



# Some Aspects of Kolmogorov-Arnold-Moser Theory and Applications

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**Habilitation Memoir to Supervise Research**

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## CHAPTER 1

### Introduction

The Kolmogorov-Arnold-Moser (KAM) theory stands as a cornerstone in the realm of dynamical systems, illuminating the persistence of quasi-periodic orbits within nearly integrable Hamiltonian systems. Originating from the combined intellectual prowess of A. N. Kolmogorov, V. I. Arnold and J. K. Moser, this groundbreaking theory has fundamentally altered our comprehension of stability and chaos across various scientific disciplines, particularly in mathematical physics. The emergence of KAM theory in the mid-20th century marked a pivotal shift in the study of complex systems, resolving longstanding puzzles regarding the solar system's stability and broadening its impact far beyond its initial scope.

#### 1. Background and history

Introduced in the mid-20th century, KAM theory not only solved perplexing questions about the stability of our solar system but also extended its influence across a broad spectrum of scientific inquiries. The historical trajectory of KAM theory is rooted in Kolmogorov's foundational theorem [79] presented in 1954, subsequently expanded by Arnold [3] and Moser [102]. Through their combined efforts, they crafted a comprehensive framework that outlines the specific conditions needed to maintain stability amidst slight disturbances. This intellectual heritage and the precision of their mathematical approach highlight the critical importance of KAM theory across the domains of both theoretical and applied mathematics.

The historical roots of KAM theory are deeply entwined with the problem of celestial mechanics, specifically the stability of the solar system, a question that dates back to Newton and was famously pondered by Poincaré. At its core, KAM theory addresses a fundamental question arising in Dynamical Systems and in particular in Celestial Mechanics [51, 27]: under what conditions can a system, subject to small perturbations, maintain its quasi-periodic motions? This question is not only central to the theory itself but also resonates with the broader quest to comprehend the long-term behavior of complex systems. In a more precise way, it is formulated as follows.

Given a completely integrable Hamiltonian dynamical system written in action-angle coordinates  $(\theta, I) \in \mathbb{T}^n \times \mathbb{R}^n$  of the form

$$\theta' = \omega(I), \quad I' = 0,$$

where  $\omega$  denotes an analytic function on  $\mathbb{T}^n$ . For each  $I_0$ , the manifold  $\mathbb{T}^n \times \{I_0\}$  is invariant and the motion on it is a constant rotation of angle  $\omega(I_0)$ . In nature, these systems are rather rare but one encounters small perturbations of them of the form

$$\theta' = \omega(I) + \varepsilon f(I, \theta), \quad I' = \varepsilon g(I, \theta), \quad (1)$$

with  $f, g$  analytic functions and  $\varepsilon > 0$  a small constant. Essentially, KAM theorem states that if the system is non-degenerate in some sense then there exists a large (in measure) compact set  $K$  such that for all  $I \in K$ , the system (1) possesses an invariant manifold which is diffeomorphic to a torus the dynamical system on which is conjugated to the rotation of angle  $\tilde{\omega}(I)$  on that torus. In some sense, a lot of the invariant tori  $\mathbb{T}^n \times I$  “survive” under a small non-degenerate perturbation.

The fundamental idea behind the KAM theory revolves around the stability of dynamical systems subjected to small perturbations. More generally, for a standard system, often referred to as being in “normal form”, the theory proposes the construction of a transformation, or a series of converging transformations, to conjugate it to a similar normal form. This process involves constructing a sequence of conjugate transformations, during which it becomes crucial to invert several linear operators and perform quantitative estimations. A key challenge in this construction is the control of the so-called “small divisors”, which is essential for ensuring the convergence of the transformation sequence. Often, meeting the small divisor conditions requires the careful selection of parameters. This delicate balance aims to demonstrate that despite the introduction of small perturbations, the system retains a significant measure of its original quasi-periodic orbits, thereby exhibiting long-term stability. In essence, KAM theory addresses how systems that are nearly integrable can remain stable and persist in their quasi-periodic behavior when small perturbations are applied, provided certain mathematical conditions are met. This insight has profound implications, suggesting that order and predictability in complex dynamical systems can be more resilient than previously thought, even in the face of disruptions.

The mathematical foundations of KAM theory rest on sophisticated analytical techniques that deal with the persistence of invariant tori in the phase space of Hamiltonian systems when subjected to small perturbations. This involves a delicate interplay between geometry, analysis, and algebra, showcasing the elegance and complexity of Hamiltonian dynamics. The notion that most, but not all, tori survive under perturbation leads to a nuanced understanding of stability and provides a gateway to the study of chaotic behavior (such as Arnold diffusion) in systems where the KAM conditions are not met.

The principal results of KAM theory have far-reaching implications across various domains of physics and beyond, underscoring the theory’s versatility and impact. This appears in particular in conjugacy problem to normal forms of vector fields at a fixed point (e.g., [25, 113]), in interval exchange maps (e.g., [95]), in spectral theory of Schrödinger operator (e.g., [37, 41, 42]), as well as in nonlinear Hamiltonian PDEs (e.g., [20, 46, 109]). These results furnish a rigorous framework for predicting the stability of motion in a wide array of systems, from atomic to celestial scales.

## 2. Works presented for the habilitation

As this memoir unfolds, it will explore the historical evolution, mathematical structure, and the critical achievements of KAM theory, with a special focus on its applications in spectral theory of Schrödinger operator, Hamiltonian PDEs and Cauchy-Riemann geometry. These explorations aim to not only highlight the theory’s foundational significance in Dynamical Systems but also to showcase its transformative influence on our understanding of the cosmos and the inherent order within it.

This memoir stems from the following works since my dissertation of Phd, and it will highlight the coherence of these works.

### **About quasi-periodic Schrödinger operators (presented in Chapter 2)**

(Z1) Ballistic motion in one-dimensional quasi-periodic discrete Schrödinger equation. *Commun. Math. Phys.* 2016, vol. 347, 511–549.

(Z2) Ballistic transport and absolute continuity of one-frequency quasi-periodic Schrödinger operators. *Commun. Math. Phys.* 2017, vol. 351, 877–921 (with Z. Zhang).

(Z3) Dispersive estimate for quasi-periodic Schrödinger operators on 1-d lattices. *Advances in Mathematics* 2020, vol. 366, 107071 (with D. Bambusi).

### **About long-time behaviors in Hamiltonian PDEs (presented in Chapter 3)**

(Z4) Symplectic Normal Form and Growth of Sobolev Norm. arXiv:2312.16492 (with Z. Liang and J. Luo).

(Z5) Almost reducibility and oscillatory growth of Sobolev norms. *Advances in Mathematics* 2024, vol. 436, 109417 (with Z. Liang and Q. Zhou).

### About singularities in Cauchy–Riemann geometry (presented in Chapter 4)

(Z6) Geometry of hyperbolic Cauchy–Riemann singularities and KAM-like theory for holomorphic involutions. *Math. Ann.* 2023, vol. 386, 587–672 (with L. Stolovitch).

### About rigidity of diffeomorphisms (presented in Chapter 5)

(Z7) Local rigidity of actions of isometries on compact real analytic Riemannian manifolds. arXiv:2312.07045 (with L. Stolovitch).

In (Z1) – (Z3), we introduce two renowned KAM schemes by Eliasson [41] (as presented in the version by Hadj Amor [61]) and by Avila-Fayad-Krikorian [10] for quasi-periodic  $SL(2, \mathbb{R})$  cocycles. These schemes are directly relevant to the spectral theory of quasi-periodic Schrödinger operators. Specifically, we explore transport properties associated with the absolutely continuous spectrum.

The work (Z4) offers a classification of normal forms for the  $sp(n, \mathbb{R})$  linear system for any  $n \in \mathbb{N}$ , as given by Hörmander [67]. This classification aims to support the establishment of a general KAM theorem for the  $sp(n, \mathbb{R})$  linear system. In (Z5), we revisit Eliasson’s KAM scheme [41] regarding the almost reducibility of the  $sl(2, \mathbb{R})$  linear system. Both findings are applied, through a classical-quantum correspondence, to examine the long-term behaviors in Sobolev spaces of solutions to a class of Hamiltonian PDEs.

Furthermore, we are committed to advancing the KAM theory into new fields of research. In (Z6), we establish a non-standard KAM theorem for holomorphic involutions, highlighting geometric characteristics of the hyperbolic Cauchy-Riemann singularity. In (Z7), by employing a KAM scheme, we demonstrate the analytic rigidity of group actions by isometries on compact Riemannian manifolds, extending Arnold’s linearization result [2] for circle diffeomorphisms.

## 3. Index of notations

Let us introduce several notations which appear frequently in this memoir.

(1) The notations “ $\lesssim$ ” and “ $\gtrsim$ ” are used to denote upper and lower bounds, respectively, by a positive constant which is not necessarily to be explicitly presented. Specifically, for non-negative values  $a$  and  $b$ , the inequality  $a \lesssim b$  (or  $a \gtrsim b$ ) indicates the existence of a constant  $c > 0$  such that  $a \leq cb$  (or  $a \geq cb$ , respectively). The expression  $a \simeq b$  signifies that both  $a \lesssim b$  and  $a \gtrsim b$  hold true.

(2) With the torus  $\mathbb{T}^d := \mathbb{R}^d / \mathbb{Z}^d$  of dimension  $d \in \mathbb{N}^*$ , let  $C_h^\omega(\mathbb{T}^d, \bullet)$ ,  $h > 0$  be the space of real-analytic  $\bullet$ -valued functions on  $\mathbb{T}^d$  which are holomorphic in the complex neighborhood of torus  $\mathbb{T}_h^d := \{z \in \mathbb{C}^d : \Re z \in \mathbb{T}^d, |\Im z| < h\}$ , equipped with the norm  $\|V\|_h := \sup_{z \in \mathbb{T}_h^d} |V(z)|_\bullet$  if  $V \in C_h^\omega(\mathbb{T}^d, \bullet)$ . Here  $\bullet$  denotes some vector space or matrix space. We also define  $\|V\|_{\mathbb{T}^d} := \sup_{z \in \mathbb{T}^d} |V(z)|_\bullet$ .

Let  $C_b^0(\mathbb{R}, \bullet)$  be the space of uniformly bounded continuous  $\bullet$ -valued functions on  $\mathbb{R}$ , equipped with the norm  $\|U\|_{C_b^0} := \sup_{t \in \mathbb{R}} \|U(t)\|_\bullet$  for  $U \in C_b^0(\mathbb{R}, \bullet)$ . Let  $C_b^1(\mathbb{R}, \bullet)$  be the subspace of  $C_b^0(\mathbb{R}, \bullet)$  whose entries are all  $C^1$  with uniformly bounded derivatives on  $\mathbb{R}$ , equipped with the norm  $\|U\|_{C_b^1} := \sup_{t \in \mathbb{R}} (\|U(t)\| + \|U'(t)\|)$  for  $U \in C_b^1(\mathbb{R}, \bullet)$ .

(3) Let  $\mathrm{Sp}(n, \mathbb{R})$  (resp.  $\mathfrak{sp}(n, \mathbb{R})$ ) be the symplectic group (resp. symplectic Lie algebra) of  $2n \times 2n$  real matrices with the matrix norm defined by

$$\|\mathbb{A}\| = \sum_{i,j=1}^{2n} |a_{ij}|, \quad \mathbb{A} = (a_{ij})_{i,j=1}^{2n} \in \mathrm{Sp}(n, \mathbb{R}) \quad (\text{or } \mathfrak{sp}(n, \mathbb{R})). \quad (2)$$

Moreover, we define  $\{\mathbf{e}_1, \dots, \mathbf{e}_{2n}\}$  the canonical basis of column vectors in  $\mathbb{R}^{2n}$ , the  $n \times n$  identity matrix  $I_n$  and the  $2n \times 2n$  standard symplectic matrix

$$\mathbb{J}_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = (-\mathbf{e}_{n+1}, \dots, \underset{\substack{\uparrow \\ n^{\text{th}}}}{-\mathbf{e}_{2n}}, \mathbf{e}_1, \dots, \mathbf{e}_n) \in \mathrm{Sp}(n, \mathbb{R}).$$

In the case  $n = 1$ , we also denote  $\mathrm{Sp}(1, \mathbb{R})$  (resp.  $\mathfrak{sp}(1, \mathbb{R})$ ) as  $\mathrm{SL}(2, \mathbb{R})$  (resp.  $\mathfrak{sl}(2, \mathbb{R})$ ). Let  $\mathrm{SO}(2, \mathbb{R})$  be the special orthogonal group in  $\mathbb{R}^2$ , composed of

$$R_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

## Time evolution of quasi-periodic Schrödinger operators

The primary aim of the studies in this chapter is to enhance our comprehension of the dynamics related to the Almost Mathieu Operator. This operator is frequently referenced within the realm of mathematical physics and in the analysis of quasi-periodic systems. It is pivotal across various domains, including quantum mechanics and spectral theory, particularly in examining the Anderson localization phenomena.

The Almost Mathieu Operator  $H_{\theta,\alpha,\lambda} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ , where  $\lambda, \theta \in \mathbb{R}$ ,  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , is articulated as

$$(H_{\theta,\alpha,\lambda}\psi)_n = -(\psi_{n+1} + \psi_{n-1}) + 2\lambda \cos(\theta + n\alpha)\psi_n, \quad n \in \mathbb{Z}. \quad (3)$$

In the context of spectral theory, Jitomirskaya [70] illustrated the “metal-insulator transition”, that is, the classification of spectral measure into three types based on the magnitude of the coupling constant  $\lambda$ , for almost all  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and almost all  $\theta \in \mathbb{R}$ . This finding is strongly supported by certain experiments, for instance, [69]. Specifically, for  $\alpha = \frac{\sqrt{5}-1}{2}$  and an initial wave packet represented by a  $\delta$ -type function  $u$ , the asymptotic expansion of the wave packet width

$$r^{(1)}(t) := \left( \sum_{n \in \mathbb{Z}} n^2 |(e^{-itH_{\theta,\alpha,\lambda}}u)_n|^2 \right)^{\frac{1}{2}},$$

can be approximated as  $r^{(1)}(t) \sim t^A$ , with three distinct regimes identified corresponding to the types of spectral measure. The theoretical outcomes and associated experimental data regarding the metal-insulator transition are summarized in the ensuing table.

	Spectral type of $H_{\theta,\alpha,\lambda}$ ([70])	growth rate $r^{(1)}(t) \sim t^A$ ([69])
$ \lambda  > 1$	purely point spectrum	localized regime, $A = 0$
$ \lambda  = 1$	purely singular continuous spectrum	sub-diffusive, $A \approx \frac{1}{2}$
$ \lambda  < 1$	purely absolutely continuous spectrum	ballistic regime, $A = 1$

(4)

The localized regime is defined by the boundedness of the norm  $r^{(1)}(t)$  and is rigorously identified as *Dynamical Localization* in various contexts. Conversely, the unbounded growth of  $r^{(1)}(t)$ , especially with a precise linear growth rate, has remained unproven for an extended duration. To address this gap, our initial focus has been on examining the time evolution associated with Schrödinger operators possessing an absolutely continuous (abbreviated as a.c.) spectrum.

**Key words:** Ballistic transport, Dispersive estimate, Absolutely continuous spectrum, Almost reducibility, Rotation reducibility

### 1. Ballistic transport, dispersion and absolute continuity

Typically, for a self-adjoint operator  $H$  on  $\ell^2(\mathbb{Z})$  characterized by an a.c. spectrum, the time evolution  $e^{-itH}$  of the associated dynamics demonstrates the optimal transport

property. This phenomenon, referred to as “delocalization”, is commonly characterized in two distinct manners: as *ballistic transport* and as *dispersion*.

**1.1. Ballistic transport.** Considering the *ballistic upper bound* (also known as the Lieb-Robinson bound [85], rigorously established in works such as [1, 32]), namely, the weighted  $\ell^2$ -norm

$$\|e^{-itH}u\|_1 := \left( \sum_{n \in \mathbb{Z}} n^2 |(e^{-itH}u)_n|^2 \right)^{\frac{1}{2}} \quad \text{provided} \quad \sum_{n \in \mathbb{Z}} n^2 |u_n|^2 < \infty \quad (5)$$

which stipulates that  $\|e^{-itH}u\|_1 \lesssim t$  for  $t > 0$ , an analogous asymptotic lower bound is anticipated for the time evolution associated with an a.c. spectrum. This expectation draws inspiration from the metal-insulator transition. A time-averaged analysis, as proposed by the Guarneri-Combes-Last theorem [82], reveals that in the context of an a.c. spectrum, for any sufficiently localized initial state  $u \neq 0$  within  $\ell_{ac}^2(\mathbb{Z})$ , the subspace of  $\ell^2(\mathbb{Z})$  related to the a.c. spectrum,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \|e^{-itH}u\|_1 dt \gtrsim T.$$

The operator  $H$ , or its time evolution  $e^{-itH}$ , is said to exhibit *ballistic transport* if

$$\|e^{-itH}u\|_1 \simeq t, \quad \text{provided } u \in \ell^2(\mathbb{Z}) \setminus \{0\} \text{ with } \sum_{n \in \mathbb{Z}} n^2 |u_n|^2 < \infty,$$

or, more strongly, for every  $s > 0$ ,

$$\|e^{-itH}u\|_s := \left( \sum_{n \in \mathbb{Z}} n^{2s} |(e^{-itH}u)_n|^2 \right)^{\frac{1}{2}} \simeq t^s, \quad u \in \ell^2(\mathbb{Z}) \setminus \{0\} \text{ with } \sum_{n \in \mathbb{Z}} n^{2s} |u_n|^2 < \infty.$$

Asch-Knauf [4] conjectured that the a.c. spectrum of a self-adjoint operator suggests the ballistic transport of its associated time evolution. The readers can refer to a recent survey by Damanik-Malinovitch-Young [33] for more detailed descriptions on this phenomenon.

As an illustrative example of an a.c. spectrum, consider the one-dimensional (1-D) Schrödinger operator with a  $p$ -periodic potential  $H_V$ , where  $p \in \mathbb{N}^*$ , defined on  $\ell^2(\mathbb{Z})$  as

$$(H_V \psi)_n = -(\psi_{n+1} + \psi_{n-1}) + V_n \psi_n, \quad \text{with } V_{n+p} = V_n, \quad \forall n \in \mathbb{Z}, \quad (6)$$

Damanik-Lukic-Yessen [32] demonstrated that this operator exhibits ballistic transport. This naturally raises the question of whether similar conclusions can be drawn for the Almost Mathieu Operator or the general 1-D Schrödinger operator with a quasi-periodic potential.

Let us examine the 1-D quasi-periodic Schrödinger operator  $H_{\theta, \alpha, V}$  on  $\ell^2(\mathbb{Z})$ , defined by

$$(H_{\theta, \alpha, V} \psi)_n = -(\psi_{n+1} + \psi_{n-1}) + V(\theta + n\alpha) \psi_n, \quad n \in \mathbb{Z}, \quad (7)$$

where  $V \in C_h^\omega(\mathbb{T}^d)$  for some  $h > 0$ , and  $\alpha \in \mathbb{R}^d$  is rationally independent. Compared to the periodic Schrödinger operator (6), a significant challenge in the quasi-periodic context is that the spectrum typically forms a Cantor set [5, 7] and is independent of  $\theta$ . The principal findings are encapsulated in the subsequent two theorems.

**Theorem 2.1** (Zhao [122]) Consider the operator  $H = H_{\theta, \alpha, V}$  defined as in (7) with  $\alpha \in \mathbb{R}^d$  satisfying a Diophantine condition, i.e., there exist  $\gamma > 0$  and  $\tau > d - 1$  such that

$$\inf_{j \in \mathbb{Z}} |\langle k, \alpha \rangle - j| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}. \quad (8)$$

If  $\|V\|_h := \varepsilon_0$  is sufficiently small (depending on  $h, d, \tau, \gamma$ ), then  $H_{\theta, \alpha, V}$  exhibits ballistic transport for any  $\theta \in \mathbb{T}^d$ . More precisely, provided any non-vanishing  $u \in \ell^2(\mathbb{Z})$  with

$\sum_n n^2 |u_n|^2 < \infty$ , there exist a constant  $C > 0$ , depending on  $\varepsilon_0$ ,  $\theta$  and  $u$ , and a numerical constant  $0 < \zeta < 1$ , such that

$$\liminf_{t \rightarrow \infty} \frac{\|e^{-itH}u\|_1}{t} \geq \frac{C}{1 + \varepsilon_0^\zeta}, \quad \limsup_{t \rightarrow \infty} \frac{\|e^{-itH}u\|_1}{t} \leq \frac{C}{1 - \varepsilon_0^\zeta}.$$

**Remark 2.2** It was well-known that the discrete Schrödinger operator with small quasi-periodic potential has purely a.c. spectrum. Since we did not find an exact proof<sup>1</sup>, we have provided one in [122].

**Theorem 2.3** (Zhang-Zhao [121]) Consider the operator  $H = H_{\theta, \alpha, V}$  defined as in (7) with  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$  and  $V \in C^\omega(\mathbb{T}, \mathbb{R})$  such that  $H_{\theta, \alpha, V}$  has purely a.c. spectrum for a.e.  $\theta \in \mathbb{T}$ . Given any non-vanishing  $u \in \ell^2(\mathbb{Z})$  with  $\sum_n n^2 |u_n|^2 < \infty$ , for a.e.  $\theta \in \mathbb{T}$ , for any  $\eta > 0$ ,

$$\lim_{t \rightarrow \infty} \frac{\|e^{-itH}u\|_1}{t^{1-\eta}} = \infty.$$

**Remark 2.4** In alternative frameworks, ballistic transport is characterized through *transport exponents*

$$\beta_u^+(s) := \limsup_{t \rightarrow \infty} \frac{\ln(\|e^{-itH}u\|_s)}{s \ln(t)}, \quad \beta_u^-(s) := \liminf_{t \rightarrow \infty} \frac{\ln(\|e^{-itH}u\|_s)}{s \ln(t)}, \quad s > 0,$$

for a self-adjoint operator  $H$  (refer to [34], for example). The operator  $H$  is said to exhibit *ballistic transport* if  $\beta_u^\pm(s) = 1$  for any non-zero, well-localized  $u \in \ell^2(\mathbb{Z})$  and for any  $s > 0$ . From this perspective, ballistic transport for the Almost Mathieu Operator with  $|\lambda| < 1$  follows as a consequence, as indicated by Theorem 2.3.

**Remark 2.5** The results of Theorem 2.1 and 2.3 have been further refined by Ge-Kachkovskiy [54], whose work establishes that  $\|e^{-itH}u\|_1$  demonstrates precise linear growth.

**1.2. Dispersion.** For a self-adjoint operator  $H$  acting on an  $\ell^2$  space, the classical RAGE theorem [31] significantly highlights how the long-term behavior of the time evolution  $e^{-itH}$  is intrinsically linked to the spectral characteristics of  $H$ . The RAGE theorem elucidates how the distribution of the conserved  $\ell^2$ -norm diverges based on the spectral type of  $H$ . In particular, the point spectrum is often associated with bound states, where dynamical localization is observed. This implies that if the initial wave packet is exponentially localized, then any polynomially weighted  $\ell^2$ -norm remains uniformly bounded w.r.t.  $t$ , meaning for any  $s > 0$ ,

$$\sup_{t \in \mathbb{R}} \sum_{n \in \mathbb{Z}} n^{2s} |(e^{-itH}u)_n|^2 < \infty \quad \text{provided} \quad |u_n| \lesssim e^{-c|n|},$$

for some constant  $c > 0$ . This indicates that the wave packet remains tightly concentrated over time. Conversely, as demonstrated in Section 1, if the spectrum is a.c., the weighted  $\ell^2$ -norm experiences polynomial growth w.r.t.  $t$ , which is indicative of a delocalization phenomenon. This naturally raises the question of how the wave packet's shape evolves over time, especially when it does not remain tightly concentrated.

For the  $\nu$ -dimensional free Schrödinger operator on  $\ell^2(\mathbb{Z}^\nu)$ , an elementary instance of an a.c. spectrum is given by

$$(-\Delta\psi)_n = - \sum_{|m-n|=1} \psi_m, \quad m \in \mathbb{Z}^\nu,$$

<sup>1</sup>There are relative proofs by Eliasson [41] for the 1-D continuous situation, and by Avila [6] for the one-frequency situation.

where Stefanov-Kevrekidis [111] demonstrated the  $\ell^1 - \ell^\infty$  dispersion: for every  $t \in \mathbb{R}$ ,

$$\|e^{it\Delta}u\|_{\ell^\infty} \lesssim \sqrt{1+t^2}^{-\frac{\nu}{3}} \|\psi\|_{\ell^1}, \quad \forall u \in \ell^1(\mathbb{Z}^\nu). \quad (9)$$

This result implies that a wave packet, initially localized in  $\ell^1$ , disperses over time, approaching infinity as time progresses towards infinity. Additionally, the decay rate of  $t^{-\frac{\nu}{3}}$  has been established as optimal.

For the 1-D Schrödinger operator with a periodic potential, as defined earlier, a similar dispersive estimate has been documented [100, 101]. Further studies have been conducted for 1-D Schrödinger operators with pointwise decaying potentials [78, 107]. Exploring the 1-D quasi-periodic Schrödinger operator  $H = H_{\theta,\alpha,V}$ , as introduced earlier, poses a greater challenge, especially since the dispersion of operators with Cantor spectra has not been thoroughly examined.

**Theorem 2.6** (Bambusi-Zhao [19]) Consider the operator  $H = H_{\theta,\alpha,V}$  defined as in (7) with  $\alpha \in \mathbb{R}^d$  satisfying a Diophantine condition (8). If  $\|V\|_h := \varepsilon_0$  is sufficiently small (depending on  $h, d, \tau, \gamma$ ), then, for any  $\theta \in \mathbb{T}^d$ , any  $t \in \mathbb{R}$ , we have

$$\|e^{-itH}\psi\|_{\ell^\infty} \leq K_0 |\ln(\varepsilon_0)|^{ad(\ln \ln(2+\langle t \rangle))^2} \|\psi\|_{\ell^1} \langle t \rangle^{-\frac{1}{3}}, \quad \forall \psi \in \ell^1(\mathbb{Z}), \quad (10)$$

with  $\langle t \rangle := \sqrt{1+t^2}$  and two absolute constants  $a, K_0$ . In particular, given any  $0 < \zeta < \frac{1}{3}$ , there exists a constant  $K_1 = K_1(\varepsilon_0, \zeta)$  such that for any  $\theta \in \mathbb{T}^d$ , any  $t \in \mathbb{R}$ ,

$$\|e^{-itH}\psi\|_{\ell^\infty} \leq K_1 \|\psi\|_{\ell^1} \langle t \rangle^{-\zeta}, \quad \forall \psi \in \ell^1(\mathbb{Z}).$$

## 2. KAM schemes for quasi-periodic $\text{SL}(2, \mathbb{R})$ -cocycles

The proofs of Theorems 2.1, 2.3, and 2.6 hinge on the spectral characteristics of the operator  $H = H_{\theta,\alpha,V}$ , as delineated in (7). Specifically, a key aspect of our analysis involves examining the asymptotic behaviors of the generalized eigenvectors  $\psi$  of  $H$ , which satisfy  $H\psi = E\psi$  for some  $E \in \Sigma(H)$ , the spectrum of  $H$ . These vectors are elucidated via *quasi-periodic Schrödinger cocycle*

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} -E + V(\theta + n\alpha) & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \psi_{n-1} \end{pmatrix}, \quad E \in \Sigma(H). \quad (11)$$

Expanding this framework, we explore the *quasi-periodic  $\text{SL}(2, \mathbb{R})$ -cocycle*: for  $\alpha \in \mathbb{T}^d$  linearly independent and  $A(\cdot) \in C^\omega(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ ,

$$\begin{aligned} (\alpha, A) : \mathbb{T}^d \times \mathbb{C}^2 &\rightarrow \mathbb{T}^d \times \mathbb{C}^2 \\ (\theta, v) &\mapsto (\theta + \alpha, A(\theta)v) \end{aligned} \quad (12)$$

Our discussion on KAM schemes in this chapter draws from the pioneering efforts of Eliasson [41], Hadj Amor [61] and Avila-Fayad-Krikorian [10], aiming to “simplify” the aforementioned discrete dynamical system.

**2.1. Almost reducibility.** Observing the asymptotic behavior in the dynamical system (12) becomes straightforward if the  $\text{SL}(2, \mathbb{R})$ -matrix is constant, that is, it does not depend on the phase  $\theta$ . A phase-dependent cocycle  $(\alpha, A(\cdot))$ , where  $A(\cdot) \in C^\omega(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ , is deemed *reducible* if there exists  $Z \in C^\omega(\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$  and  $B \in \text{SL}(2, \mathbb{R})$  such that

$$Z(\cdot + \alpha)^{-1}A(\cdot)Z(\cdot) = B.$$

Naturally, a KAM scheme aims to progressively eliminate the phase-dependent component, converging to the aforementioned identity in the limit. However, achieving reducibility is not always possible due to the inherent challenges posed by small divisors in the KAM construction.

As a response to these challenges, the concept of *almost reducibility* was introduced by Eliasson [41] for quasi-periodic linear systems and has since been adopted in various other dynamical systems contexts [8, 9, 43]. A quasi-periodic  $\mathbf{SL}(2, \mathbb{R})$ -cocycle  $(\alpha, A(\cdot))$  is considered *almost reducible* if there are sequences  $Z_l(\cdot) \in C^\omega(\mathbb{T}^d, \mathbf{SL}(2, \mathbb{R}))$ ,  $A_l \in \mathbf{SL}(2, \mathbb{R})$ , and  $F_l(\cdot) \in C^\omega(\mathbb{T}^d, \mathfrak{gl}(2, \mathbb{R}))$ , with  $A_l + F_l(\cdot) \in C^\omega(\mathbb{T}^d, \mathbf{SL}(2, \mathbb{R}))$  and  $\|F_l\|_{\mathbb{T}^d} \rightarrow 0$ , such that

$$Z_l(\theta + \alpha)^{-1}A(\theta)Z_l(\theta) = A_l + F_l(\theta).$$

The essence of almost reducibility lies in conjugating the time-dependent system arbitrarily close to a constant through a potentially divergent sequence of transformations  $\{Z_l(\cdot)\}$ .

For an  $\mathbf{SL}(2, \mathbb{R})$ -cocycle that is nearly constant and has a Diophantine frequency vector, the KAM scheme, attributable to Hadj Amor [61], builds upon the foundational work of Eliasson [41] concerning the reducibility and almost reducibility of the quasi-periodic  $\mathfrak{sl}(2, \mathbb{R})$ -linear system, which significantly parallels the analysis of the continuous Schrödinger operator on  $\mathbb{R}$ .

More specifically, for the cocycle  $(\alpha, A_0 + F_0(\cdot))$ , where  $\alpha \in \mathbb{R}^d$  meets the Diophantine condition (8),  $A_0 \in \mathbf{SL}(2, \mathbb{R})$ , and  $A_0 + F_0 \in C_{h_0}^\omega(\mathbb{T}^d, \mathbf{SL}(2, \mathbb{R}))$  with  $\|F_0\|_{h_0}$  sufficiently small and  $h_0 > 0$ , Hadj Amor [61] demonstrated that such a cocycle is always almost reducible. Furthermore, it becomes reducible if its *fibered rotation number*  $\rho = \rho(\alpha, A_0 + F_0(\cdot))$  - a classical concept established by Johnson-Moser [71] and Herman [66] - is *Diophantine w.r.t.*  $\alpha$ , that is, there exist constants  $K, \sigma > 0$  ensuring

$$\left| \rho - \frac{\langle k, \alpha \rangle}{2} \right| \geq \frac{K}{|k|^\sigma}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}, \quad (13)$$

or *rational w.r.t.*  $\alpha$ , i.e.,

$$\rho = \frac{\langle k_*, \alpha \rangle}{2} \quad \text{for some } k_* \in \mathbb{Z}^d.$$

Consequently, this establishes that the quasi-periodic Schrödinger cocycle is reducible for almost every  $E \in \mathbb{R}$ . If not, it is deemed almost reducible.

The arithmetic condition on the fibered rotation number is crucial within the KAM scheme. Suppose that  $\|F_0\|_{h_0} =: \varepsilon_0$  is adequately small. Let  $\{h_l\}_{l \in \mathbb{N}}$  be a monotonically decreasing sequence of positive numbers where  $h_{l+1} \leq (1 - \frac{1}{4^l})h_l$ . We define sequences  $\{\varepsilon_l\}_{l \in \mathbb{N}}$  and  $\{N_l\}_{l \in \mathbb{N}}$  as follows:

$$\varepsilon_{l+1} = \varepsilon_l^{\frac{201}{200}}, \quad N_l = \frac{4^l}{50h_l} |\ln \varepsilon_l|.$$

Assuming we reach a particular  $l$ -th step in the KAM iteration with the cocycle  $(\alpha, A_l + F_l(\cdot))$ , where  $A_l \in \mathbf{SL}(2, \mathbb{R})$  with eigenvalues  $e^{\pm i\xi_l}$ ,  $\pm i\xi_l \in \mathbb{R} \cup i\mathbb{R}$ , and  $A_l + F_l(\cdot) \in C_{h_l}^\omega(\mathbb{T}^d, \mathbf{SL}(2, \mathbb{R}))$  with  $\|F_l\|_{h_l} < \varepsilon_l$ . The goal at this stage is to achieve the conjugation

$$Z_l(\theta + \alpha)^{-1}(A_l + F_l(\theta))Z_l(\theta) = A_{l+1} + F_{l+1}(\theta), \quad (14)$$

by constructing an appropriate transformation  $Z_l \in C_{h_l}^\omega(2\mathbb{T}^d, \mathbf{SL}(2, \mathbb{R}))$  so that  $A_{l+1} \in \mathbf{SL}(2, \mathbb{R})$  and  $A_{l+1} + F_{l+1}(\cdot) \in C_{h_{l+1}}^\omega(\mathbb{T}^d, \mathbf{SL}(2, \mathbb{R}))$  with  $\|F_{l+1}\|_{h_{l+1}} < \varepsilon_{l+1}$ . This step crucially involves the removal of the ‘‘principal part’’ of  $F_l$ . Specifically, through the Fourier expansion

$$F_l(\theta) = \sum_{k \in \mathbb{Z}^d} \hat{F}_l(k) e^{2\pi i \langle k, \theta \rangle}, \quad \hat{F}_l(k) \in \mathfrak{gl}(2, \mathbb{R}),$$

we decompose  $F_l$  into  $\Pi_{N_l} F_l + \Pi_{N_l}^\perp F_l$ , where

$$(\Pi_{N_l} F_l)(\theta) := \sum_{\substack{k \in \mathbb{Z}^d \\ |k| \leq N_l}} \hat{F}_l(k) e^{2\pi i \langle k, \theta \rangle}, \quad (\Pi_{N_l}^\perp F_l)(\theta) := \sum_{\substack{k \in \mathbb{Z}^d \\ |k| > N_l}} \hat{F}_l(k) e^{2\pi i \langle k, \theta \rangle}.$$

By leveraging the exponential decay w.r.t.  $|k|$  of the matrix-valued Fourier coefficients  $\hat{F}_l(k)$ , the “residual part”  $\Pi_{N_l}^\perp F_l$  ensures that  $|\Pi_{N_l}^\perp F_l|_{h_{l+1}} \leq \varepsilon_l^{\frac{3}{2}}$ , identifying the principal part of  $F_l$  as  $\Pi_{N_l} F_l$ .

**(Non-resonance step)** If  $\xi_l$  fulfills the *second Melnikov condition*

$$|2\xi_l - \langle k, \alpha \rangle| \geq \varepsilon_l^{\frac{1}{15}}, \quad \forall k \in \mathbb{Z}^d \text{ with } 0 < |k| \leq N_l, \quad (15)$$

then, by examining the matrix-valued Fourier coefficients of  $\Pi_{N_l} F_l$ , we can identify some  $Y_l \in C_{h_l}^\omega(2\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  with  $\|Y_l\|_{h_l} < \varepsilon_l^{\frac{1}{2}}$ , such that

$$Y_l(\theta + \alpha)A_l - A_l Y_l(\theta) = A_l \left( (\Pi_{N_l} F_l)(\theta) - \hat{F}_l(0) \right).$$

This enables the conjugation (14) via the near-identity  $Z_l = e^{Y_l} \in C_{h_l}^\omega(2\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$ , accompanied by some  $A_{l+1} \in \text{SL}(2, \mathbb{R})$  that adheres to  $\|A_{l+1} - A_l\| < \varepsilon_l^{\frac{1}{2}}$ .

It is worth noting that for the constant cocycle  $(\alpha, A_l)$ , if  $\xi_l \in \mathbb{R}$ , then the fibered rotation number  $\rho_{(\alpha, A_l)} = \xi_l \bmod \pi$ , and the second Melnikov condition (15) acts as a truncated Diophantine condition (13). In a parameterized KAM scheme that ensures the non-degeneracy of  $\xi_l$  w.r.t. the parameter (such as  $E$  in (11)), the conventional KAM step is executed as described, except it excludes a subset of parameters failing to meet the condition (15). This excluded parameter subset measures  $\lesssim N_l \varepsilon_l^{\frac{1}{15}}$ , thereby directing the scheme towards reducibility for a significant-measure subset of parameters, as demonstrated by Dinaburg-Sinai [37].

**(Resonance step)** Assuming the second Melnikov condition (15) is not satisfied, and given the Diophantine nature of the base frequency  $\alpha \in \mathbb{R}^d$ , there exists a unique  $k_l \in \mathbb{Z}^d$  with  $0 < |k_l| \leq N_l$  for which

$$|2\xi_l - \langle k_l, \alpha \rangle| < \varepsilon_l^{\frac{1}{15}}. \quad (16)$$

For  $A_l = C_{A_l} \begin{pmatrix} e^{i\xi_l} & 0 \\ 0 & e^{-i\xi_l} \end{pmatrix} C_{A_l}^{-1}$  with  $C_{A_l} \in \text{SL}(2, \mathbb{R})$ , let us define

$$H_{n_l}^{A_l}(\cdot) := C_{A_l} \begin{pmatrix} e^{\frac{i}{2}\langle n_l, \cdot \rangle} & 0 \\ 0 & e^{-\frac{i}{2}\langle n_l, \cdot \rangle} \end{pmatrix} C_{A_l}^{-1}.$$

Considering the cocycle  $(\alpha, \tilde{A}_l + \tilde{F}_l(\cdot))$  with

$$\tilde{A}_l := H_{n_l}^{A_l}(\theta + \alpha)^{-1} A_l H_{n_l}^{A_l}(\theta) = C_{A_l} \begin{pmatrix} e^{i(\xi_l - \frac{\langle n_l, \alpha \rangle}{2})} & 0 \\ 0 & e^{-i(\xi_l - \frac{\langle n_l, \alpha \rangle}{2})} \end{pmatrix} C_{A_l}^{-1},$$

$$\tilde{F}_l(\theta) := H_{n_l}^{A_l}(\theta + \alpha)^{-1} F_l(\theta) H_{n_l}^{A_l}(\theta),$$

the new Melnikov condition is met for  $\tilde{\xi}_l := \xi_l - \frac{\langle n_l, \alpha \rangle}{2}$ , indicating

$$\left| 2\tilde{\xi}_l - \langle k, \alpha \rangle \right| \geq \varepsilon_l^{\frac{1}{15}}, \quad \forall k \in \mathbb{Z}^d \text{ with } 0 < |k| \leq N_l.$$

Therefore, the standard KAM step can be applied to  $(\alpha, \tilde{A}_l + \tilde{F}_l(\cdot))$ , resulting in the existence of  $\tilde{Z}_l = e^{\tilde{Y}_l}$  with  $\tilde{Y}_l \in C_{h_l}^\omega(2\mathbb{T}^d, \text{SL}(2, \mathbb{R}))$  and  $A_{l+1} \in \text{SL}(2, \mathbb{R})$  such that  $\|A_{l+1} - \tilde{A}_l\| < \varepsilon_l^{\frac{1}{2}}$ . The conjugation (14) is achieved with  $Z(\cdot) = H_{n_l}^{A_l}(\cdot) e^{Y_l(\cdot)}$ .

The iteration process advances regardless of whether the second Melnikov condition (15) is met. In the parametrized KAM scheme described in [61], cocycles are differentiated into two categories:

- For almost every parameter, if the resonance step occurs only finitely many times during the iteration, convergence is achieved in the sequence of transformations  $Z_l$ . This results from the transformations increasingly resembling the identity matrix beyond a certain iteration step. Consequently, the cocycle  $(\alpha, A_0 + F_0(\cdot))$  is analytically reducible.
- For a set of parameters of measure zero, the resonance step recurs infinitely. In this scenario, the sequence of transformations does not converge, rendering the cocycle  $(\alpha, A_0 + F_0(\cdot))$  almost reducible.

For additional perspectives and developments in this KAM scheme regarding almost reducibility, readers may consult references such as [26, 68, 84, 103].

**2.2. Rotation reducibility.** For any irrational base frequency  $\alpha \in \mathbb{R}$ , Avila-Fayad-Krikorian [10] developed a KAM scheme specifically designed to address challenges associated with Liouvillean frequencies. In their approach, the cocycle is conjugated to a rotation, which intriguingly, does not have to remain constant.

Let us revisit some fundamental concepts related to the irrational frequency  $\alpha$ . Consider  $\{q_n\}$ , the sequence of denominators in the continued fraction expansion of  $\alpha$ . Let  $\{Q_l\} = \{q_{n_l}\}$  denote a subsequence of these denominators, with  $\{\bar{Q}_l\} = \{q_{n_{l+1}}\}$  being the subsequent terms in the original sequence. For given constants  $0 < \mathcal{A} \leq \mathcal{B} \leq \mathcal{C}$ , a pair of denominators  $(q_l, q_n)$ , where  $1 \leq l \leq n$ , is defined as a  $CD(\mathcal{A}, \mathcal{B}, \mathcal{C})$ -bridge if

$$q_l^{\mathcal{B}} \leq q_n \leq q_l^{\mathcal{C}}, \quad q_{i+1} \leq q_i^{\mathcal{A}}, \quad l \leq i \leq n-1.$$

Furthermore, given a constant  $\mathcal{D} > 0$ , a subsequence  $Q_l$  is termed  $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ -admissible if  $Q_0 = 1$ ,  $Q_k \leq \bar{Q}_{k-1}^{\mathcal{D}}$  for  $k \geq 1$ , and either  $\bar{Q}_k > Q_k^{\mathcal{A}}$  or  $(\bar{Q}_{k-1}, Q_k)$  and  $(Q_k, Q_{k+1})$  are both  $CD(\mathcal{A}, \mathcal{B}, \mathcal{C})$ -bridges.

For the  $\mathbf{SL}(2, \mathbb{R})$ -cocycle  $(\alpha, A_0 + F_0(\cdot))$ , with  $A_0 \in \mathbf{SO}(2, \mathbb{R})$  and  $\|F_0\|_{\mathbb{T}}$  being sufficiently small, let us represent  $A_0 \in \mathbf{SO}(2, \mathbb{R})$  as  $R_{\phi_0}$  where  $\phi_0 \in \mathbb{T}$ , and express  $A_0 + F_0(\cdot)$  as  $R_{\phi_0}(\text{Id} + \xi_0(\cdot))$ , with  $\xi_0(\cdot) = R_{-\phi_0}F_0(\cdot)$ . Avila-Fayad-Krikorian [10] demonstrated that if there exist  $\epsilon > 0$ ,  $\tau > 0$ , and  $0 < \nu < \frac{1}{2}$  such that the fibered rotation number of the cocycle  $\rho = \rho_{(\alpha, A_0 + F_0(\cdot))}$  fulfills

$$\inf_{j \in \mathbb{Z}} |2q_n \rho - j| > \epsilon \max \{q_{n+1}^{-\nu}, q_n^{-\tau}\}, \quad (17)$$

with  $q_n$  being the sequence of denominators of  $\alpha$ , then the cocycle  $(\alpha, A_0 + F_0(\cdot))$  is *rotation reducible*. This means that there exist  $Z \in C^\omega(\mathbb{T}, \mathbf{SL}(2, \mathbb{R}))$  and  $\phi \in C^\omega(\mathbb{T}, \mathbb{T})$  such that

$$Z(\theta + \alpha)^{-1}(A_0 + F_0(\theta))Z(\theta) = R_{\phi(\theta)}.$$

The KAM scheme intricately intertwines with the arithmetic characteristics of the fibered rotation number. For a sufficiently large  $\mathcal{A} > 0$ , an  $(\mathcal{A}, \mathcal{A}, \mathcal{A}^{22}, \mathcal{A}^{21})$ -admissible subsequence of denominators of  $\alpha$ ,  $\{Q_l\}$ , is identified, as confirmed by Lemma 3.2 in [10]. When reaching the  $l$ -th KAM iteration, the focus is on the cocycle  $(\alpha, B_l(\cdot))$ , where

$$B_l(\cdot) = R_{\phi_l(\cdot)}(\text{Id} + \xi_l(\cdot)) \in C^\omega(\mathbb{T}, \mathbf{SL}(2, \mathbb{R}))$$

with  $\phi_l \in C^\omega(\mathbb{T}, \mathbb{T})$  and  $\xi_l \in C^\omega(\mathbb{T}, \mathfrak{gl}(2, \mathbb{R}))$ , and  $\|\xi_l\|_{\mathbb{T}} < c_1 \exp(-\bar{Q}_l^{D_1})$  for some constants  $c_1, D_1 > 0$ . Importantly,  $\rho_{(\alpha, B_l(\cdot))} = \rho_{(\alpha, A_0 + F_0(\cdot))}$ . The goal at this stage is to achieve the conjugation

$$W_l(\theta + \alpha)^{-1}B_l(\theta)W_l(\theta) = R_{\phi_{l+1}(\cdot)}(\text{Id} + \xi_{l+1}(\cdot)) \quad (18)$$

via some near-identity transformation  $W_l \in C^\omega(\mathbb{T}, \mathbf{SL}(2, \mathbb{R}))$ , with  $\phi_{l+1} \in C^\omega(\mathbb{T}, \mathbb{T})$  and  $\xi_{l+1} \in C^\omega(\mathbb{T}, \mathfrak{gl}(2, \mathbb{R}))$ , ensuring  $\|\xi_{l+1}\|_{\mathbb{T}} < c_1 \exp(-\bar{Q}_{l+1}^{D_1})$ .

In alignment with Lemma 4.5 from [10], when  $Q_l$  is substantially large and if the fibered rotation number  $\rho = \rho_{(\alpha, B_l(\cdot))}$  meets the criterion

$$\inf_{j \in \mathbb{Z}} |2Q_k \rho - j| > \epsilon \max \{Q_k^{-\tau}, \bar{Q}_k^{-\nu}\}, \quad 1 \leq k \leq l+1,$$

then the  $Q_{l+1}$ -th iterate of  $(\alpha, B_l(\cdot))$ ,

$$B_l^{(Q_{l+1})}(\cdot) := B_l(\cdot + (Q_{l+1} - 1)\alpha) \cdots B_l(\cdot + \alpha) B_l(\cdot)$$

can be expressed as

$$B_l^{(Q_{l+1})}(\cdot) = R_{\phi_l^{[Q_{l+1}]}}(\text{Id} + \xi_{(Q_{l+1})}(\cdot)),$$

where  $\xi_{(Q_{l+1})}(\cdot) \in C^\omega(\mathbb{T}, \text{gl}(2, \mathbb{R}))$  and  $\phi_l^{[Q_{l+1}]}$  represents the  $Q_{l+1}$ -th Birkhoff sum of  $\phi_l(\cdot)$ , explicitly given by

$$\phi_l^{[Q_{l+1}]}(\cdot) := \sum_{k=0}^{Q_{l+1}-1} \phi_l(\cdot + k\alpha).$$

Following Proposition 4.1-(1) in [10], a near-identity transformation  $W_l \in C^\omega(\mathbb{T}, \text{SL}(2, \mathbb{R}))$  and a phase function  $\varsigma_l \in C^\omega(\mathbb{T}, \mathbb{T})$  exist such that

$$W_l(\cdot + Q_{l+1}\alpha)^{-1} B_l^{(Q_{l+1})}(\cdot) W_l(\cdot) = R_{\varsigma_l(\cdot)}(\text{Id} + \tilde{\xi}_l(\cdot)).$$

From this premise, one can deduce  $\phi_{l+1} \in C^\omega(\mathbb{T}, \mathbb{T})$  and  $\xi_{l+1} \in C^\omega(\mathbb{T}, \text{gl}(2, \mathbb{R}))$ , adhering to the necessary estimates, thus achieving the intended conjugation as outlined in (18).

The readers can refer to [10] for the details.

### 3. Application of KAM schemes

The KAM schemes outlined in Section 2 are indeed provided in parametrized versions, as already mentioned in the works of Eliasson [41] and Hadj Amor [61], which include detailed estimations on the dependency on parameters. In our work [121], we adapted the KAM scheme from [10] into a parametrized format, incorporating the necessary estimations.

Building on the generalized eigenvectors of the operator  $H = H_{\theta, \alpha, V}$ , derived via the quasi-periodic Schrödinger cocycle (11), we manage to establish a spectral transformation linked to operator  $H$  (see [28, 106]). This transformation, denoted as  $\mathcal{S}$ , maps sequences from  $\ell^2(\mathbb{Z})$  to functions in  $\mathcal{L}^2(d\varphi)$ , a space of square-integrable functions w.r.t. a suitable a.c. measure matrix  $d\varphi$ , supported on the spectrum  $\Sigma(H)$  of  $H$ . More precisely, we define

$$\begin{aligned} \mathcal{S} : \ell^2(\mathbb{Z}) &\rightarrow \mathcal{L}^2(d\varphi) \\ (q_n)_{n \in \mathbb{Z}} &\mapsto \begin{pmatrix} \sum_{n \in \mathbb{Z}} q_n \mathcal{K}_n \\ \sum_{n \in \mathbb{Z}} q_n \mathcal{J}_n \end{pmatrix}, \end{aligned}$$

where  $\mathcal{K}(E) = (\mathcal{K}_n(E))_{n \in \mathbb{Z}}$  and  $\mathcal{J}(E) = (\mathcal{J}_n(E))_{n \in \mathbb{Z}}$  are linearly independent generalized eigenvectors of  $H$ , which are  $C^1$  in  $E \in \Sigma(H)$  in the Whitney sense,  $d\varphi = \begin{pmatrix} d\varphi_{1,1} & d\varphi_{1,2} \\ d\varphi_{2,1} & d\varphi_{2,2} \end{pmatrix}$  is a  $2 \times 2$ -matrix of suitable a.c. measures supported on  $\Sigma(H)$ , and  $\mathcal{L}^2(d\varphi)$  is defined as

$$\mathcal{L}^2(d\varphi) := \left\{ \begin{pmatrix} g_1(E) \\ g_2(E) \end{pmatrix} : \sum_{j,k=1,2} \int_{\Sigma(H)} g_j(E) \bar{g}_k(E) d\varphi_{j,k} < \infty \right\}.$$

The essence of this transformation lies in its ability to convert the dynamics of the discrete Schrödinger equation under the operator  $H$  into a continuous form w.r.t. the spectral parameter  $E$ . Specifically, under the spectral transformation, the linear Schrödinger equation's propagation, described by  $e^{-itH}$ , translates into a simpler form where the time evolution directly corresponds to multiplicative factors  $e^{-iEt}$  for each spectral component

$E \in \Sigma(H)$ . More precisely, under the spectral transformation  $G(E, t) := (\mathcal{S}q)(E, t)$ , the discrete equation  $i\partial_t q(t) = Hq(t)$  is transformed into

$$i\partial_t G(E, t) = \begin{pmatrix} \sum_n (Hq)_n(t) \mathcal{K}_n(E) \\ \sum_n (Hq)_n(t) \mathcal{J}_n(E) \end{pmatrix} = \begin{pmatrix} \sum_n q_n(t) (H\mathcal{K})_n(E) \\ \sum_n q_n(t) (H\mathcal{J})_n(E) \end{pmatrix} = EG(E, t).$$

Hence  $G(E, t) = e^{-iEt}G(E, 0)$  for  $E \in \Sigma(H)$ .

**3.1. Proof of ballistic transport.** The principal approach to proving Theorem 2.1 and 2.3 revolves around translating the growth of  $\|e^{-itH}u\|_1$  into that of  $G(E, t)$ . By considering  $\partial_E G(E, t)$ , the derivative w.r.t.  $E$  in the Whitney sense, it becomes evident that

$$\partial_E G(E, t) = e^{-iEt} (-itG(E, 0) + \partial_E G(E, 0)). \quad (19)$$

To demonstrate the linear growth of  $\|e^{-itH}u\|_1$  w.r.t.  $t$ , it suffices to show that

$$\|\partial_E G(E, t)\|_{\mathcal{L}^2(d\varphi)} \lesssim \left( \sum_{n \in \mathbb{Z}} |nq_n(t)|^2 \right)^{\frac{1}{2}}, \quad (20)$$

for a certain matrix of spectral measures  $d\varphi$ . This assertion is supported by insights into the generalized eigenvectors  $(\mathcal{K}_n(E))_n$  and  $(\mathcal{J}_n(E))_n$  of  $H_{\theta, \alpha, V}$ , derived via the Schrödinger cocycle (11).

For the operator  $H$  discussed in Theorem 2.1, Lemma 4.4 in [122] elucidates that for the matrix of spectral measures defined on the spectrum  $\Sigma$  as

$$d\varphi|_{\Sigma} := \frac{1}{\pi} \begin{pmatrix} \rho'^{-1} & 0 \\ 0 & \rho'^{-1} \end{pmatrix} dE, \quad d\varphi|_{\mathbb{R} \setminus \Sigma} := 0,$$

with  $\rho$  being the fibered rotation number of the Schrödinger cocycle (11), and for  $\sigma := \frac{1}{200}$ ,

$$\left| \|\partial_E G(E, t)\|_{\mathcal{L}^2(d\varphi)} - \left( \sum_{n \in \mathbb{Z}} |nq_n(t)|^2 \right)^{\frac{1}{2}} \right| \leq \varepsilon_0^{\frac{\sigma}{4}} |q_0| + \varepsilon_0^{\frac{\sigma^2}{10}} \left( \sum_{n \in \mathbb{Z}} |nq_n(t)|^2 \right)^{\frac{1}{2}},$$

thereby fulfilling (20). This conclusion is reached through an approximation technique, wherein the variables at a given KAM iteration are piecewise smooth functions of the real parameter  $E$ .

For the operator  $H$  in Theorem 2.3, directly applying the KAM scheme of Avila-Fayad-Krikorian [10] faces challenges due to the phase-dependence in both the Schrödinger operator and the associated Schrödinger cocycle, which may not be minor. A crucial initial step involves a preliminary transformation to conjugate the cocycle, bringing it closer to a constant rotation. Avila [6] demonstrated that a purely a.c. spectrum for a one-frequency quasi-periodic Schrödinger operator indicates the almost reducibility of its associated quasi-periodic Schrödinger cocycle. More concretely, for any irrational  $\alpha$ , and for any potential  $V \in C^\omega(\mathbb{T}, \mathbb{R})$  ensuring that the operator  $H = H_{\theta, \alpha, V}$ , defined as in (7), maintains a purely a.c. spectrum for almost every  $\theta \in \mathbb{T}$ , the Schrödinger cocycle (11) can be almost reduced to some  $A_0 \in \text{SO}(2, \mathbb{R})$  for almost every  $E \in \Sigma(H)$ . This means that, for any  $\varepsilon > 0$ , there exists a transformation  $\Lambda(\cdot) \in C^\omega(\mathbb{T}, \text{SL}(2, \mathbb{R}))$  fulfilling

$$\left\| \Lambda(\cdot + \alpha)^{-1} \begin{pmatrix} -E + V(\cdot + n\alpha) & -1 \\ 1 & 0 \end{pmatrix} \Lambda(\cdot) - A_0 \right\|_h < \varepsilon \text{ for some } h > 0.$$

Given that spectral parameters  $E$  meeting the condition (17) span a positive-measure subset of  $\Sigma(H)$  - covering a substantial portion of  $\Sigma$  as  $\varepsilon \rightarrow 0$  - achieving (20) directly poses a

challenge. However, as delineated in Corollary A and B of [121], employing a suitable matrix of spectral measures, for any  $\eta > 0$ , it is established that as  $t \rightarrow \infty$ ,

$$\|\partial_E G(E, t)\|_{\mathcal{L}^2(d\varphi)} \leq t^{\frac{\eta}{2}} \left( \sum_{n \in \mathbb{Z}} |nq_n(t)|^2 \right)^{\frac{1}{2}}.$$

Combined with (19), this result implies that

$$\frac{1}{t^{1-\eta}} \left( \sum_{n \in \mathbb{Z}} |nq_n(t)|^2 \right)^{\frac{1}{2}} \rightarrow \infty,$$

thus illustrating a methodology for translating the growth of  $\|e^{-itH}u\|_1$  to that of  $G(E, t)$  under these spectral transformations.

**3.2. Proof of dispersion.** In the case where the Schrödinger cocycle (11) is phase-independent, i.e.,  $V$  is constant or specifically  $V = 0$ , we consider the corresponding Schrödinger operator, often termed the free Schrödinger operator, defined as

$$(H\psi)_n = (-\Delta\psi)_n = -(\psi_{n+1} + \psi_{n-1}), \quad n \in \mathbb{Z}, \quad (21)$$

which has the spectrum  $\Sigma(H) = [-2, 2]$ . The two linearly independent generalized eigenvectors for this operator can be explicitly given by

$$\mathcal{K}_n(E) = \cos(n\rho_0(E)), \quad \mathcal{J}_n(E) = \sin(n\rho_0(E)), \quad \rho_0(E) := \cos^{-1}\left(\frac{-E}{2}\right). \quad (22)$$

Utilizing the unitary spectral transformation

$$\mathcal{S} : (u_n)_n \mapsto \begin{pmatrix} \sum_n u_n \cos(n\rho_0(E)) \\ \sum_n u_n \sin(n\rho_0(E)) \end{pmatrix},$$

with the spectral measures matrix

$$d\varphi|_{\Sigma(H)} := \frac{1}{\pi} \begin{pmatrix} \rho'_0(E) & 0 \\ 0 & \rho'_0(E) \end{pmatrix} dE, \quad d\varphi|_{\mathbb{R} \setminus \Sigma(H)} := 0,$$

we can address the time-dependent equation in  $\mathcal{L}^2(d\varphi)$ . Applying the inverse spectral transformation allows us to recover the evolution under  $e^{it\Delta}$  for  $\psi \in \ell^1(\mathbb{Z})$ :

$$\begin{aligned} (e^{it\Delta}\psi)_n &= \sum_{m \in \mathbb{Z}} \psi_m \int_{-2}^2 e^{-iEt} \cos((m-n)\rho_0(E)) \cdot \rho'_0(E) dE \\ &= \sum_{m \in \mathbb{Z}} \psi_m \int_0^\pi e^{2it \cos(\rho_0)} \cos((m-n)\rho_0) d\rho_0. \end{aligned}$$

The 1-D dispersive estimate, such as (9), can be directly derived by applying the Van der Corput lemma (as detailed in Chapter VIII of [112]) to the oscillatory integrals over sub-intervals of  $[0, \pi]$ .

When a quasi-periodic potential is incorporated into the Schrödinger operator, as described for  $H$  in (7), the spectral transformation continues to rely on two linearly independent generalized eigenvectors,  $(\mathcal{K}_n(E))_n$  and  $(\mathcal{J}_n(E))_n$ , constructed through the quasi-periodic Schrödinger cocycle (11). For sufficiently large  $N$  (corresponding to a certain KAM step where the phase-dependent part becomes smaller than  $t^{-2}$  for a given non-zero  $t$ ), these generalized eigenvectors can be approximated as

$$\begin{pmatrix} \mathcal{K}_n^{(N)}(\theta, E) \\ \mathcal{J}_n^{(N)}(\theta, E) \end{pmatrix} = \begin{pmatrix} a_n^{(N)}(\theta, E) \cos(n\rho_N(E)) \\ a_n^{(N)}(\theta, E) \sin(n\rho_N(E)) \end{pmatrix},$$

where  $a_n^{(N)}$  and  $\rho_N$  are piecewise smooth with respect to  $E$  within  $\Sigma_N(H)$ , the “approximated spectrum” of  $H$ . Here,  $\rho_N$  serves as the approximated fibered rotation number of the Schrödinger cocycle, which tends towards the devil’s staircase as  $N \rightarrow \infty$ . In a significant portion of  $\Sigma_N(H)$ ,  $\rho_N(E)$  is nearly equivalent to  $\rho_0(E) = \cos^{-1}(-\frac{E}{2})$ , and the coefficients  $a_n^{(N)}$  are almost 1. Conversely, in the smaller segments of  $\Sigma_N(H)$ ,  $\rho'_N(E) > 0$  has an upper bound,  $|\rho''_N(E)|$  has a lower bound, and  $a_n^{(N)}$  is sufficiently small.

Given these insights, the time evolution  $e^{-itH}$  can be approximately represented as a sum of oscillatory integrals over different intervals of  $[\inf \Sigma_N(H), \sup \Sigma_N(H)]$ . By applying the Van der Corput lemma to each interval, a  $|t|^{-\frac{1}{3}}$  estimate can be derived. After considering the multiplication by the number of intervals, which is capped by  $|\ln(\varepsilon_0)|^{ad(\ln \ln(2+|t|))^2}$  for some  $a > 0$ , the dispersive estimate (10) is achieved.

#### 4. Further discussions

A natural extension of these findings involves examining the dispersion for 1-D quasi-periodic Schrödinger operators within the sub-critical regime. This regime is defined in Avila’s global theory [9] for one-frequency quasi-periodic Schrödinger operators, where ballistic transport has already been demonstrated in [121]. In this sub-critical regime, the appearance of the a.c. spectrum and the almost reducibility of the associated quasi-periodic Schrödinger cocycle are connected to a range of dynamic behaviors.

An equally critical follow-up is to demonstrate both ballistic transport and dispersion for higher-dimensional lattice Schrödinger operators that exhibit an a.c. spectrum. In cases of quasi-periodicity, exploring the almost reducibility of quasi-periodic cocycles within the broader symplectic group and employing spectral transformations in a more generalized form may provide valuable insights.

Broadening the scope, there is anticipation to uncover the relationships between the a.c. spectrum, ballistic transport, and dispersion, without resorting to specific assumptions about the self-adjoint operator.



## Long-time behaviors in Hamiltonian PDEs

Many fundamental PDEs from physics are expressible in Hamiltonian form. The resulting Hamiltonian systems, such as 1-D Korteweg-de Vries and the 1-D nonlinear Schrödinger equations, exhibit integrable structures and possess significant geometric and dynamical characteristics. Typically, it is the perturbations of these systems that emerge in practical scenarios. Our aim is to understand the extent to which the behavior of solutions to these perturbed systems mirrors that of the original integrable systems they derive from.

Our focus extends to the long-term behavior of solutions to Hamiltonian PDEs within the context of Sobolev spaces. Initiated by Bourgain [22] and further explored through a series of groundbreaking studies (e.g., [29, 60]), the analysis of solutions to Hamiltonian PDEs in Sobolev spaces has become a focal point in the realm of Mathematical Physics over recent decades. Notably, the exploration of unbounded trajectories in Sobolev spaces has revealed connections to phenomena such as weak turbulence and energy cascades, particularly within various quantum Hamiltonian frameworks.

In view of the significance of the quadratic quantum Hamiltonian in illustrating wave propagation phenomena and the connection with the coherent state, the study concentrates on Hamiltonian PDEs quantized by a quadratic polynomial with time dependent coefficients. Specifically, the work delves into the  $n$ -dimensional PDE

$$\begin{aligned} \frac{1}{2\pi i} \partial_t \psi(t, x) &= H_0(t, D, X) \psi(t, x) \\ &= (\mathcal{Q}_{\mathcal{A}(t)}(Z) + \mathcal{L}_{\ell(t)}(Z) + c(t)) \psi(t, x), \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n, \end{aligned} \quad (23)$$

where  $D = \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix} := \frac{1}{2\pi i} \nabla_x$ ,  $X := \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$  with  $X_j$  the multiplication by  $j$ -th coordinate function, i.e.,  $(X_j f)(x) = x_j f(x)$ ,  $Z := \begin{pmatrix} D \\ X \end{pmatrix}$ , and the linear operator  $H_0(t, D, X)$  in Eq. (23) is decomposed into

- *Homogeneous part*  $\mathcal{Q}_{\mathcal{A}(t)}(Z) := -\frac{1}{2} \langle Z, \mathcal{A}(t) \mathbb{J}_n Z \rangle$  with

$$\mathcal{A}(\cdot) = \begin{pmatrix} A_{11}(\cdot) & A_{12}(\cdot) \\ A_{21}(\cdot) & -A_{11}(\cdot)^* \end{pmatrix} \in C_b^0(\mathbb{R}, \text{sp}(n, \mathbb{R})), \quad \mathbb{J}_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

- *Linear part*  $\mathcal{L}_{\ell(t)}(Z) := \langle \ell(t), Z \rangle$  with  $\ell(\cdot) = (l_1(\cdot)^*, l_2(\cdot)^*)^* \in C_b^0(\mathbb{R}, \mathbb{R}^{2n})$ ,
- *Scalar part*  $c(t)$ .

Eq. (23) is quantized by the quadratic polynomial classical hamiltonian

$$h_0(t, \xi, x) = -\frac{1}{2} \langle z, \mathcal{A}(t) \mathbb{J}_n z \rangle + \langle \ell(t), z \rangle + c(t), \quad z := \begin{pmatrix} \xi \\ x \end{pmatrix} \in \mathbb{R}^{2n}, \quad (24)$$

where  $z \in \mathbb{R}^{2n}$  such that  $z^* = (\xi^*, x^*)$ , and the three terms of  $h_0(t, \xi, x)$  are also called *homogeneous part*, *linear part* and *scalar part* respectively in the classical hamiltonian sense.

As a typical example of the quantum Hamiltonian  $H_0$  and a well-known “equilibrium state”, the  $n$ -D quantum harmonic oscillator (QHO for short),

$$\mathcal{T} = \sum_{j=1}^n \mathcal{T}_j := 2\pi (\langle D, D \rangle + \langle X, X \rangle) = -\frac{1}{2\pi} \Delta + 2\pi |X|^2, \quad \mathcal{T}_j := 2\pi(D_j^2 + X_j^2), \quad (25)$$

i.e.,  $H_0(t, D, X)$  with constant coefficients  $\mathcal{A}(\cdot) = 4\pi \mathbb{J}_n$ ,  $\ell(\cdot) = 0$ ,  $c(\cdot) = 0$ , as well as their perturbations, are well investigated in many recent works and it is much related to this chapter.

**Key words:** Growth of Sobolev norm, Quadratic Hamiltonian, Symplectic normal form, Metaplectic representation, Schrödinger representation, (Almost) reducibility, Anosov-Katok construction

### 1. Reducibility – Symplectic normal forms

For a specific quantum Hamiltonian  $H_0(t, D, X)$  with constant coefficients, a straightforward approach exists for analyzing the propagators of Eq. (23). This includes, for example, Mehler’s formula for the  $n$ -D QHO  $\mathcal{T}$  defined in (25). The properties of the solutions are explored through direct computations. Consequently, to efficiently describe the behavior of solutions to the time-dependent equation, it is practical to remove the time-dependence from the original equation via an appropriate transformation. This process is referred to as *reducibility* of the quantum Hamiltonian in this chapter.

A considerable amount of prior research has focused on the reducibility of time-dependent PDEs. For 1-D QHOs, time periodic smooth perturbations were initially examined [30, 40, 47, 62, 80]. In the context of time quasi-periodic perturbations, both bounded [59, 89, 117, 118] and unbounded disturbances [11, 12, 18, 86, 91, 94] have been thoroughly investigated. The study of reducibility issues has led to the comprehensive development of infinite-dimensional KAM theory for 1-D PDEs with unbounded perturbations, as seen in works by Bambusi-Graffi [13], Kuksin [81], and Liu-Yuan [93]. Regarding higher-dimensional PDEs, the concept of reducibility was pioneered by Eliasson-Kuksin [45, 46] for the quasi-periodic Schrödinger equation. For higher-dimensional QHOs, references [58, 88, 90] discuss bounded potentials, and [14] examines unbounded perturbations, including Hamiltonian PDEs quantized by quadratic polynomials. Through the lens of reducibility, these results offer insights into the long-term behavior of solutions (at least in terms of upper bounds of Sobolev norms) and the spectral characteristics of the Floquet operator.

As deduced from a quantization argument, the reducibility of the quadratic quantum Hamiltonian (23) is closely linked to the reducibility of the corresponding finite-dimensional system. To fully comprehend the dynamics of solutions to Eq. (23), it is crucial to understand the precise forms of the reduced constant equations, termed here as the *quantum normal forms* (QNFs). In essence, a comprehensive classification of quantum normal forms provides a complete depiction of the long-term behavior of solutions.

Let us give the precise definition of reducibility for finite-dimensional and infinite-dimensional systems. The affine system

$$z' = \mathcal{A}(t)z + \ell(t), \quad \mathcal{A}(\cdot) \in C_b^0(\mathbb{R}, \text{sp}(n, \mathbb{R})), \quad \ell(\cdot) \in C_b^0(\mathbb{R}, \mathbb{R}^{2n})$$

is called *reducible*, if there exist  $\mathcal{B} \in \text{sp}(n, \mathbb{R})$ ,  $l \in \mathbb{R}^{2n}$  and  $S(\cdot) \in C_b^1(\mathbb{R}, \text{Sp}(n, \mathbb{R}))$ ,  $p(\cdot) \in C_b^1(\mathbb{R}, \mathbb{R}^{2n})$ , such that it is conjugated to  $y' = \mathcal{B}y + l$  under the change of variable  $z = S(t)y + p(t)$ . The quadratic quantum Hamiltonian with time dependent coefficients

$$\frac{1}{2\pi i} \partial_t \psi = H(t, Z) \psi$$



**QNF2** For  $n$  even,  $\mathcal{R}(Z) = \mathcal{R}_2^{(n)}(Z) = \mathcal{Q}_{\mathcal{B}_2^{(n)}(\lambda_1, \lambda_2)}$ ,  $\lambda_1, \lambda_2 > 0$ ,

$$\mathcal{B}_2^{(n)}(\lambda_1, \lambda_2) := \left( \begin{array}{ccc|ccc} A_2 & & & & & \\ -I_2 & A_2 & & & & \\ & \ddots & \ddots & & & \\ & & -I_2 & A_2 & & \\ \hline & & & -A_2^* & I_2 & \\ & & & & \ddots & \ddots \\ & & & & & -A_2^* & I_2 \\ & & & & & & -A_2^* \end{array} \right), \quad A_2 := \begin{pmatrix} -\lambda_1 & -\lambda_2 \\ \lambda_2 & -\lambda_1 \end{pmatrix}.$$

**QNF3** For  $n \in \mathbb{N}^*$ , for  $\mathcal{R}(Z) = \mathcal{R}_3^{(n)}(Z) = \mathcal{Q}_{\mathcal{B}_3^{(n)}(\mu, \gamma)}$ ,  $\gamma = \pm 1$ ,  $\mu > 0$ , where

$$\mathcal{B}_3^{(n)}(\mu, \gamma) := \gamma \left( \begin{array}{ccc|ccc} & & & & & -1 & \mu \\ & & & & & \ddots & \ddots \\ & & & & & -1 & \mu \\ \hline & & & & \mu & & \\ & & & -\mu & 1 & & \\ & & \ddots & \ddots & & & \\ -\mu & 1 & & & & & \end{array} \right).$$

**QNF4** For  $n \in \mathbb{N}^*$ ,  $\mathcal{R}(Z) = \mathcal{R}_4^{(n)}(Z) = \mathcal{Q}_{\mathcal{B}_4^{(n)}(\gamma)} + \mathcal{L}_{l_4^{(n)}(\iota_1)}$ ,  $\gamma = \pm 1$ ,  $\iota_1 \in \mathbb{R}$ , with

$$\mathcal{B}_4^{(n)}(\gamma) := \gamma \left( \begin{array}{ccc|ccc} 0 & & & & & \\ -1 & 0 & & & & \\ & \ddots & \ddots & & & \\ & & -1 & 0 & & \\ \hline & & & 0 & 1 & \\ & & & & \ddots & \ddots \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{array} \right), \quad l_4^{(n)}(\iota_1) := \iota_1 \mathbf{e}_1 \in \mathbb{R}^{2n}.$$

**QNF5** For  $n$  odd,  $\mathcal{R}(Z) = \mathcal{R}_5^{(n)}(Z) = \mathcal{Q}_{\mathcal{B}_5^{(n)}} + \mathcal{L}_{l_5^{(n)}(\iota_1, \iota_2)}$ ,  $\iota_1, \iota_2 \in \mathbb{R}$ , with

$$\mathcal{B}_5^{(n)} := \left( \begin{array}{ccc|ccc} 0 & & & & & \\ -1 & 0 & & & & \\ & \ddots & \ddots & & & \\ & & -1 & 0 & & \\ \hline & & & 0 & 1 & \\ & & & & \ddots & \ddots \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{array} \right), \quad l_5^{(n)}(\iota_1, \iota_2) := \iota_1 \mathbf{e}_1 + \iota_2 \mathbf{e}_{2n} \in \mathbb{R}^{2n}.$$

**QNF6** (Decomposable case) For  $n \in \mathbb{N}^*$  with  $n \geq 2$ ,  $\mathcal{R}(Z) = \mathcal{R}_6^{(n)}(Z)$  is composed as a direct sum of operators from the previously mentioned five cases, each operating in a lower-dimensional space. More precisely, there is a mutually disjoint partition

$\bigcup_{j \in \Lambda} \mathcal{I}_j = \{1, \dots, n\}$  with  $\#\mathcal{I}_j = m_j < n$  and  $k_j \in \{1, \dots, 5\}$  such that

$$\mathcal{R}_6^{(n)}(Z) = \sum_{j \in \Lambda} \mathcal{R}_{k_j}^{(m_j)}(Z_{\mathcal{I}_j}), \quad Z_{\mathcal{I}_j} := \begin{pmatrix} (D_l)_{l \in \mathcal{I}_j} \\ (X_l)_{l \in \mathcal{I}_j} \end{pmatrix}.$$

Moreover, for Eq. (23) with non-vanishing initial condition  $\psi(0) \in \mathcal{H}^s(\mathbb{R}^n)$ ,  $s > 0$ , there exists a constant  $C > 1$ , and growth rates (GRs)  $g_k^{(n)}(s, t)$ ,  $1 \leq k \leq 6$ , such that, corresponding to QNF1 – QNF6,

$$C^{-1}g_k^{(n)}(s, t) \leq \|\psi(t)\|_{\mathcal{H}^s(\mathbb{R}^n)} \leq Cg_k^{(n)}(s, t), \quad |t| \rightarrow \infty.$$

More precisely,

**GR1** For  $\mathcal{R}(Z) = \mathcal{R}_1^{(n)}(Z)$ ,  $g_1^{(n)}(s, t) = |t|^{(n-1)s} e^{\lambda s |t|}$ .

**GR2** For  $n$  even,  $\mathcal{R}(Z) = \mathcal{R}_2^{(n)}(Z)$ ,  $g_2^{(n)}(s, t) = |t|^{(\frac{n}{2}-1)s} e^{\lambda_1 s |t|}$ .

**GR3** For  $\mathcal{R}(Z) = \mathcal{R}_3^{(n)}(Z)$ ,  $g_3^{(n)}(s, t) = |t|^{(n-1)s}$ .

**GR4** For  $\mathcal{R}(Z) = \mathcal{R}_4^{(n)}(Z)$ ,  $g_4^{(n)}(s, t) = |t|^{(2n-1)s} + |\iota_1|^s |t|^{2ns}$ .

**GR5** For  $n$  odd,  $\mathcal{R}(Z) = \mathcal{R}_5^{(n)}(Z)$ ,  $g_5^{(n)}(s, t) = |t|^{(n-1)s} + (|\iota_1| + |\iota_2|)^s |t|^{ns}$ .

**GR6** (Decomposable case) For  $n \geq 2$ ,  $\mathcal{R}(Z) = \mathcal{R}_6^{(n)}(Z)$ ,  $g_6^{(n)}(s, t) = \sum_{j \in \Lambda} g_{k_j}^{(m_j)}(s, t)$ .

The work outlined in this section diverges from a KAM approach. In the case of 1-D systems, we draw upon the KAM scheme developed by Eliasson [41] for quasi-periodic  $\text{sp}(1, \mathbb{R})$ -linear systems, which is fundamentally based on the classification of normal forms within  $\text{sp}(1, \mathbb{R})$ . However, when considering quasi-periodic systems in higher dimensions - that is,  $\text{sp}(n, \mathbb{R})$ -linear systems with  $n \geq 2$  - the anticipation of a KAM scheme remains, aimed at leveraging the concepts of reducibility and almost reducibility under specific arithmetic conditions. To achieve this objective, acquiring a thorough understanding of the possible classifications of normal forms is crucial. This knowledge not only aids in extending the KAM framework to accommodate higher-dimensional quasi-periodic systems but also in exploring the intricacies involved in their dynamic stability and structural characteristics. The exploration of normal forms in higher dimensions presents a significant step towards realizing a comprehensive KAM scheme that encompasses a broader spectrum of complex systems.

Based on the classifications presented in Theorem 3.3, stability of solutions to Eq. (23) within the Sobolev space, which is characterized by the boundedness of the solution's Sobolev norm, can be observed exclusively under specific conditions. These conditions are either for QNF3 with  $n = 1$ , or for QNF5 with  $n = 1$  and both  $\iota_1$  and  $\iota_2$  equal to zero. This corresponds to:

$$\mathcal{R}(Z) = \pm \frac{\mu}{2} (D_1^2 + X_1^2), \quad \mu > 0, \quad \text{or} \quad \mathcal{R}(Z) = 0,$$

or in the case of QNF6, which represents a direct sum of the aforementioned 1-D QNFs. Consequently, it appears that stability in the Sobolev space manifests primarily as a 1-D phenomenon.

**Theorem 3.4** Assume that the affine system  $z' = \mathcal{A}(t)z + \ell(t)$  is reducible with  $\mathcal{A}(\cdot) \in C_b^0(\mathbb{R}, \text{sp}(n, \mathbb{R}))$  and  $\ell(\cdot) \in C_b^0(\mathbb{R}, \mathbb{R}^{2n})$  as in Eq. (23). Given  $s > 0$ , if the solution to Eq. (23) satisfies  $\sup_{t \in \mathbb{R}} \|\psi(t)\|_{\mathcal{H}^s(\mathbb{R}^n)} < \infty$ , then Eq. (23) is reduced to the constant Hamiltonian

$$\mathcal{R}(Z) = \sum_{j=1}^n c_j (D_j^2 + X_j^2), \quad c_j \in \mathbb{R}.$$

**1.2. Metaplectic and Schrödinger representations.** The classical-quantum correspondence facilitated by Weyl quantization is well-established. More specifically, for any given symbol  $h = h(\xi, x)$ , where  $\xi, x \in \mathbb{R}^n$  and  $n \geq 1$ , the Weyl operator  $h^W$  associated with  $h$  is defined as follows:

$$(h^W u)(x) = \int_{\xi, y \in \mathbb{R}^n} e^{2\pi i \langle \xi, x-y \rangle} h\left(\xi, \frac{x+y}{2}\right) u(y) dy d\xi, \quad \forall u \in L^2(\mathbb{R}^n). \quad (29)$$

Notably, if  $h$  is a quadratic polynomial in  $(\xi, x)$ , then the corresponding Weyl operator  $h^W$  becomes a quadratic polynomial in  $(D, X)$  after symmetrization. It allows us to express the quantization in a concrete way via Metaplectic and Schrödinger representations.

Let us recall briefly the definitions of these representations. The readers can refer to [53] for detailed presentations and properties.

Given the real  $(2n+1)$ -D Heisenberg group  $\mathbf{H}_n$  endowed with the group law: for  $(p, q), (p', q') \in \mathbb{R}^n \times \mathbb{R}^n$ ,  $\tau, \tau' \in \mathbb{R}$ ,

$$(p, q, \tau)(p', q', \tau') = (p + p', q + q', \tau + \tau' + \frac{1}{2}(\langle p, q' \rangle - \langle q, p' \rangle)),$$

the Schrödinger representation  $\rho$  of  $\mathbf{H}_n$  is an irreducible unitary representation of  $\mathbf{H}_n$  on  $L^2(\mathbb{R}^n)$ , defined by

$$(\rho(p, q, \tau)u)(x) = e^{2\pi i(\tau + \langle q, x \rangle + \frac{1}{2}\langle p, q \rangle)} u(x + p), \quad u \in L^2(\mathbb{R}^n). \quad (30)$$

Given  $\mathbb{A} \in \text{Sp}(n, \mathbb{R})$ , let  $T_{\mathbb{A}}(p, q, \tau) := \left( \mathbb{A} \begin{pmatrix} p \\ q \end{pmatrix}, \tau \right)$ . By Stone-von Neumann theorem, there is a unitary operator  $\mathcal{M}(\mathbb{A})$  on  $L^2(\mathbb{R}^n)$  such that

$$\rho \circ T_{\mathbb{A}}(X) = \mathcal{M}(\mathbb{A}) \circ \rho(X) \circ \mathcal{M}(\mathbb{A})^{-1}, \quad X \in \mathbf{H}_n. \quad (31)$$

The Metaplectic representation  $\mathcal{M}$  of  $\text{Sp}(n, \mathbb{R})$  is a double-valued unitary representation of  $\text{Sp}(n, \mathbb{R})$  on  $L^2(\mathbb{R}^n)$  such that (31) is satisfied for any  $X \in \mathbf{H}_n$ .

For the Schrödinger representation  $\rho$  defined as in (30), since the variable  $\tau$  always acts in a straightforward manner, it is often convenient to omit it entirely. Consequently, we redefine  $\rho$  on  $\mathbb{R}^{2n}$  as

$$\left( \rho \begin{pmatrix} p \\ q \end{pmatrix} u \right)(x) = e^{2\pi i \langle q, x \rangle + \pi i \langle p, q \rangle} u(x + p), \quad \forall \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbb{R}^{2n}, \quad u \in L^2(\mathbb{R}^n). \quad (32)$$

For specific matrices  $\mathbb{A} \in \text{Sp}(n, \mathbb{R})$ , explicit formulas of the Metaplectic representation can be provided (Theorem 4.51 & 4.53 in [53]): for  $\mathbb{A} = \begin{pmatrix} A & B \\ C & F \end{pmatrix} \in \text{Sp}(n, \mathbb{R})$ ,  $u \in L^2(\mathbb{R}^n)$ ,

(i) if  $\det A \neq 0$ , then

$$(\mathcal{M}(\mathbb{A})u)(x) = (\det A)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{2\pi i(-\frac{1}{2}\langle x, CA^{-1}x \rangle + \langle \xi, A^{-1}x \rangle + \frac{1}{2}\langle \xi, A^{-1}B\xi \rangle)} \hat{u}(\xi) d\xi,$$

(ii) If  $\det B \neq 0$ , then

$$(\mathcal{M}(\mathbb{A})u)(x) = i^{\frac{n}{2}} (\det B)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{2\pi i(-\frac{1}{2}\langle x, FB^{-1}x \rangle + \langle y, B^{-1}x \rangle - \frac{1}{2}\langle y, B^{-1}Ay \rangle)} u(y) dy.$$

The above oscillatory integrals presented do not provide an explicit description for arbitrary  $A \in \text{Sp}(n, \mathbb{R})$ . This is due to the prevalence of  $2n \times 2n$  symplectic matrices whose  $n \times n$  blocks are all singular. This issue is further explored and settled in Section 3 of [87].

It is shown in Section 3 of [87] that, for  $s > 0$  and  $u \in \mathcal{H}^s \setminus \{0\}$ , for  $\mathbb{A} \in \text{Sp}(n, \mathbb{R})$ ,

$$\|\mathbb{A}\|^{-s} \|u\|_{\mathcal{H}^s} \lesssim \|\mathcal{M}(\mathbb{A})u\|_{\mathcal{H}^s} \lesssim \|\mathbb{A}\|^s \|u\|_{\mathcal{H}^s}, \quad \|\mathcal{M}(\mathbb{A})u\|_s \gtrsim_u \|\mathbb{A}\|^s, \quad (33)$$

and, for  $w \in \mathbb{R}^{2n}$ ,  $\|w\|^{-s}\|u\|_{\mathcal{H}^s} \lesssim \|\rho(w)u\|_{\mathcal{H}^s} \lesssim \|w\|^s\|u\|_{\mathcal{H}^s}$ . These estimates are crucial for the  $\mathcal{H}^s$ –estimates of the  $L^2$ –unitary transformations in the reducibility argument, demonstrating the  $\mathcal{H}^s$ –equivalence between the original time-dependent equation and the reduced equation. Conversely, the lower bound in the second inequality of (33) is applied to the time propagator of the homogeneous quadratic Hamiltonian PDE: for  $\mathcal{A} \in \text{sp}(n, \mathbb{R})$ ,

$$\|e^{t\mathcal{A}}\|^s \lesssim_u \|\mathcal{M}(e^{t\mathcal{A}})u\|_{\mathcal{H}^s} \lesssim \|e^{t\mathcal{A}}\|^s\|u\|_{\mathcal{H}^s}, \quad u \in \mathcal{H}^s \setminus \{0\},$$

which essentially contributes to the growth of the Sobolev norm for the solution to the reduced Hamiltonian PDE.

**1.3. Proof of Theorem 3.3.** The reducibility assumption of the affine system

$$z' = \mathcal{A}(t)z + \ell(t), \quad \mathcal{A}(\cdot) \in C_b^0(\mathbb{R}, \text{sp}(n, \mathbb{R})), \quad \ell(\cdot) \in C_b^0(\mathbb{R}, \mathbb{R}^{2n}), \quad (34)$$

means that there exist  $U(\cdot) \in C_b^1(\mathbb{R}, \text{Sp}(n, \mathbb{R}))$  and  $v(\cdot) \in C_b^1(\mathbb{R}, \mathbb{R}^{2n})$  such that, via the transformation  $z = U(t)(w + v(t))$ , the system (34) is conjugated to the affine system with constant coefficients  $w' = \mathcal{B}w + l$ . The reducibility of the quantum Hamiltonian (23) is achieved in two steps, which is summarized in the following diagram.

Affine system	Hamiltonian PDE	
$z' = \mathcal{A}(t)z + \ell(t)$	$\longrightarrow$	$\frac{1}{2\pi i} \partial_t \psi = (\mathcal{Q}_{\mathcal{A}(t)} + \mathcal{L}_{\ell(t)})\psi$
$z = U(t)y$	$\downarrow$	$\psi = \mathcal{M}(U(t))\phi$
$y' = \mathcal{B}y + \tilde{\ell}(t)$	$\longrightarrow$	$\frac{1}{2\pi i} \partial_t \phi = (\mathcal{Q}_{\mathcal{B}} + \mathcal{L}_{\tilde{\ell}(t)})\phi$
$y = w + v(t)$	$\downarrow$	$\phi = \rho(v(t))\varphi$
$w' = \mathcal{B}w + l$	$\longrightarrow$	$\frac{1}{2\pi i} \partial_t \varphi = (\mathcal{Q}_{\mathcal{B}} + \mathcal{L}_l)\varphi$

The growth rates of Sobolev norms are computed through the representations and Baker-Campbell-Hausdorff formulas.

## 2. Almost reducibility – Oscillatory growth of Sobolev norms

The GRs, which present steady growth, listed in Theorem 3.3 are shown through reducibility of quantum Hamiltonians. If the system is not reducible, almost reducibility is also useful to see the long-time behaviors of solutions of Hamiltonian PDEs. Eliasson [43] showed almost reducibility for the quasi-periodic linear wave equation, through which a log-log-bound on Sobolev norms of solutions is obtained. Similar idea was employed by Bambusi and his collaborators [15, 16, 17] for time dependent Schrödinger equations, with a more and more regularized remaining part along with the iteration step, instead of the usual asymptotic smallness assumption. A  $t^\epsilon$ –upper bound, for arbitrary  $\epsilon > 0$ , on Sobolev norms of solutions is obtained through such an argument.

Besides, another method often used to achieve unbounded trajectories involves constructing specific perturbations that induce infinite growth. For a 1-D QHO, Delort [35] (also see Maspero [97]) designed a time-periodic order-zero pseudo-differential operator as the perturbation, leading to  $t^{\frac{5}{2}}$  polynomial growth in  $\mathcal{H}^s$ –norms. In an abstract setting, Maspero [98, 99] further explored time-periodic perturbations and provided conditions under which certain solutions exhibit such polynomial growth. For a 2-D QHO, Faou-Raphaël [48] embedded Arnold diffusion into an infinite-dimensional quantum system, constructing a time-decaying potential that resulted in logarithmic growth of Sobolev norms over time. Similarly, Thomann [116], based on studies of linear Lowest Landau Level equations with a

time-dependent potential [110], designed a perturbation that projects onto Bargmann-Fock space, resulting in polynomial growth of Sobolev norms over time for some traveling wave solutions. Bourgain [23] and Haus-Maspero [63] also demonstrated logarithmic growth of Sobolev norms in various settings, including for linear Schrödinger equations with quasi-periodic potentials and semiclassical anharmonic oscillators with regular time-dependent potentials.

Let us consider the 1-D QHO with a time quasi-periodic quadratic perturbation,

$$\frac{1}{2\pi i} \partial_t \psi = (\mathcal{T}_1 + \mathcal{Q}_{P(\alpha t)}) \psi, \quad (36)$$

where  $\mathcal{T}_1$  is the 1-D QHO defined as in (25),  $P(\cdot) \in C_h^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  for some  $h > 0$ , and  $\alpha \in \mathbb{R}^d$  satisfying the Diophantine condition, i.e., there exist  $\gamma > 0$  and  $\tau > d - 1$  such that

$$\inf_{j \in \mathbb{Z}} |\langle k, \alpha \rangle - j| \geq \frac{\gamma}{|k|^\tau}, \quad \forall k \in \mathbb{Z}^d \setminus \{0\}. \quad (37)$$

Eq. (36) is regarded as Eq. (23) in 1-D homogeneous situation:  $\frac{1}{2\pi i} \partial_t \psi = \mathcal{Q}_{\mathcal{A}(t)} \psi$ , with  $\mathcal{A}(t) = \mathbb{J}_1 + P(\alpha t)$ . Our main result is stated as follows.

**Theorem 3.5** (Liang-Zhao-Zhou [92]) Consider Eq. (36) with  $P \in C^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  and  $\alpha \in \mathbb{R}^d$  satisfying the Diophantine condition (37). If  $\|P\|_h$  is sufficiently small (depending on  $h, d, \tau, \gamma$ ), then Eq. (36) is almost reducible, i.e., for every  $j \in \mathbb{N}^*$ , there exists an equation

$$\frac{1}{2\pi i} \partial_t \psi_j = \mathcal{Q}_{L_j + P_{j+1}(\alpha t)} \psi_j, \quad (38)$$

where  $L_j \in \mathfrak{sl}(2, \mathbb{R})$  and  $P_{j+1} \in C^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  with  $\|P_{j+1}\|_{\mathbb{T}^d} \rightarrow 0$  as  $j \rightarrow \infty$ , such that Eq. (36) is conjugated to Eq. (38) via a time quasi-periodic  $L^2$ -unitary transformation

$$\psi(t) = \mathcal{U}_j(t) \psi_j(t).$$

If  $\|P\|_h$  is sufficiently small as above and Eq. (36) is non-reducible, then for the solution  $\psi(t)$  to Eq. (36) with  $\psi(0) \in \mathcal{H}^{s+2}$ ,  $s > 0$ , we have

$$\lim_{t \rightarrow +\infty} \frac{\|\psi(t)\|_{\mathcal{H}^s}}{t^s} = 0. \quad (39)$$

Moreover, for any  $s > 0$  and non-vanishing  $\psi(0) \in \mathcal{H}^{s+2}$ , given  $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  with

$$f(t) \rightarrow \infty, \quad f(t) = o(t^s), \quad t \rightarrow \infty,$$

there exists  $P \in C^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  such that for the solution  $\psi$  to Eq. (36), we have

$$\limsup_{t \rightarrow +\infty} \frac{\|\psi(t)\|_{\mathcal{H}^s}}{f(t)} = \infty, \quad \liminf_{t \rightarrow +\infty} \|\psi(t)\|_{\mathcal{H}^s} < \infty. \quad (40)$$

**Remark 3.6** For  $f(t) = o(t^s)$  which may grow with a rate arbitrarily “close” to  $t^s$ , e.g.,

$$f(t) = \frac{t^s}{\log \log \log(e^e + t)},$$

according to (40), there exist two sequences of moments  $\{T_j\}$ ,  $\{t_j\}$ , both of which tending to  $\infty$ , and a constant  $c$ , depending on  $s$  and  $\psi(0)$ , such that

$$\|\psi(T_j)\|_{\mathcal{H}^s} \geq f(T_j) = \frac{T_j^s}{\log \log \log(e^e + T_j)}, \quad \|\psi(t_j)\|_{\mathcal{H}^s} \leq c, \quad \forall j \in \mathbb{N}, \quad (41)$$

which gives an oscillatory growth of Sobolev norms for the solution to Eq. (36), and shows the optimality of the  $o(t^s)$ -upper bound (39).

**2.1. Almost reducibility of quantum Hamiltonian.** With the classical-quantum correspondence given in (35), the reducibility and almost reducibility of the 1-D homogeneous quantum Hamiltonian (36) relies on those of the quasi-periodic linear system  $(\alpha, \mathbb{J}_1 + P(\cdot))$ :

$$y'(t) = (\mathbb{J}_1 + P(\alpha t))y(t), \quad \mathbb{J}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P(\cdot) \in C^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R})). \quad (42)$$

For the linear system  $(\alpha, \mathbb{J}_1 + P(\cdot))$  with  $P(\cdot)$  sufficiently small, it is shown by Eliasson [41] that  $(\alpha, \mathbb{J}_1 + P(\cdot))$  is reducible if the fibered rotation number  $\rho = \rho_{(\alpha, \mathbb{J}_1 + P)}$  is Diophantine or rational w.r.t.  $\alpha$ . For the case that  $\rho$  is Liouville w.r.t.  $\alpha$ , the reducibility is sometimes not expected, since, in the KAM scheme aiming for reducibility, it is possible to meet the resonances infinitely many times, and the corresponding renormalization, which is not close to identity, makes the convergence of sequence of transformation unrealizable. In this situation, we can only expect the almost reducibility. This KAM scheme, which is summarized in the following proposition, is essentially parallel with that introduced in Section 2.1.

**Proposition 3.7** ([41, 84]) If  $\|P\|_{\mathbb{T}^d}$  is sufficiently small, then the following holds.

- (1) The linear system  $(\alpha, \mathbb{J}_1 + P(\cdot))$  is almost reducible, i.e., there exist sequences  $\{L_j\} \subset \mathfrak{sl}(2, \mathbb{R})$  and  $\{P_j\} \subset C^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  with  $\|P_j\|_{\mathbb{T}^d} \rightarrow 0$ , such that the system (42) is conjugated to

$$y'_j(t) = (L_j + P_{j+1}(\alpha t))y_j(t)$$

via a time quasi-periodic change of variables  $y = U_j(t)y_j$ .

- (2) If  $(\alpha, J + P(\cdot))$  is reducible, then there exists  $N \in \mathbb{N}^*$  such that, for  $j \geq N$ ,

$$U_j(t) = U_N(t) \text{ with } \sup_{t \in \mathbb{R}} \|U_N(t)\| < \infty, \quad L_j = L_N, \quad P_{j+1} = 0. \quad (43)$$

Otherwise,  $\sup_{t \in \mathbb{R}} \|U_j(t)\| \rightarrow \infty$  as  $j \rightarrow \infty$ , and there exists  $\{\varepsilon_j\}_{j \in \mathbb{N}} \subset (0, 1)$  satisfying

(44) such that (45) is satisfied.

- (3) If  $(\alpha, J + P(\cdot))$  is non-reducible, then  $\sup_{t \in \mathbb{R}} \|U_j(t)\| \rightarrow \infty$  as  $j \rightarrow \infty$ , and there is a sequence  $\{\varepsilon_j\}_{j \in \mathbb{N}}$  with

$$0 < \varepsilon_{j+1} < \varepsilon_j \exp \left\{ -\frac{h}{2} \varepsilon_j^{-\frac{1}{18\tau}} \right\}, \quad (44)$$

such that

$$\sup_{t \in \mathbb{R}} \|U_j(t)\| \leq |\ln \varepsilon_j|^{2\tau}, \quad \|L_j\| < \varepsilon_j^{\frac{1}{16}}, \quad \det(L_j) > \frac{\gamma^2}{|\ln \varepsilon_{j+1}|^{2\tau}}, \quad \|P_{j+1}\|_{\mathbb{T}^d} < \varepsilon_{j+1}. \quad (45)$$

The conjugation between the homogeneous quantum Hamiltonians is translated through the Metaplectic representation from that between classical Hamiltonian systems:

$$\begin{aligned} (\alpha, \mathbb{J}_1 + P(\cdot)) &\longrightarrow \frac{1}{2\pi i} \partial_t \psi = \mathcal{Q}_{\mathbb{J}_1 + P(\alpha t)} \psi \\ \downarrow U_j(t) & \qquad \qquad \qquad \downarrow \mathcal{U}_j(t) = \mathcal{M}(U_j(t)) \quad . \\ (\alpha, L_j + P_{j+1}(\cdot)) &\longrightarrow \frac{1}{2\pi i} \partial_t \psi_j = \mathcal{Q}_{L_j + P_{j+1}(\alpha t)} \psi_j \end{aligned} \quad (46)$$

The  $o(t^s)$  upper-bound (39) is computed through Metaplectic representation (see Proposition 4.1 of [92]).

## 2.2. Anosov-Katok construction for non-reducible quantum Hamiltonian.

Recall  $s > 0$  and  $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$  with  $f(t) = o(t^s)$  given in Theorem 3.5. Let us define

$$g(t) := 1 - \frac{\ln(f(t))}{s \ln(t)}.$$

Choose a sequence  $\{k_j\} \subset \mathbb{Z}^d$  with  $\langle k_j, \alpha \rangle > 0$  such that  $|k_j|$  and  $\langle k_j, \alpha \rangle$  decays sufficiently fast w.r.t.  $j$ . More precisely, let  $k_{j+1} \in \mathbb{Z}^d$  satisfy

$$|k_{j+1}| > e^{|k_j|} + 10, \quad (47)$$

and, with  $h > 0$  given in Eq. (36), with  $T_j \approx \frac{5\pi}{2\langle k_{j+1}, \alpha \rangle}$ ,

$$\langle k_{j+1}, \alpha \rangle^{\frac{g(T_j)}{2}} < \langle k_j, \alpha \rangle^{\frac{g(T_{j-1})}{2}} e^{-33h|k_j|} \exp \left\{ -\langle k_j, \alpha \rangle^{-\left(1 + \frac{1}{367}\right)} \right\}. \quad (48)$$

We also define sequences  $\{\varphi_j\}_{j \in \mathbb{N}^*}, \{\lambda_j\}_{j \in \mathbb{N}^*} \subset (0, 1)$  and  $\{z_j\}_{j \in \mathbb{N}^*} \subset (1, \infty)$  by

$$\varphi_j = \langle k_{j+1}, \alpha \rangle^{\frac{3}{4}g(T_j)}, \quad \lambda_j := \sqrt{\varphi_j^2 - \langle k_{j+1}, \alpha \rangle^2}, \quad z_j = \sqrt{\frac{\varphi_j + \lambda_j}{\langle k_{j+1}, \alpha \rangle}}. \quad (49)$$

We have the fibred Anosov-Katok construction for the quasi-periodic linear system  $(\alpha, \mathbb{J}_1 + P(\cdot))$ , which leads to a concrete non-reducibility argument.

**Proposition 3.8** [92] For any  $h > 0, \varepsilon > 0$ , there exists  $P \in C_h^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  satisfying  $\|P\|_h < \varepsilon$ , such that, for every  $j \in \mathbb{N}^*$ , the quasi-periodic linear system  $(\alpha, \mathbb{J}_1 + P(\cdot))$  is conjugated to the linear system  $\dot{y}_j(t) = (L_j + P_{j+1}(\alpha t))y_j(t)$ , via the time quasi-periodic transformation  $y(t) = U_j(t)y_j(t)$  with

$$U_1(t) = R_{\langle k_0 + k_1, \alpha \rangle t}, \quad U_j(t) = R_{\langle k_0 + k_1, \alpha \rangle t} \prod_{n=1}^{j-1} \left( \begin{pmatrix} z_n & 0 \\ 0 & z_n^{-1} \end{pmatrix} R_{\langle k_{n+1}, \alpha \rangle t} \right), \quad j \geq 2, \quad (50)$$

and  $L_j \in \mathfrak{sl}(2, \mathbb{R}), P_{j+1} \in C^\omega(\mathbb{T}^d, \mathfrak{sl}(2, \mathbb{R}))$  satisfying

$$L_j = \begin{pmatrix} 0 & \varphi_j + \lambda_j \\ -(\varphi_j - \lambda_j) & 0 \end{pmatrix}, \quad \|P_{j+1}\|_{\mathbb{T}^d} < \langle k_{j+2}, \alpha \rangle^{\frac{g(T_{j+1})}{2}}. \quad (51)$$

In view of (50), we see that  $\sup_{t \in \mathbb{R}} \|U_j(t)\| \rightarrow \infty$  as  $j \rightarrow \infty$ , which means the non-reducibility. According to the classical-quantum correspondance (46), it gives a non-reducible time quasi-periodic quantum Hamiltonian. It is shown in Proposition 4.2 and 4.3 of [92] that, for any  $\psi(0) \in \mathcal{H}^{s+2}$ ,

$$\frac{\|\psi(T_j)\|_{\mathcal{H}^s}}{f(T_j)} \rightarrow \infty, \quad \|\psi(4T_j)\|_{\mathcal{H}^s} \lesssim 1,$$

which implies the oscillatory behavior (40).

## 3. Further discussions

It is worthy further exploiting the almost reducibility in quantum Hamiltonians and its related long-time dynamics. The almost reducibility for the higher-dimensional quadratic quantum Hamiltonians is still unknown, since the corresponding higher-dimensional classical system ( $\mathrm{Sp}(n, \mathbb{R})$ -cocycle or  $\mathfrak{sp}(n, \mathbb{R})$ -linear system) has not yet been well investigated.

The asymptotic estimate presented in (28) is anticipated to be valid in more expansive contexts, extending beyond the limits of the reducibility assumption and the quadratic polynomial form of the Hamiltonian. Let us consider two illustrative examples in the 1-D framework.

**Example 1.** Corresponding to the oscillatory growth with an optimal  $o(t^s)$ -upper bound in Theorem 3.5, it has already been shown by Eliasson [41] that, for the non-reducible quasi-periodic  $\mathfrak{sl}(2, \mathbb{R})$ -linear system, there exist oscillatory unbounded solutions with an  $o(t)$ -upper bound (refer to Theorem A2 and B2 in [41]).

**Example 2.** In the case of a 1-D QHO perturbed by a specific time-periodic order-0 pseudo-differential operator:

$$\frac{1}{2\pi i} \partial_t \psi = \left( \frac{1}{4\pi} \mathcal{T} + h^W(t, D, X) \right) \psi, \quad h(t, \xi, x) = \epsilon \cos(2t) \eta(\xi, x) \frac{x\xi}{x^2 + \xi^2},$$

where  $\eta \in C^\infty(\mathbb{R}^2, \mathbb{R}_+)$  is defined such that  $\eta(\xi, x)$  equals 0 when  $x^2 + \xi^2 \leq \frac{1}{4}$  and 1 when  $x^2 + \xi^2 \geq \frac{1}{2}$ , and  $h^W$  is the Weyl quantization of the classical Hamiltonian  $h$  on  $\mathbb{R}^2$  (refer to (29)). Maspero [99] has demonstrated that if  $\epsilon > 0$  is sufficiently small, then  $h^W$  acts as a transporter, implying the existence of a solution  $\psi(t) \in \mathcal{H}^s$  for  $s > 0$ , which satisfies  $\|\psi(t)\|_s \sim t^{\frac{s}{2}}$  as  $t \rightarrow \infty$ , within the context of this paper. Conversely, the corresponding classical system

$$x' = \xi + \partial_\xi h(t, \xi, x), \quad \xi' = -x - \partial_x h(t, \xi, x), \quad x, \xi \in \mathbb{R}, \quad (52)$$

permits a solution for  $t \geq \epsilon^{-1}$  of the form

$$x(t) = \sqrt{\epsilon} \cos(t) \sqrt{t + \frac{\sin(4t)}{4}}, \quad \xi(t) = -\sqrt{\epsilon} \sin(t) \sqrt{t + \frac{\sin(4t)}{4}}.$$

Therefore, the quantity  $\sqrt{x^2(t) + \xi^2(t)}$  scales approximately as  $(\epsilon t)^{\frac{1}{2}}$  when  $t \rightarrow \infty$ , representing the optimal growth rate for solutions to the system (52).



## Geometry of hyperbolic Cauchy-Riemann singularities

In the 1960s, Bishop [21] revealed intriguing analytical characteristics of certain real submanifolds within complex spaces, specifically those exhibiting singularities in their Cauchy-Riemann structures. Such singularities can lead to the existence of substantial open sets, which allow for the extension of any analytic function defined on the submanifold, independent of the function itself. Bishop's focus was particularly on the higher-order perturbations of the quadric surface  $\mathcal{Q}_\gamma \subset \mathbb{C}^2$ , defined by

$$\mathcal{Q}_\gamma : z_2 = Q_\gamma(z_1, \bar{z}_1) = |z_1|^2 + \gamma(z_1^2 + \bar{z}_1^2), \quad 0 \leq \gamma \leq \infty. \quad (53)$$

In the 1980s, Moser-Webster [105] furthered this research by demonstrating that examining this geometric structure is tantamount to studying an associated dynamical system. Specifically, they investigated the higher-order perturbations of a pair of linear involutions for  $0 < \gamma < \infty$ :

$$\tau_1 : \begin{cases} \xi' = \lambda\eta \\ \eta' = \lambda^{-1}\xi \end{cases}, \quad \tau_2 : \begin{cases} \xi' = \lambda^{-1}\eta \\ \eta' = \lambda\xi \end{cases},$$

where  $\lambda$  is determined as the root of the quadratic equation  $\gamma\lambda^2 - \lambda + \gamma = 0$ . Gong-Stolovitch [56, 57] later extended these findings significantly by exploring dynamics in higher dimensions, demonstrating the versatility and depth of these analytical properties in complex geometrical structures.

The quadric surface  $\mathcal{Q}_\gamma$ , defined earlier, exhibits a unique CR (Cauchy-Riemann) singularity at the origin when  $\gamma \neq \frac{1}{2}$ . This is because the surface is entirely real (with  $d = 0$ ) at all points except at the origin, where the tangent space becomes a complex line  $z_2 = 0$  (hence  $d = 1$ ). This particular condition indicates that the surface transitions from being purely real to having a complex tangent space at a singular point, highlighting the singularity's distinctive nature.

For cases where  $0 < \gamma < \frac{1}{2}$ , the singularity is classified as *elliptic*. In this context, the term “elliptic” refers to the nature of the singularity, which influences the geometric and analytical properties of the quadric surface around that point. Moser-Webster's groundbreaking work [105] on the analysis of such surfaces involved exploring higher-order analytic perturbations of elliptic quadrics  $\mathcal{Q}_\gamma$ . They successfully demonstrated that these submanifolds are holomorphically equivalent to a *normal form*, a simplified model that still retains the essential geometric and analytical characteristics of the original structure. The normal form is given by

$$z_2 = |z_1|^2 + (\gamma + \epsilon \operatorname{Re}(z_2)^s)(z_1^2 + \bar{z}_1^2) \quad \text{for some } \epsilon \in \{-1, 0, 1\} \text{ and } s \in \mathbb{N}^* \cup \{\infty\}. \quad (54)$$

This normal form is significant because it provides a clearer understanding of the geometric features intrinsic to the quadric surface.

In the context of the *hyperbolic* case, where the higher-order analytic perturbation of  $\mathcal{Q}_\gamma$  is considered for  $\gamma > \frac{1}{2}$ , the dynamics and geometry present more complex challenges compared to the elliptic case. Unlike the elliptic case, where Moser-Webster [105] established conditions under which the perturbed quadrics are holomorphically equivalent to a standard normal form, the hyperbolic scenario reveals that such a straightforward equivalence may not always exist for some analytic perturbations of  $\mathcal{Q}_\gamma$ . Gong's contribution

[55] to understanding the hyperbolic case came with the clarification that if a higher-order analytic perturbation of  $\mathcal{Q}_\gamma$  is *formally* equivalent to  $\mathcal{Q}_\gamma$  itself - that is, equivalent under a transformation expressed by formal power series - and if this perturbation meets a specific *Diophantine condition* related to  $\gamma$ , then the perturbation is indeed holomorphically equivalent to the quadric surface. Klingenberg's work [77] further explores the implications of the Diophantine condition within the realm of hyperbolic quadrics. He demonstrated that for any higher-order analytic perturbation  $M$  of the hyperbolic quadric  $\mathcal{Q}_\gamma$ , assuming the Diophantine condition related to  $\gamma$  is satisfied, there will always exist a holomorphic curve that intersects  $M$  along two distinct transversal totally real curves.

The objective of this study is to demonstrate that *non-degenerate analytic perturbations* of hyperbolic quadrics - which are perturbations not formally equivalent to the original quadrics - contain a significant number of analytic hyperbolas. In simpler terms, we aim to show that there is a substantial compact subset  $\mathcal{K} \subset \mathbb{R}$  with a positive measure. Within this subset, for every  $\omega \in \mathcal{K}$ , there exists a holomorphic curve  $\mathcal{S}_\omega$  that intersects the non-degenerate analytic perturbation  $M$  along two real curves. These curves are then holomorphically mapped to the two branches of the real hyperbola  $\xi\eta = \omega$ , near the origin.

To achieve this, we develop a KAM theory tailored for a pair of holomorphic involutions near a fixed point (designated as 0) in  $\mathbb{C}^2$ . These involutions are interchanged through conjugation with an anti-holomorphic involution.

**Key words:** Hyperbolic Cauchy-Riemann singularity, Holomorphic involution, Formal normal form, Non-degenerate perturbation

## 1. Geometry of hyperbolic Cauchy-Riemann singularities

Consider  $M$  as a higher-order analytic perturbation of  $\mathcal{Q}_\gamma \subset (\mathbb{C}^2, 0)$ , where  $\gamma > \frac{1}{2}$ , defined by

$$M : z_2 = |z_1|^2 + \gamma(z_1^2 + \bar{z}_1^2) + O^3(z_1, \bar{z}_1). \quad (55)$$

This manifold is linked to a local dynamical system within  $(\mathbb{C}^2, 0)$ , denoted as  $\{\tau_1^o, \tau_2^o, \rho\}$ , where  $\tau_1^o$  and  $\tau_2^o$  are local holomorphic involutions that fix the point 0 and  $\rho$  is an anti-holomorphic involution. These satisfy the conditions:

$$\tau_j^o \circ \tau_j^o = \text{Id}, \quad \tau_2^o = \rho \circ \tau_1^o \circ \rho.$$

Such a set of involutions  $\{\tau_1^o, \tau_2^o, \rho\}$  fully defines the holomorphic equivalence class of the submanifold  $M$ , as illustrated in Proposition 1.1 of [105] or Proposition 2.8 of [56]. Additionally, the biholomorphism germ  $\sigma_o := \tau_1^o \circ \tau_2^o$  proves to be significant. In optimal local holomorphic coordinates  $(\xi, \eta)$ , we have  $\rho(\xi, \eta) = (\bar{\xi}, \bar{\eta})$ ,

$$\tau_1^o(\xi, \eta) = \begin{pmatrix} e^{\frac{1}{2}\lambda}\eta + p^o(\xi, \eta) \\ e^{-\frac{1}{2}\lambda}\xi + q^o(\xi, \eta) \end{pmatrix}, \quad (56)$$

$$\tau_2^o(\xi, \eta) = (\rho \circ \tau_1^o \circ \rho)(\xi, \eta) = \begin{pmatrix} e^{-\frac{1}{2}\lambda}\eta + \bar{p}^o(\xi, \eta) \\ e^{\frac{1}{2}\lambda}\xi + \bar{q}^o(\xi, \eta) \end{pmatrix}, \quad (57)$$

with  $e^{\pm\frac{1}{2}\lambda}$  being roots of the quadratic equation  $\gamma X^2 - X + \gamma = 0$ . Moreover,  $\sigma_o(\xi, \eta)$  is represented as:

$$\sigma_o(\xi, \eta) = \begin{pmatrix} e^{i\lambda}\xi + f^o(\xi, \eta) \\ e^{-i\lambda}\eta + g^o(\xi, \eta) \end{pmatrix}.$$

Here,  $\bar{h}$  denotes  $\bar{h}(\xi, \eta) := \sum_{k,l \geq 0} \bar{h}_{k,l} \xi^k \eta^l$  if  $h(\xi, \eta) := \sum_{k,l \geq 0} h_{k,l} \xi^k \eta^l$ , with  $p^o, q^o, f^o, g^o$  being germs of holomorphic functions of order  $\geq 2$  at the origin (meaning both the function and its first derivative vanish at 0).

We assume that the submanifold  $M$  (or their associated involutions  $\tau_1, \tau_2$ ) is *non-exceptional*, meaning that  $e^{\frac{i}{2}\lambda}$  is not a root of unity. In this case, Moser-Webster showed (see Lemma 3.2 and Theorem 3.4 of [105]) that there exists a *formal transformation*  $\hat{\Psi}$  satisfying  $\hat{\Psi} \circ \rho = \rho \circ \hat{\Psi}$  such that

$$\hat{\tau}_1 := (\hat{\Psi}^{-1} \circ \tau_1^o \circ \hat{\Psi})(\xi, \eta) = \begin{pmatrix} \Lambda(\xi\eta)\eta \\ \Lambda^{-1}(\xi\eta)\xi \end{pmatrix}, \quad (58)$$

$$\hat{\tau}_2 := (\hat{\Psi}^{-1} \circ \tau_2^o \circ \hat{\Psi})(\xi, \eta) = \begin{pmatrix} \Lambda(\xi\eta)^{-1}\eta \\ \Lambda(\xi\eta)\xi \end{pmatrix}, \quad (59)$$

$$\hat{\sigma} := (\hat{\Psi}^{-1} \circ \sigma_o \circ \hat{\Psi})(\xi, \eta) = \begin{pmatrix} \hat{M}(\xi\eta)\xi \\ \hat{M}(\xi\eta)^{-1}\eta \end{pmatrix}. \quad (60)$$

Here,  $\Lambda(z)$  and  $\hat{M}(z)$  are *formal power series* of  $z$  and satisfy:

$$\Lambda(z)\bar{\Lambda}(z) = 1, \quad \hat{M}(z) = \Lambda(z)^2, \quad \Lambda(0) = e^{\frac{i}{2}\lambda}, \quad \hat{M}(0) = e^{i\lambda}.$$

If  $\Lambda(z) = \Lambda(0)$ , then  $\{\tau_1^o, \tau_2^o\}$  is formally linearizable by a formal transformation that commutes with  $\rho$ . Hence,  $M$  is formally equivalent to the quadric  $\mathcal{Q}_\gamma$ . Gong's theorem [55] asserts that, if  $\lambda$  satisfies a *Diophantine condition*, i.e., there are  $r, c > 0$ , such that

$$|e^{ik\lambda} - 1| \geq \frac{c}{k^r}, \quad k \in \mathbb{N}^*,$$

then the submanifold is actually holomorphically equivalent to the quadric  $\mathcal{Q}_\gamma$  near the origin. We consider the situation where  $M$  is not formally equivalent to the quadric  $\mathcal{Q}_\gamma$ . Our main result is stated as follows.

**Theorem 4.1** (Stolovitch-Zhao [114]) Assume that the non-exceptional real-analytic surface  $M$  given in (55) is not formally equivalent to  $\mathcal{Q}_\gamma$ . There exists a non-constant Whitney smooth family of holomorphic hyperbolas on  $M$  in a neighborhood of the origin. More precisely, there exists a compact set  $\mathcal{K} \subset \mathbb{R}$  of positive measure such that for all  $\omega \in \mathcal{K}$ , there exists a holomorphic curve  $\mathcal{S}_\omega$ , depending smoothly on  $\omega$  in the sense of Whitney, that intersects the submanifold  $M$  along two real curves which are simultaneously holomorphically mapped to the two branches of the hyperbolas  $\xi\eta = \omega$ .

**Remark 4.2** This contrasts with the elliptic case treated by Moser-Webster [105]. Indeed, as in (54), in the holomorphic normalizing coordinates, there is a real-analytic family of holomorphic curves  $\mathcal{S}_c : z_2 = c$  for  $c$  in a real neighborhood of the origin, and for every  $c$ ,  $\mathcal{S}_c$  intersects  $M$  along the ellipse  $c = |z_1|^2 + (\gamma + \epsilon c^s)(z_1^2 + \bar{z}_1^2)$ .

## 2. KAM scheme for involutions

As mentioned above, the triple  $\{\tau_1^o, \tau_2^o, \rho\}$  given in (56) and (57) completely characterizes the holomorphic equivalent class of the submanifold  $M$  given in (55). Hence, it is essential to conjugate the pair of involutions  $\{\tau_1^o, \tau_2^o\}$  to a normal form in some suitable sense via a transformation commuting with  $\rho$ .

We assume that  $\sigma_o, \tau_1^o, \tau_2^o$  are defined in  $|\xi|, |\eta| < r$  for some  $0 < r < \frac{1}{4}$ , and

- (1)  $\lambda \in [0, 4\pi[$  with  $\frac{\lambda}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$ ,
- (2)  $p^o$  and  $q^o$  are convergent power series on  $\{|\xi|, |\eta| < r\}$  of order  $\geq 2$ , i.e.,

$$p^o(\xi, \eta) = \sum_{\substack{l+j \geq 2 \\ l, j \geq 0}} \check{p}_{l,j}^o \xi^l \eta^j, \quad q^o(\xi, \eta) = \sum_{\substack{l+j \geq 2 \\ l, j \geq 0}} \check{q}_{l,j}^o \xi^l \eta^j,$$

with coefficients  $\check{p}_{l,j}^o, \check{q}_{l,j}^o \in \mathbb{C}$ .

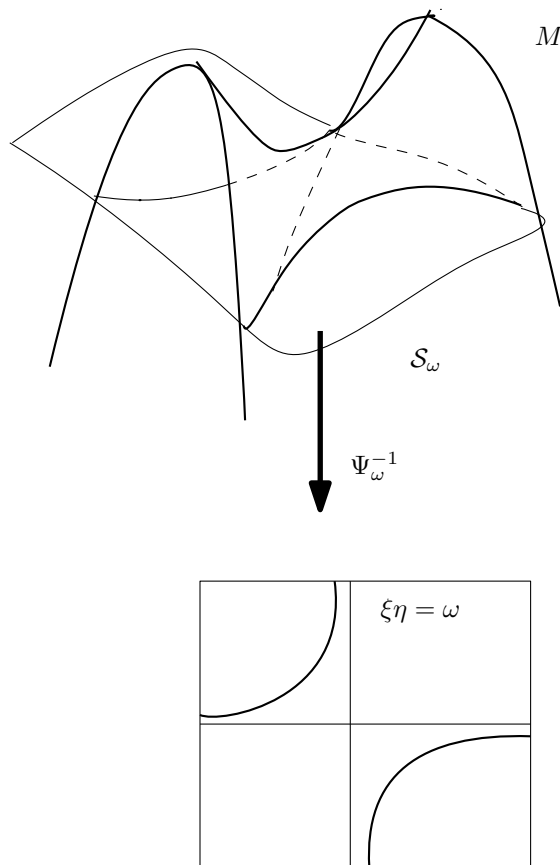


FIGURE 1. Holomorphic hyperbola : Intersection of  $M$  by a holomorphic curve.

It is easy to verify that  $\sigma_o$  is *reversible* w.r.t. the involution  $\rho$ , i.e.,  $\sigma_o^{-1} = \rho \circ \sigma_o \circ \rho$ .

Since  $\frac{\lambda}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$ ,  $e^{\frac{i}{2}\lambda}$  is not a root of unity. Then (58) and (59) hold. Let  $\hat{\Psi}$  be the unique normalized formal transformation together with the formal power series  $\Lambda = \Lambda(z)$ . We assume that  $\Lambda(z)$  is not constant. Let  $s \in \mathbb{N}^*$  be the smallest positive integer such that  $\Lambda^{(s)}(0) \neq 0$ . More precisely, we assume

$$\Lambda(z) = e^{\frac{i}{2}\lambda} + \sum_{j \geq s} \tilde{C}_j z^s, \quad \tilde{C}_s \neq 0. \quad (61)$$

For  $\omega \in \mathbb{R}$ ,  $r > 0$ , let  $\mathcal{C}_\omega^r := \{(\xi, \eta) \in \mathbb{C}^2 : \xi\eta = \omega, |\xi|, |\eta| < r\}$ . The following theorem shows that there is a family of invariant closed curves for the involutions  $\tau_j^o$  and the reversible map  $\sigma_o$  in any neighborhood of the origin, which enables us to obtain the result on the geometry of real-analytic surfaces with a hyperbolic CR singularity.

**Theorem 4.3** (Stolovitch-Zhao [114]) There exists a sufficiently small  $R = R(\lambda, r, s) > 0$  such that there is a compact set  $\mathcal{O}_\infty \subset ]-R^2, R^2[$  satisfying

$$\frac{\text{Leb}(\mathcal{O}_\infty)}{2R^2} \rightarrow 1, \quad R \rightarrow 0, \quad (62)$$

such that for any  $\omega \in \mathcal{O}_\infty$ , one can find  $\mu_\omega \in \mathbb{R}$  and a bounded holomorphic transformation  $\Psi_\omega : \mathcal{C}_\omega^R \rightarrow \mathbb{C}^2$  with  $\Psi_\omega \circ \rho = \rho \circ \Psi_\omega$ , such that, on  $\mathcal{C}_\omega^R$  :

$$\begin{aligned} (\Psi_\omega^{-1} \circ \tau_1^o \circ \Psi_\omega)(\xi, \eta) &= \begin{pmatrix} e^{\frac{i}{2}\mu_\omega \eta} \\ e^{-\frac{i}{2}\mu_\omega \xi} \end{pmatrix}, & (\Psi_\omega^{-1} \circ \tau_2^o \circ \Psi_\omega)(\xi, \eta) &= \begin{pmatrix} e^{-\frac{i}{2}\mu_\omega \eta} \\ e^{\frac{i}{2}\mu_\omega \xi} \end{pmatrix}, \\ (\Psi_\omega^{-1} \circ \sigma_o \circ \Psi_\omega)(\xi, \eta) &= \begin{pmatrix} e^{i\mu_\omega \xi} \\ e^{-i\mu_\omega \eta} \end{pmatrix}, & (\xi, \eta) &\in \mathcal{C}_\omega^R. \end{aligned}$$

In other words,  $\tau_1^o, \tau_2^o$  and  $\sigma_o$  have  $\Psi_\omega(\mathcal{C}_\omega^R)$  as holomorphic invariant set and their restrictions to it are conjugate to the restrictions to  $\mathcal{C}_\omega^R$  of linear maps defined above.

Suppose that we arrive at a certain KAM step, with a pair of holomorphic involutions

$$\tau_1 : \begin{cases} \xi' = e^{\frac{i}{2}\alpha(\xi\eta)}\eta + p(\xi, \eta) \\ \eta' = e^{-\frac{i}{2}\alpha(\xi\eta)}\xi + q(\xi, \eta) \end{cases}, \quad \tau_2 = \rho \circ \tau_1 \circ \rho, \quad \rho : (\xi, \eta) \mapsto (\bar{\xi}, \bar{\eta})$$

and the reversible map

$$\sigma = \tau_1 \circ \tau_2 : \begin{cases} \xi' = e^{i\alpha(\xi\eta)}\eta + f(\xi, \eta) \\ \eta' = e^{-i\alpha(\xi\eta)}\xi + g(\xi, \eta) \end{cases},$$

defined on the ‘‘crown’’  $\mathcal{C}_{\omega, \beta}^r := \{|\xi\eta - \omega| < \beta, |\xi|, |\eta| < r\}$  with some sufficiently small  $\beta > 0$  and  $\omega$  belonging to some parameter set  $\in \mathcal{O} \subset ]-r^2, r^2[$ . Assume that the following conditions are satisfied.

- (Non-degeneracy): For every  $\omega \in \mathcal{O}$ ,  $\alpha(\xi\eta)$  is real-analytic on  $\mathcal{C}_{\omega, \beta}^r$ , and there exists  $s \in \mathbb{N}^*$  such that  $|\alpha^{(s)}(\omega)| > \frac{1}{2}$ .
- (Smallness): For the unique decomposition

$$p(\xi, \eta) = p^{0,0}(\xi\eta) + \sum_{l \geq 1} p^{l,0}(\xi\eta)\xi^l + \sum_{j \geq 1} p^{0,j}(\xi\eta)\eta^j,$$

we have that

$$\|p\|_{\omega, \beta, r} := \sum_{l, j=0} \sup_{|\xi\eta - \omega| < \beta} |p^{l,j}(\xi\eta)| r^{l+j} < \varepsilon \sim \beta^{40s}, \quad \|q\|_{\omega, \beta, r} < \varepsilon$$

- (Skew-term):  $\|e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p\|_{\omega, \beta, r} < \varepsilon^{\frac{3}{2}}$ .

For some  $r_+ \in ]0, r[$ ,  $\beta_+ = \beta^{\frac{5}{4}}$ ,  $\varepsilon_+ = \varepsilon^{\frac{5}{4}}$ , and

$$\omega \in \mathcal{O}_+ := \left\{ \omega \in \mathcal{O} \cap ]-r_+^2, r_+^2[ : |e^{ik\alpha(\omega)} - 1| > \varepsilon^{\frac{1}{64s}}, \quad 1 \leq |k| \leq K \sim |\ln \varepsilon| \right\},$$

through the approximated solutions to the cohomological equations

$$\begin{aligned} e^{i\alpha(\xi\eta)}u(\xi, \eta) - u(e^{i\alpha(\xi\eta)}\xi, e^{-i\alpha(\xi\eta)}\xi) &= f^{0,0}(\xi\eta) + \sum_{2 \leq l \leq K} f^{l,0}(\xi\eta)\xi^l + \sum_{1 \leq j \leq K} f^{0,j}(\xi\eta)\eta^j, \\ e^{-i\alpha(\xi\eta)}v(\xi, \eta) - v(e^{i\alpha(\xi\eta)}\xi, e^{-i\alpha(\xi\eta)}\xi) &= f^{0,0}(\xi\eta) + \sum_{1 \leq l \leq K} g^{l,0}(\xi\eta)\xi^l + \sum_{2 \leq j \leq K} g^{0,j}(\xi\eta)\eta^j, \end{aligned}$$

we are able to construct a transformation

$$\psi(\xi, \eta) = (\text{Id} + \mathcal{U})(\xi, \eta) = \begin{pmatrix} \xi + u(\xi, \eta) \\ \eta + v(\xi, \eta) \end{pmatrix}, \quad (63)$$

with  $\|u\|_{\omega, \beta_+, r_+}, \|v\|_{\omega, \beta_+, r_+} < \varepsilon^{\frac{49}{50}}$  and, because of the skew-term condition,

$$\|\eta u + \xi v\|_{\omega, \beta_+, r_+} < \varepsilon^{\frac{61}{32}} + \varepsilon^{-\frac{1}{16}} \|e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p\|_{\omega, \beta, r} < \varepsilon^{\frac{11}{8}},$$

such that  $\tau_1^+ := \psi^{-1} \circ \tau_1 \circ \psi : \begin{cases} \xi' = e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta + p_+(\xi, \eta) \\ \eta' = e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi + q_+(\xi, \eta) \end{cases}$  on  $\mathcal{C}_{\omega, \beta_+}^{r_+}$  with  $\alpha_+$  similar to  $\alpha$  and

$$\|p_+\|_{\omega, \beta_+, r_+}, \|q_+\|_{\omega, \beta_+, r_+} \lesssim \varepsilon^{\frac{61}{32}} + \|\eta u + \xi v\|_{\omega, \beta_+, r_+} < \varepsilon_+.$$

To shown the skew-term for the new pair of involutions  $\{\tau_1^+, \tau_2^+ = \rho \circ \tau_1^+ \circ \rho\}$ , we need to control the part

$$\begin{pmatrix} (e^{\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v)} - e^{\frac{i}{2}\alpha(\xi\eta)})\eta \\ (e^{-\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v)} - e^{-\frac{i}{2}\alpha(\xi\eta)})\xi \end{pmatrix} \quad (64)$$

of  $\begin{pmatrix} p_+ \\ q_+ \end{pmatrix}$ , since, through the transformation (63),  $\begin{pmatrix} e^{\frac{i}{2}\alpha(\xi\eta)}\eta \\ e^{-\frac{i}{2}\alpha(\xi\eta)}\xi \end{pmatrix}$  becomes

$$\begin{pmatrix} e^{\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v+uv)}(\eta + v) \\ e^{-\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v+uv)}(\xi + u) \end{pmatrix}.$$

A straightforward computation on the skew-term of (64) shows that

$$\begin{aligned} & e^{\frac{i}{2}\alpha(\xi\eta)}\eta \cdot \left( e^{-\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v)} - e^{-\frac{i}{2}\alpha(\xi\eta)} \right) \xi + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi \cdot \left( e^{\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v)} - e^{\frac{i}{2}\alpha(\xi\eta)} \right) \eta \\ &= (\xi\eta) \cdot \left( -\frac{i}{2}\alpha'(\xi\eta)(\eta u + \xi v) \right) + (\xi\eta) \cdot \left( \frac{i}{2}\alpha'(\xi\eta)(\eta u + \xi v) \right) + O^2(\eta u + \xi v) \\ &= O^2(\eta u + \xi v) \end{aligned}$$

The cancellation shows that the skew-term property will propagate with the KAM step.

The detailed statement of the KAM scheme and the demonstrations are given in Section 4 and 7 of [114].

### 3. Further discussions

Theorem 4.3 elucidates the convergence to the normal form via conjugation on “most” sub-manifolds within the vicinity of the origin. These sub-manifolds are associated with a large family of parameters, and their invariance hints at the geometric characteristics of  $M$  defined in (55), as delineated in Theorem 4.1. Consequently, it arouses curiosity regarding the behavior of the involutions on other sub-manifolds, where such convergence is not demonstrated, along with their geometric implications. As highlighted in Chapter 1, this inquiry could unveil connections to chaotic dynamics, such as Arnold diffusion, and unearth rich geometric insights.

Moreover, the exploration into the geometry of higher-order holomorphic perturbations within higher-dimensional quadrics,

$$Q_{\gamma_1, \dots, \gamma_n} : z_{2k} = |z_{2k-1}|^2 + \gamma_k(z_{2k-1}^2 + \bar{z}_{2k-1}^2), \quad k = 1, \dots, n,$$

presents itself as particularly captivating. With the premise that at least one  $\gamma_k$  exceeds  $\frac{1}{2}$ , the geometric attributes and the dynamics of the corresponding holomorphic involutions of the quadric remain an open field of discovery.

## Local analytic rigidity of actions of isometries

Since the 1960s, the *linearization theory* for circle diffeomorphisms has stood out as a pivotal field within dynamical systems. This theory poses the question: given a circle diffeomorphism  $F = R_\alpha + f : \mathbb{T} \curvearrowright$  with  $f \in C^r(\mathbb{T})$ ,  $r = \mathbb{N}^* \cup \{\infty, \omega\}$  and  $R_\alpha : x \mapsto x + \alpha$  represents a rotation by angle  $\alpha \in \mathbb{T}$ , can  $F$  be conjugated to  $R_\alpha$  through some  $C^r$  transformation, that is, does there exist a transformation  $h \in C^r(\mathbb{T})$  such that  $F = h \circ R_\alpha \circ h^{-1}$ ?

The progress in this area significantly hinges on the arithmetic nature of the rotation angle  $\alpha$  and the rotation number the *rotation number*  $\rho$  of  $F$  defined as the uniform limit:

$$\rho(F) := \lim_{n \rightarrow \infty} \frac{\tilde{F}^n(x) - x}{n} \pmod{\mathbb{Z}}, \quad (65)$$

where  $\tilde{F} : \mathbb{R} \curvearrowright$  satisfies  $F \circ \Pi = \Pi \circ \tilde{F}$ ,  $\Pi(x) := x \pmod{\mathbb{Z}}$ . Introduced by Poincaré, the rotation number is a foundational concept in circle diffeomorphism theory, famously known for its existence as a limit that is independent of  $x \in \mathbb{R}$ .

For a single circle diffeomorphism, the first important result on the linearization is that of Arnold [2], who showed the *local* linearization of the analytic diffeomorphism  $F = R_\alpha + f$ , i.e.,  $f$  is supposed to be small in the sense of  $C^\omega$ , under the assumption that  $\rho(F) = \alpha$  satisfies the *Diophantine condition*: there exist  $c, \tau > 0$  s.t.

$$\inf_{l \in \mathbb{Z}} |n\alpha - l| \geq \frac{c}{|n|^\tau}, \quad \forall n \in \mathbb{Z} \setminus \{0\}.$$

Later Herman [65] proved a smooth version (as well as a differentiable version) of Arnold's local result under the same Diophantine condition, and also showed the optimality of the Diophantine condition. It is one of the great achievements of Herman [65] and Yoccoz [119] to have proved that, under the Diophantine condition, one can in fact get a *global* linearization result, i.e. no smallness assumption of  $f$  for  $F = R_\alpha + f$  (see also Khanin-Sinai [75] and Katznelson-Ornstein [73, 74]). The Diophantine condition is indeed not optimal in the analytic case. For the analytic circle diffeomorphism, Yoccoz [120] showed the local and global linearization under the optimal arithmetic conditions, i.e., the local linearization under the *Brjuno* condition, and the global linearizations under the *Herman* condition. Both conditions strictly contain the Diophantine condition. The readers can refer to [120] for more details, and to [44] for the contributions of Herman and Yoccoz made in this field.

Besides the linearization of a single circle diffeomorphism, it is also important to investigate the *simultaneous linearization* for finitely many commuting circle diffeomorphisms  $F_j = R_{\alpha_j} + f_j \in C^r(\mathbb{T})$ , i.e.,  $F_j = h \circ R_{\alpha_j} \circ h^{-1}$  for some  $h \in C^r(\mathbb{T})$ . It can be interpreted as the “rigidity” of an action of abelian group by rotations. For finitely many commuting smooth diffeomorphisms, Moser [104] showed the local simultaneous linearization under the assumption that  $\rho(F_j) = \alpha_j$  satisfies the *simultaneous Diophantine* condition: there exist  $c, \tau > 0$  s.t.

$$\inf_{l \in \mathbb{Z}} \max_j |n\alpha_j - l| \geq \frac{c}{|n|^\tau}, \quad \forall n \in \mathbb{Z} \setminus \{0\}. \quad (66)$$

Later, under the same assumption, the global simultaneous linearization was shown by Fayad-Khanin [49].

The advancements in the linearization of circle diffeomorphisms, now an integral part of rigidity theory, have evolved in tandem with the developments in KAM theory and renormalization theory. These progresses foster the expectation that rigidity, whether local or global, is attainable in broader contexts where KAM theory remains applicable.

In pursuit of this broader application, we explore a general compact real-analytic Riemannian manifold  $M$  (analogous to the circle) endowed with a group of isometries defined by a finite set of generators and relations, serving the function of rotations in this analogy. A significant challenge we face is the lack of a direct counterpart to the concept of rotation number, a pivotal element in KAM schemes for circle diffeomorphisms. Additionally, a more general form of the arithmetic conditions, which are naturally given through the angle of rotation when we consider the circle diffeomorphisms.

In this work, we introduce a tailored notion of ‘‘Diophantiness’’ for the isometries, intricately linked to the geometry and metric of the manifold  $M$ , with the spectrum of the Laplace-Beltrami operator playing a critical role. This redefined criterion, substituting the traditional ‘‘rotation number of the perturbation,’’ is articulated as the capacity to conjugate the perturbation arbitrarily closely to the original isometries. In other contexts, this might be expressed as the perturbation being *almost conjugated* to its unperturbed form. The essence is to demonstrate that one can effectively achieve a genuine analytic conjugacy between the unperturbed and the small enough perturbed actions.

**Key words:** Local rigidity, Group action by isometries, Grauert tube, Diophantine condition, Almost conjugation,

## 1. Actions of isometries on compact Riemannian manifold

Let  $G$  be a finitely presented group  $G$  with  $S = \{\gamma_1, \dots, \gamma_k\}$  the generators and  $R = \{\mathcal{W}_1, \dots, \mathcal{W}_p\}$  the relations. More precisely, each  $\mathcal{W}_i$  is a finite word in an alphabet of the  $k$  letters in  $S$  equal to  $e$ , the identity of  $G$ , i.e., for  $j = 1, \dots, p$ ,

$$e = \mathcal{W}_j = \gamma_{l_1^{(j)}} \cdots \gamma_{l_{m_j}^{(j)}}, \quad 1 \leq l_1^{(j)}, \dots, l_{m_j}^{(j)} \leq k.$$

Let  $(M, g)$  be a real-analytic compact Riemannian manifold with an analytic Riemannian metric  $g$ . Define the  $G$ -action by analytic isometries  $\pi : G \rightarrow \text{Isom}(M, g) \subset \text{Diff}^\omega(M)$ , i.e., for  $\gamma \in G$ ,  $\pi(\gamma)$  is an analytic diffeomorphism of  $M$  which preserves the distance induced by the Riemannian metric  $g$ . This action by isometries on  $M$  induces an action on the tangent bundle  $TM$ , and it gives rise to the Hochschild complex of cochains of  $L^2$  vector fields :

$$C^0(\Gamma, L^2(M, TM)) \xrightarrow{d_0} C^1(\Gamma, L^2(M, TM)) \xrightarrow{d_1} C^2(\Gamma, L^2(M, TM)) \xrightarrow{d_2} \dots \quad (67)$$

One can identify  $C^0(G, L^2(M, TM))$  with  $L^2(M, TM)$ ,  $C^1(G, L^2(M, TM))$  with maps from  $S$  to  $L^2(M, TM)$ , or equivalently  $L^2(M, TM)^k$ , and  $C^2(G, L^2(M, TM))$  with maps from  $R$  to  $L^2(M, TM)$ , or equivalently  $L^2(M, TM)^p$ . The homomorphisms  $d_0$  and  $d_1$  can be respectively defined as

$$d_0 v = (v - \pi(\gamma_l)_* v)_{1 \leq l \leq k}, \quad v \in L^2(M, TM),$$

$$d_1 V = \left( \sum_{1 \leq z \leq m_j} \left( D\pi \left( \prod_{i=1}^{z-1} \gamma_{l_i^{(j)}} \right) \cdot V_{l_z^{(j)}} \right) \circ \left( \prod_{i=z}^{m_j} \pi \left( \gamma_{l_i^{(j)}} \right) \right) \right)_{1 \leq j \leq p}, \quad V \in (L^2(M, TM))^k.$$

Both homomorphisms are defined once we have the  $G$ -action by isometries on  $M$ . For example, for a family of finitely many rotations on the circle  $M = \mathbb{T}$ ,

$$R_{\alpha_l} : \mathbb{T} \rightarrow \mathbb{T} \\ x \mapsto x + \alpha_l, \quad l = 1, \dots, m, \quad (68)$$

which is a  $\mathbb{Z}^m$ -actions by rotations, we define  $d_0$  and  $d_1$ , in the above sense, as

$$(d_0 v)(x) = (v(x) - v(x \pm \alpha_j))_j, \quad v \in L^2(\mathbb{T}),$$

$$(d_1 V)(x) = (V_i(x + \alpha_j) - V_i(x) - V_j(x + \alpha_i) + V_j(x))_{i,j}, \quad V \in L^2(\mathbb{T})^m,$$

where, based on the commutativity relation of  $\mathbb{Z}^m$ , the 1-cochain  $C^1(\mathbb{Z}^m, L^2(\mathbb{T}))$  can be simplified as  $L^2(\mathbb{T})^m$  and the 2-cochain  $C^2(\mathbb{Z}^m, L^2(\mathbb{T}))$  is identified with  $L^2(\mathbb{T}, \mathfrak{so}(n))$  (see [104]).

We then define the self-adjoint box operator  $\square : L^2(M, TM)^k \rightarrow L^2(M, TM)^k$ , which is fundamental for our purpose, by

$$\square = d_0 \circ d_0^* + d_1^* \circ d_1 : L^2(M, TM)^k \rightarrow L^2(M, TM)^k,$$

with the adjoint being defined upon the  $L^2$ -scalar product on  $M$ . Relating the spectral properties of  $\square$  to that of the Laplace-Beltrami operator on the tangent bundle  $\Delta_{TM}$ , the ‘‘Diophantine condition’’ of the  $G$ -action by isometries is defined as follows. According to Peter-Weyl decomposition [123]  $L^2(M, TM) = \bigoplus_{j \geq 0} E_{\lambda_j}$ , with  $\{\lambda_j\}_{j \in \mathbb{N}}$  the eigenvalues of  $|\Delta_{TM}|^{\frac{1}{2}}$  satisfying

$$0 = \lambda_0 < \lambda_1 < \dots \rightarrow \infty,$$

and  $E_{\lambda_j}$  the associated eigenspaces, we define  $\square_j = \square \circ \mathbb{P}_{E_{\lambda_j}^k}$ , the projection onto  $E_{\lambda_j}^k$ . The  $G$ -action by isometries  $\pi$  is called *Diophantine* if there exists  $\sigma > 0$  and  $\tau \geq 0$  such that, any non-zero eigenvalue  $\mu_j$  of  $\square_j$  satisfies

$$\mu_j \geq \frac{\sigma}{(1 + \lambda_j)^\tau}, \quad \forall j \in \mathbb{N}. \quad (69)$$

Similar definition was given by Dolgopyat [38] for subset of the group  $G$ . As a simple example, for the  $\mathbb{Z}^m$ -action by rotations (68), we have  $\lambda_j = j$  and  $\mu_j = 4 \sum_{l=1}^m \sin^2(\pi j \alpha_l)$ . It is shown by Moser [104] that the Diophantine condition (69) of this action by rotations is guaranteed by the simultaneous Diophantine assumption (66) of  $\{\alpha_j\}$ .

Consider a  $G$ -action by analytic diffeomorphisms  $\pi_0 : \Gamma \rightarrow \text{Diff}^\omega(M)$ , which is a perturbation of the  $G$ -action  $\pi$ , written as

$$\pi_0(\gamma) = \text{Exp}\{P_0(\gamma)\} \circ \pi(\gamma), \quad \gamma \in S,$$

with  $P_0(\gamma) \in L^2(M, TM)$  sufficiently small in a suitable sense, where  $\text{Exp}$  is the exponential map defined upon the Riemannian connection (see [64]). Given  $0 < \zeta < 1$ ,  $\pi_0$  is said to be  $\zeta$ -formally conjugate to  $\pi$  on  $M$ , if, for any  $\varepsilon > 0$ , there exists  $y^\varepsilon \in \text{Vect}^\omega(M, TM)$  with  $\|y^\varepsilon\|_{C^1} < \zeta$  such that

$$\text{Exp}\{y^\varepsilon\}^{-1} \circ \pi_0(\gamma) \circ \text{Exp}\{y^\varepsilon\} = \text{Exp}\{z^\varepsilon(\gamma)\} \circ \pi(\gamma), \quad \forall \gamma \in S,$$

for some  $z^\varepsilon : \Gamma \rightarrow \text{Vect}^\omega(M, TM)$  satisfying  $\|z^\varepsilon\|_{S, C^1} < \varepsilon$ , where  $\|\cdot\|_{C^1}$  denotes the  $C^1$ -norm on  $M$ , and  $\|\cdot\|_{S, C^1}$  denotes the sum of  $C^1$ -norms over the generators of  $G$  and their inverses. The main result on the analytic rigidity of the  $G$ -action by isometries on  $M$  is stated as follows.

**Theorem 5.1** (Stolovitch-Zhao [115]) Let  $\pi$  be a Diophantine  $G$ -action by analytic isometries on  $M$ . Assume that  $\dim \text{Ker} \square < +\infty$ . Let  $\pi_0$  be an analytic  $G$ -action by diffeomorphisms on  $M$  which is sufficiently close to  $\pi$ . If  $\pi_0$  is almost conjugated to  $\pi$ , then  $\pi_0$  is analytically conjugated to  $\pi$ .

As a particularly interesting example of local analytic rigidity, we have the following theorem, which can be seen as an analytic version of Fisher’s “local rigidity” result [52][Theorem 1.1] of Diophantine  $G$ -action by analytic isometries.

**Theorem 5.2** (Stolovitch-Zhao [115]) Let  $\pi$  be a Diophantine  $G$ -action by analytic isometries on  $M$  as above. Assume that the first cohomology group  $H^1(G, L^2(M, TM)) := \text{Ker } d_1 / \text{Im } d_0$  of the complex (67) vanishes. Then any small enough analytic perturbation  $\pi_0$  of  $\pi$  is analytically conjugate to  $\pi$ .

The following theorem, a corollary of Theorem 5.1, can be seen as an analytic version of results by Moser [104], Karaliolios [72] and Petkovic [108] relative to simultaneous conjugacy of a commutative family of perturbations of rotations on the torus to rotations. Let  $\mathcal{G} = \{e_1, \dots, e_m\}$  be the canonical basis of  $\mathbb{Z}^m$ .

**Theorem 5.3** (Stolovitch-Zhao [115]) Let  $\pi$  be Diophantine  $\mathbb{Z}^m$ -action by rotations on the torus  $\mathbb{T}^d$ : Let  $\alpha_i \in \mathbb{R}^d$  be the rotation vector of the rotation  $\pi(e_i)$ . Assume there exist  $c, \tau > 0$ , such that for all  $(\mathbf{k}, l) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{0\}$ ,

$$\max_{1 \leq i \leq m} |\langle \mathbf{k}, \alpha_i \rangle - l| \geq \frac{c}{|\mathbf{k}|^\tau}.$$

Then any small enough analytic perturbation  $\pi_0$  (isotopic to Id) of  $\pi$  such that, for each  $i$ , the rotation vector  $\alpha_i$  belongs to the convex hull of rotation set of  $\pi(e_i)$ , is analytically conjugate to  $\pi$ .

## 2. KAM scheme for group actions

For the  $G$ -action  $\pi_0$  by analytic diffeomorphisms, a perturbation of the  $G$ -action by isometries  $\pi$ , written as

$$\pi_0(\gamma) = \text{Exp}\{P_0(\gamma)\} \circ \pi(\gamma), \quad \gamma \in S,$$

where  $P_0 : S \rightarrow H_{r_0}^2(M, TM)$ , the *Hardy space* on the *Grauert tube*  $M_{r_0}$  for some sufficiently small  $r_0$ , with sufficiently small

$$\|P_0\|_{S, H_{r_0}^2} = \sum_{\gamma \in G} \|P_0(\gamma)\|_{H_{r_0}^2} =: \varepsilon_0.$$

Theorem 5.1 and 5.2 are shown through a KAM scheme for the  $G$ -action  $\pi_0 = \text{Exp}\{P_0\} \circ \pi$ . This relies on a “Fourier analysis” on the manifold  $M$ , as well as the Grauert tube  $M_{r_0}$ , drawing upon the foundational work of Boutet de Monvel [24] (see also [83]).

For a given decaying positive sequence  $\{r_j\}$ , with  $r_j \rightarrow \frac{r_0}{2}$  as  $j \rightarrow \infty$ , suppose that we have arrived at the  $j$ -th KAM step with the  $G$ -action by analytic diffeomorphisms

$$\pi_j(\gamma) = \text{Exp}\{P_j(\gamma)\} \circ \pi(\gamma), \quad \gamma \in S,$$

where  $P_j : S \rightarrow H_{r_j}^2(M, TM)$  with  $\|P_j\|_{S, H_{r_j}^2} < \varepsilon_0^{(\frac{5}{4})^j}$ . The aim of this KAM step is to construct a near-identity  $\text{Exp}\{w_j\} \in \text{Diff}^\omega(M)$  with some  $w_j \in H_{r_{j+1}}^2(M, TM)$  such that

$$\text{Exp}\{w_j\}^{-1} \circ \text{Exp}\{P_j(\gamma)\} \circ \pi(\gamma) \circ \text{Exp}\{w_j\} = \text{Exp}\{P_{j+1}(\gamma)\} \circ \pi(\gamma), \quad \forall \gamma \in S,$$

with  $P_{j+1} : S \rightarrow H_{r_{j+1}}^2(M, TM)$  with  $\|P_{j+1}\|_{S, H_{r_{j+1}}^2} < \varepsilon_0^{(\frac{5}{4})^{j+1}}$ . As in many classical KAM theorems, the construction of  $w_j$  relies on the elimination of the truncation of

$$(P_j(\gamma))_{\gamma \in G} \in (H_{r_j}^2(M, TM))^k \subset (L^2(M, TM))^k,$$

which is naturally divided into two parts according to the orthogonal decomposition

$$(L^2(M, TM))^k = \text{Ker} \square \oplus \text{Im} \square.$$

For the part in  $\text{Im}\square$ , which is usually regarded as the sum of “non-resonant terms”, the invertibility of the box operator  $\square$  and the Diophantine condition (69) provide the existence and suitable estimation of  $y_j$ .

As for the part in  $\text{Ker}\square$ , the sum of “resonant terms”, it is not expected to be eliminated through some conjugation. The main idea is to show that this part of  $(P_j(\gamma))_{\gamma \in G}$  is indeed much smaller than the total size of  $P_j$ . This is realized under the hypothesis that  $\pi_0$  is formal conjugated to  $\pi$  and  $\dim \text{Ker}\square < \infty$  (as in Theorem 5.1) or that the first cohomology group  $H^1(G, L^2(M, TM))$  vanishes (as in Theorem 5.2). In fact, both hypotheses play the similar role of the assumption that “the rotation number equals to the angle of rotation” in the KAM scheme for circle diffeomorphisms.

### 3. Further discussions

This local result may pave the way toward understanding global rigidity on the manifold. The study of rigidity theory for diffeomorphisms on compact manifolds remains a pivotal area in the field of dynamical systems today, as highlighted in various studies [50, 76]. Broadly, the challenge lies in identifying conditions under which two topologically conjugated maps are also conjugated smoothly or analytically, without necessitating their proximity to each other.

A landmark in this domain is Herman’s theorem [65], which addresses the linearization of smooth circle diffeomorphisms characterized by Diophantine rotation numbers. Currently, conclusive results on global rigidity are confined to the one-dimensional (1-D) scenario. Khanin’s conjecture, proposed at ICM 2018 [76], suggests that global rigidity extends beyond these parameters. Although there have been advancements [36, 39], achieving a comprehensive understanding that aligns with Khanin’s conjecture remains an ongoing endeavor.



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