Algebras of partial functions

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Overview of talk

- Introductory part

- Complete representations
- Finite representation property
- Multiplace functions
- Partial operations (from separation logic)

Definitions

Definition

A **partial function** on a set X is a subset f of $X \times X$ satisfying

$$(x, y) \in f \text{ and } (x, z) \in f \implies y = z$$

There are various 'concrete' operations on partial functions (composition, intersection...)

Definition

An **algebra of partial functions** of the signature σ is:
an algebra of the signature σ whose

- elements are partial functions on some set X
- symbols are interpreted as the intended operations

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Definition

Let $\mathfrak{A}$ be an algebra of the signature σ.
A **representation** of $\mathfrak{A}$ is a isomorphism from $\mathfrak{A}$ to an algebra of partial functions

Operations

Composition

$$f\,;g = \{(x,z) \in X^2 \mid \exists y \in X : (x,y) \in f \text{ and } (y,z) \in g\}$$

Intersection

$$f \cdot g = \{(x,y) \in X^2 \mid (x,y) \in f \text{ and } (x,y) \in g\}$$

Domain

$$\mathrm{D}(f) = \{(x,x) \in X^2 \mid \exists y \in X : (x,y) \in f\}$$

Range

$$\mathrm{R}(f) = \{(y,y) \in X^2 \mid \exists x \in X : (x,y) \in f\}$$

Zero $0 = \emptyset$

Identity $1' = \{(x,x) \in X^2\}$

Antidomain $\mathrm{A}(f) = \{(x,x) \in X^2 \mid \nexists y \in X : (x,y) \in f\}$

Preferential union

$$(f \sqcup g)(x) = \begin{cases} f(x) & \text{if } f(x) \text{ defined} \\ g(x) & \text{if } f(x) \text{ undefined, but } g(x) \text{ defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Questions asked

About the class of representable algebras:

- axiomatisable by first-order logic?
- simplest fragment for axiomatisation? (equations? quasiequations? universal sentences?)
- same questions for *finite* axiomatisations
- is the equational theory decidable? what is the computational complexity?
- same for quasiequational theory, etc.

About finite algebras:

- is representability decidable? (what is the computational complexity?)

Complete representation for $\{;, \cdot, A\}$

Definition

A representation θ of $\mathfrak{A}$ is **join complete** if for any $S \subseteq \mathfrak{A}$

$$\bigvee S \text{ exists} \implies \theta(\bigvee S) = \bigcup \theta[S]$$

The representation is **meet complete** if for any *nonempty* $S \subseteq \mathfrak{A}$

$$\bigwedge S \text{ exists} \implies \theta(\bigwedge S) = \bigcap \theta[S]$$

Not always equivalent

Boolean/relation algebras: join complete $\equiv$ meet complete
bounded distributive lattices: join complete $\not\equiv$ meet complete

Complete representation for $\{;, \cdot, \mathsf{A}\}$

Theorem

The class of $\{;, \cdot, \mathsf{A}\}$-algebras completely representable by partial functions is axiomatised by an $\forall\exists\forall$-sentence, but not by any $\exists\forall\exists$-theory

Axiomatisation has three parts

- equational axiomatisation of (plain) representability
- assertion that algebra is atomic
- assertion that for any a, b, c

$$c \geq a \,;\, x \text{ for all atoms } x \leq b \quad \Longrightarrow \quad c \geq a \,;\, b$$

Non-axiomatisability part proved using three-round back-and-forth game on two Boolean algebras

Complete representation for $\{;, \cdot, A\}$

Definition

A representation θ with base X is **atomic** if

$$\text{for all } x \in X \text{ there is an atom } a \text{ with } x \in \theta(a)$$

For representations: complete $\equiv$ atomic

For an algebra, having an atomic representation implies the algebra is atomic

but (in this case) it is strictly stronger...

Complete representation for $\{;, \cdot, \mathsf{A}\}$

is representable, **is** atomic, **no** atomic representation

Example

The following concrete algebra of partial functions, $\mathfrak{F}$.

Base: disjoint union of a one element set, $\{p\}$, and $\mathbb{N}_\infty \coloneqq \mathbb{N} \cup \{\infty\}$

Let $\mathcal{S}$ be all the subsets of $\mathbb{N}_\infty$ that are either finite and do not contain ∞, or cofinite and contain ∞.

The elements of $\mathfrak{F}$ are:

1. Restrictions of the identity to $A \cup B$ where $A \subseteq \{p\}$ and $B \in \mathcal{S}$.
2. The function f, defined only on p and taking p to ∞.

Complete representation for $\{;, \cdot, \mathsf{A}\}$

The representation:
For each $a \in \mathfrak{A}$, let $\theta(a)$ be the following partial function on $\mathsf{At}(\mathfrak{A})$.

$$\theta(a)(x) = \begin{cases} x \mathbin{;} a & \text{if } x \mathbin{;} a \neq 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then θ is a complete representation of $\mathfrak{A}$ by partial functions, with base $\mathsf{At}(\mathfrak{A})$.

Finite representation property for $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$

Definition

(For a specified notion of representation) a signature has the **finite representation property** if

every finite and representable algebra has a representation on a finite base

Example

For representation by *binary relations*, the relation algebra signature does *not* have the finite representation property.

Refuted by Tarski's 'point algebra'

Finite representation property for $\{;, \cdot, \mathsf{D}, \mathsf{R}\}$

Theorem

For representation by partial functions the signature $\{;, \cdot, \mathsf{D}, \mathsf{R}\}$ *has the finite representation property*

signatures without R are easy
Hirsch, Jackson, and Mikulás (2016) gave positive answer for $\{;, \mathsf{D}, \mathsf{R}\}$
...and posed the question for $\{;, \cdot, \mathsf{D}, \mathsf{R}\}$

Finite representation property for $\{;, \cdot, \mathrm{D}, \mathrm{R}\}$

The proof:

1. view representation as edge-labelled graph
2. show label of reflexive edge determines the ‘present’ and ‘future’ of the point
3. construct finite representation inductively from these pieces, working from latest to earliest
   —make sure enough is added at each level to ensure the induction goes through!

Multiplace functions

Definition

An n-ary partial function f is a subset of $X^{(n+1)}$ satisfying

$$(x_1, \ldots, x_n, y) \in f \text{ and } (x_1, \ldots, x_n, z) \in f \implies y = z$$

Intersection, preferential union, zero

as usual

Composition $\langle\ \rangle;$ (n+1)-ary

$\boldsymbol{f} \,;\, g = \{(\boldsymbol{x}, z) \in X^{n+1} \mid \exists \boldsymbol{y} \in X^n : (\boldsymbol{x}, y_i) \in f_i \text{ for each } i \text{ and } (\boldsymbol{y}, z) \in g\}$

Domain

$\mathrm{D}_i(f) = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid \exists y \in X : (\boldsymbol{x}, y) \in f\}$

Identity

$\pi_i = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid \exists B \in P :\ x_1, \ldots, x_n \in B\}$

Antidomain

$\mathrm{A}_i(f) = \{(\boldsymbol{x}, x_i) \in X^{n+1} \mid \exists B \in P :\ x_1, \ldots, x_n \in B \text{ and } \nexists y \in X : (\boldsymbol{x}, y) \in f\}$

Multiplace functions

Theorem

- *For* $\{\langle\ \rangle ;, \mathsf{A}_i\}$
  the class of algebras representable by n-ary partial functions is axiomatised by a finite number of quasiequations
- *For* $\{\langle\ \rangle ;, \mathsf{A}_i, \cdot\}$, $\{\langle\ \rangle ;, \mathsf{A}_i, \sqcup\}$, $\{\langle\ \rangle ;, \mathsf{A}_i, \cdot, \sqcup\}$,
  the class of algebras representable by n-ary partial functions is axiomatised by a finite number of equations

(for $\{\langle\ \rangle ;, \mathsf{A}_i\}$ the representation class is a *proper* quasivariety)

Multiplace functions

Assuming $\mathfrak{A}$ validates the relevant axioms

Lemma

Let U be an ultrafilter of A*-elements of* $\mathfrak{A}$
Write $[a]$ for the $\sim_U$-equivalence class of an element $a \in \mathfrak{A}$.
Let $X := \{[a] \mid a \in \mathfrak{A}\} \setminus \{[0]\}$ and for each $b \in \mathfrak{A}$ let $\theta_U(b)$ be the partial function from X^n to X given by

$$\theta_U(b)\colon ([a_1], \ldots, [a_n]) \mapsto \begin{cases} [\langle a_1, \ldots, a_n\rangle \,;\, b] & \text{if this is not equal to } [0] \\ \text{undefined} & \text{otherwise} \end{cases}$$

Then the image of θ is an algebra of n-ary partial functions and θ is a homomorphism of $\{\langle\ \rangle ;, \mathsf{A}_i\}$-algebras

If a is inequivalent to both 0 *and b then θ_U separates a from b.*

Multiplace functions

Theorem

For each of

- $\{\langle\ \rangle;, \mathsf{A}_i\}$
- $\{\langle\ \rangle;, \mathsf{A}_i, \cdot\}$
- $\{\langle\ \rangle;, \mathsf{A}_i, \sqcup\}$
- $\{\langle\ \rangle;, \mathsf{A}_i, \cdot, \sqcup\}$

the equational theory of the algebras representable by n-ary partial functions is coNP*-complete*

Proof idea:
show that when an equation $s = t$ is refuted on an algebra $\mathfrak{F}$ of partial functions
by an assignment $f_1, \ldots, f_m$ to the variables in the equation
we can restrict the base of $\mathfrak{F}$ to a set of size linear in the length of $s = t$
and the algebra generated by the restrictions of $f_1, \ldots, f_m$ still refutes $s = t$

Partial operations from separation logic

Separating conjunction $*$

$h, s \models \varphi * \psi$ if and only if there exist h_1, h_2 with disjoint domains, such that $h = h_1 \cup h_2$ and $h_1, s \models \varphi$ and $h_2, s \models \psi$

Definition

Given two partial functions f and g the **domain-disjoint union** $f \overset{\bullet}{\smile} g$ equals $f \cup g$ if the domains of f and g are disjoint, else it is undefined.

Given two sets S and T the **disjoint union** $S \overset{\bullet}{\cup} T$ equals $S \cup T$ if $S \cap T = \emptyset$, else it is undefined

Separating implication $-*$

Definition

The **subset complement** $S \overset{\bullet}{\setminus} T$ equals $S \setminus T$ if $T \subseteq S$, else it is undefined

$h, s \models \varphi -* \psi$ if and only if for all h_1, h_2 such that $h = h_2 \overset{\bullet}{\setminus} h_1$ we have $h_1, s \models \varphi$ implies $h_2, s \models \psi$.

Partial operations from separation logic

Note: we insist 'concrete' algebras are closed under any partial operations. E.g. in a $\overset{\bullet}{\cup}$-partial algebra of sets $\mathfrak{A}$, if $S, T \in \mathfrak{A}$ and $S \overset{\bullet}{\cup} T$ exists, then $S \overset{\bullet}{\cup} T \in \mathfrak{A}$

Theorem

For the signatures $(\overset{\bullet}{\cup})$, $(\overset{\bullet}{\setminus})$, *and* $(\overset{\bullet}{\cup}, \overset{\bullet}{\setminus})$, *the class of partial algebras representable as sets is first-order axiomatisable*

For the signatures $(\overset{\bullet}{\smile})$, $(\overset{\bullet}{\setminus})$, *and* $(\overset{\bullet}{\smile}, \overset{\bullet}{\setminus})$, *the class of partial algebras representable as partial functions is first-order axiomatisable*

Theorem

None of the above classes is finitely axiomatisable

Partial operations from separation logic

The axiomatisability proofs:

1. show the class is pseudoelementary
2. show the class is closed under ultraroots

The non-axiomatisability proofs:
—show the complement class is not closed under ultraproducts

1. describe a sequence of algebras
2. show each is not representable
3. show an ultraproduct of them *is* representable