-contest, we just have to split its set of right sequents into those belonging to O and those belonging to O'. Roughly speaking, a 1-contest is just a 0-contest where O and her sequents has been split in two parts. More precisely, we may build a groupoid of split 0-contests, where by split 0-contests, we mean a contest equipped with a distinguished subset of its set of right sequents (the subset of *truly* right sequents). And the folding upgrades into an equivalence from the groupoid of 1-contests to the one of split 0-contests.

6.3. **Upgrading our half game.** Now we want to upgrade our half game as a game where positions are 1-contests. We have to define "full" moves among 1-contests, which we may view as split 0-contests. We simply take, for moves from c to d, moves among the corresponding 0-contests $fold(c)$ and $fold(d)$ compatible in the natural sense with the splitting of right sequents. More precisely each move among 0-contests entails a map among sets of indices, and we simply require this map to be compatible with the splitting, sending truly right sequents to truly right sequents and vice-versa. In this way, we obtain a bipartite graphoid of 1-contests which we call the full game.

6.4. **Adequacy of our full game.** The adequacy of our full game is asserted by the following statement:

Proposition. A 1-contest C is true (resp. false) iff $fold(C)$ is true (resp. false).

This is because a move at $fold(C)$ can always be lifted as a move at C.

7. ASSEMBLING THE PUZZLE: THE CATEGORICAL POINT OF VIEW

In this section, in order to have a game-theoretic cut-elimination, we upgrade our game into a categorical one. For this, following [9], we have to understand our 1-contests as morphisms in a one-way category enlarging the graphoid of 0-contests.

7.1. **A quick review of categorical game theory.** In this subsection we briefly review what we need from categorical game theory [8, 9].

A one-way category is a category where objects, also called ports, have signs, and there is no morphism from an odd to an even object. A categorical game $G := (\mathfrak{C}, \mathcal{M}^P, \mathcal{M}^O)$ consists of a one-way category $\mathfrak{C}$ equipped with two sets $\mathcal{M}^P$ and $\mathcal{M}^O$ of odd morphisms, called P-moves and O-moves respectively. The parity extends in the obvious way to morphisms, which are called positions. If the position is even, O has to play, otherwise P has to play. In an even position p, O may either post-compose p with an O-move, that is to say choose an O-move m and replace p by $m \circ p$, or pre-compose p with a P-move, that is to say choose a P-move m and replace p by $p \circ m$. In an odd position p, P has to decompose p either in the form $m \circ q$ with m a P-move, or in the form $q \circ m$ with m an O-move, and replace p by q. There is one more rule, the so-called switching rule, which says that O cannot switch neither from the source to the target nor the other way around. Winning positions are those where O has no more move available. There is a category S of strategies, the objects of which are even ports, and a morphism from a to b in S is a morphism p in C equipped with a strategy saying how P will play against O starting form this position p. For details, see [8]

7.2. **Occurrences of a contest.** In the construction of our category of contests, we will need the set of occurrences of a contest which we now define. The set of occurrences of the contest $c : (I, L, R, \ell, r, f, t)$ depends only on the family (I, f): it is the set $Occ(c)$ of pairs (i, o) where o is an occurrence in $f(i)$ (here we see formulas in MALL as trees). The set $Occ(c)$ comes equipped with a map, called *index*, to I, and a map, called *subformula*, to MALL.

7.3. **The category of contests.** We are now ready to define our category Co of contests. The objects of Co are 0-contests. We now have to define the set of morphisms from $c := (I, L, R, \ell, r, f, t)$ to $c' := (I', L', R', \ell', r', f', t')$.

We start by defining a premorphism from c to c' to be a pair of a morphism m between the underlying 0-trees and a relation ρ between $Occ(c)$ and $Occ(c')$. There is an obvious composition of premorphisms yielding a category PCo.

We then define a morphism from c to c' to be a premorphism (m, ρ) satisfying the following three compatibility conditions:

- the morphism m should be compatible with the tokens in the following sense:
  - if both tokens are on the LHS, we require $t' = m_*(t)$
  - if both tokens are on the RHS, we require $t = m^*(t')$
  - if t is on the RHS and t' on the LHS, and I or I' is not empty, we require (t, t') to be in the heart of $I \times I'$
  - finally we forbid the case where t is on the LHS while t' is on the RHS.
- the relation ρ should be compatible with formulas, namely it should be contained in the locus of pairs (o, o') where the subformula for o and the subformula for o' agree
- the relation ρ should be compatible with the heart, in the sense that its projection into $I \times I'$ should be made of pairs (i, i') of indices where i and i' have the same image (by q) in the heart of m.

Since identities are obviously morphisms, in order to prove that these morphisms define a subcategory Co of PCo, we only have to check the following :

Lemma: The composite (as premorphisms) of two morphisms is again a morphism.

7.4. **Upgrading the half moves.** Here we show by some examples how the moves of our half game can be upgraded as odd morphisms in Co. These moves are defined up to isomorphism: we declare some morphisms as moves and it should be understood that morphisms which are isomorphic to these moves are themselves moves.

We describe for instance the passive P-moves. This concerns an odd 0-contest $C := (I, L, R, \ell, r, \phi, t)$ where we have an index i with $\phi(i)$ negative and $\ell(i) = t$. After the passive move, we get (up to isomorphism) the even 0-contest $C' := (I, L, R\ell, r, \phi, r(i))$. Since we are speaking of a backward move, we have to describe a morphism from C' to C: for this, we have to specify

- an application from L to L: we take the identity;
- an application from R to R: we take again the identity;
- a relation on $Occ(\phi) \times Occ(\phi)$: we take the identity again.

Now we turn to moves corresponding to the introduction rule for multiplicative conjunction. In order to build such a move, we start from an even 0-contest $C := (I, L, R, \ell, r, f, t)$ with two distinct indices i and j in I with the same image t in R, and we build a new 0-contest $D = (J, M, S, m, s, g, u)$ as follows :

- J is obtained from I by identifying i and j;
- For k different from i and j, $g(k)$ is $f(k)$, while $g(\{i,j\})$ is $f(i) \otimes f(j)$;
- M is obtained from L by identifying the images of i and j, and m is the map induced by ℓ;
- S is R and s is the map induced by r.
- Finally, u is the image in M of the class of i and j.

Our move N is the morphism from C to D represented as follows:

- $N_* : L \to M$ is the quotient map;
- $N^* : S \to R$ is the identity;
- ρ_N is the union of the set of diagonal pairs $((j,o),(j,o))$ for $j \neq i, j$, with the set of pairs $((i,o),(\{i,j\},1.o))$ and the set of pairs $((j,o),(\{i,j\},2.o))$.

We leave to the reader the description of the other moves.

7.5. **The connection with the full game.** The adequacy of our categorical game is ensured by the following:

Proposition. Each move among positions in our categorical game induces a move in the full game between the underlying 1-contests. Conversely, given a position p in the categorical game, each full move at the underlying 1-contest lifts as a categorical move at p.

As a consequence, a position in the categorical game is true iff the underlying 1-contest is true in the full game.

8. Conclusion and future work

We have presented a categorical game for $MALL$. Our solution has some strong points which we enumerate:

- In playing, P can follow exactly the proof he had in mind, each move "is" the application of an introduction rule, or forces its application by O. This is probably the main progress we wanted to reach with respect to the previous solutions [10, 2]. Furthermore, P may insist on the order on moves as far as they concern the same sequent; while O controls the interleaving of the proofs of different sequents.
- The linear negation is a categorical involution of our game.
- As far as a single formula is concerned, who starts has no consequence, since the player facing a negative formula may only pass the token.
- There is no infinite play; this feature is not so surprising on the $MALL$ fragment. One extremely serious task is to extend it to exponentials. (Extending the game to exponentials without this feature is immediate).

One drawback of our solution (and of all other existing solutions) is that the game is for cut-free logic. In a future contribution, we will upgrade the present game into a game with cut moves.

References

[1] Samson Abramsky and Radha Jagadeesan. Games and full completeness for multiplicative linear logic. *Journal of Symbolic Logic*, 59(2):543–574, June 1994.

[2] Samson Abramsky, Radha Jagadeesan, and Pasquale Malacaria. Full abstraction for PCF. *Information and Computation*, 163(2):409–470, December 2000.
[3] Samson Abramsky and Paul-André Melliès. Concurrent games and full completeness. In *Proceedings of the fourteenth annual symposium on Logic In Computer Science*, pages 431–442, Trento, July 1999. IEEE, IEEE Computer Society Press.
[4] M. Barbut. Partitions d'un ensemble fini : leur treillis (cosimplexe) et leur représentation géométrique. pages p. 5–22, 1968.
[5] Jean-Yves Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
[6] Jean-Yves Girard. Locus solum: From the rules of logic to the logic of rules. *Mathematical Structures in Computer Science*, 11(3):301–506, June 2001.
[7] Jean-Yves Girard. From foundations to ludics. *Bulletin of Symbolic Logic*, 9(2):131–168, 2003.
[8] Michel Hirschowitz. *Jeux abstraits et composition catégorique*. Thèse de doctorat, Université Paris VII, December 2004. Available at `http://www.pps.jussieu.fr/∼mh/tez/fvf.ps`.
[9] Michel Hirschowitz, Tom Hirschowitz, and André Hirschowitz. A theory for game theories. dec 2007. Available at `http://www.pps.jussieu.fr/˜mh/catgames.pdf`.
[10] Martin Hyland and Luke Ong. On full abstraction for PCF. *Information and Computation*, 163(2):285–408, December 2000.
[11] Giorgi Japaridze. A constructive game semantics for the language of linear logic. *Ann. Pure Appl. Logic*, 85(2):87–156, 1997.
[12] Paul-André Melliès. Asynchronous games 3 : An innocent model of linear logic. In *CTCS*, 2004.
[13] Paul-André Melliès. Asynchronous games 4 : A fully complete model of propositional linear logic. 2005.

E-mail address: `michel.hirschowitz@[cea.fr,lix.polytechnique.fr]`
E-mail address: `ah@math.unice.fr`